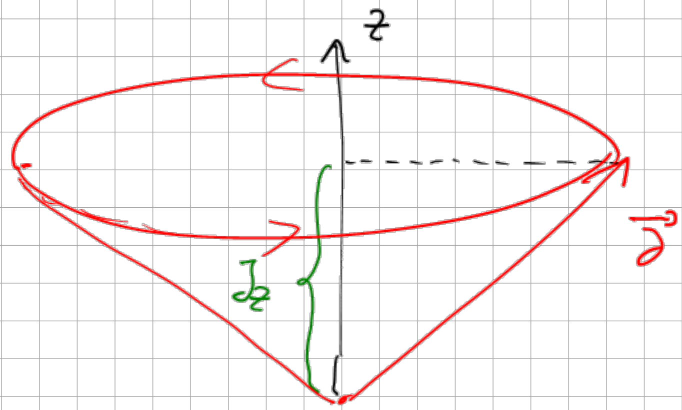
Summary:

semiclassical vector-model of angular momenta:

- classically: all angular moment possible
- quantum mechanically: two quantization conditions
  > length: $|\vec{J}| = \hbar\sqrt{J(J+1)}$
  > projection upon z-axis: $J_z = m\hbar$
- arise from $[\hat{\vec{J}}^2, \hat{J}_z]_- = 0$

  but: $[\hat{J}_j, \hat{J}_k]_- = i\hbar\,\varepsilon_{jkl}\,\hat{J}_l$



$\vec{J}$ is precessing around z-axis with fixed J_z projection

11.4 Matrix Elements:

assumption: $|j,m\rangle$ with $\hat{\vec{J}}^2|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle$

$\hat{J}_z|j,m\rangle = \hbar m\,|j,m\rangle$

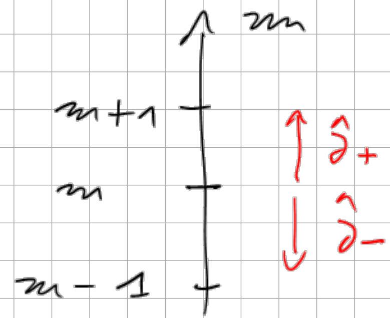
$\Rightarrow$ orthonormality: $\langle j',m'|j,m\rangle = \delta_{jj'}\,\delta_{m'm}$

$\langle j',m'|\hat{\vec{J}}^2|j,m\rangle = \hbar^2 j(j+1)\,\delta_{jj'}\,\delta_{mm'}$

$\langle j',m'|\hat{J}_z|j,m\rangle = \hbar m\,\delta_{jj'}\,\delta_{mm'}$

now: $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$

$\hat{J}_\pm|j,m\rangle \sim |j,m\pm 1\rangle$



$\rightarrow$ now: normalization constants determined

$$\langle j,m|\underbrace{\hat{J}_+^{+}}_{=\hat{J}_-}\hat{J}_+|j,m\rangle \overset{\text{last time}}{=} \langle j,m|\hat{\vec{J}}^2-\hat{J}_z^2-\hbar\hat{J}_z|j,m\rangle = \hbar^2\{j(j+1)-m^2-m\}$$

$$\hat{J}_+|j,m\rangle = c^{(+)}_{jm}|j,m+1\rangle \Rightarrow \langle j,m|\hat{J}_+^{+}\hat{J}_+|j,m\rangle = |c^{(+)}_{jm}|^2$$

$$\Rightarrow c^{(+)}_{jm} = \hbar\sqrt{(j-m)(j+m+1)}\cdot\underbrace{e^{i\alpha}}_{=1}$$

analogously: $\hat{J}_-|j,m\rangle = c^{(-)}_{jm}|j,m-1\rangle \Rightarrow c^{(-)}_{jm} = \hbar\sqrt{(j+m)(j-m+1)}\,\underbrace{e^{i\alpha}}_{=1}$

$$\Rightarrow \hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm1\rangle$$

$$\langle j',m'|\hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\;\delta_{j,j'}\,\delta_{m',m\pm1}$$

11.5 Example: Spin 1/2

$j=\frac{1}{2}$, $m=\pm\frac{1}{2}$ $\Rightarrow$ basis states $|\underbrace{\tfrac{1}{2}}_{=j},\underbrace{\pm\tfrac{1}{2}}_{=m}\rangle$

non-vanishing matrix elements:

$$\langle \tfrac{1}{2},\tfrac{1}{2}|\hat{J}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar \qquad \hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad \hat{J}_x = \tfrac{1}{2}(\hat{J}_+ + \hat{J}_-)$$

$$\langle \tfrac{1}{2},-\tfrac{1}{2}|\hat{J}_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar \qquad \hat{J}_- = \hat{J}_x - i\hat{J}_y \quad\Rightarrow\quad \hat{J}_y = \tfrac{1}{2i}(\hat{J}_+ - \hat{J}_-)$$

$$\langle \tfrac{1}{2},\pm\tfrac{1}{2}|\hat{J}_z|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle = \pm\frac{\hbar}{2}$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|,\ \langle\tfrac{1}{2},-\tfrac{1}{2}|\right)\begin{pmatrix}\tfrac{1}{2}(\hat{J}_+ + \hat{J}_-)\\ \tfrac{1}{2i}(\hat{J}_+ - \hat{J}_-)\\ \hat{J}_z\end{pmatrix}\begin{pmatrix}|\tfrac{1}{2},\tfrac{1}{2}\rangle\\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle\end{pmatrix}$$

$$
= \begin{pmatrix}
\begin{pmatrix} \langle \tfrac12, \tfrac12 | \tfrac12 (\cancel{\hat J_+} + \cancel{\hat J_-}) | \tfrac12, \tfrac12 \rangle & \langle \tfrac12, \tfrac12 | \tfrac12 (\hat J_+ + \cancel{\hat J_-}) | \tfrac12, -\tfrac12 \rangle \\ \langle \tfrac12, -\tfrac12 | \tfrac12 (\cancel{\hat J_+} + \hat J_-) | \tfrac12, \tfrac12 \rangle & \langle \tfrac12, -\tfrac12 | \tfrac12 (\cancel{\hat J_+} + \cancel{\hat J_-}) | \tfrac12, -\tfrac12 \rangle \end{pmatrix} \\
\begin{pmatrix} 0 & \langle \tfrac12, \tfrac12 | \tfrac{1}{2i} (-\hat J_-) | \tfrac12, -\tfrac12 \rangle \\ \langle \tfrac12, -\tfrac12 | \tfrac{1}{2i} (-\hat J_-) | \tfrac12, \tfrac12 \rangle & 0 \end{pmatrix} \\
\begin{pmatrix} \langle \tfrac12, \tfrac12 | \hat J_z | \tfrac12, \tfrac12 \rangle & 0 \\ 0 & \langle \tfrac12, -\tfrac12 | \hat J_z | \tfrac12, -\tfrac12 \rangle \end{pmatrix}
\end{pmatrix}
$$

$$
= \frac{\hbar}{2} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \frac{\hbar}{2} \vec\sigma = \vec S \quad \text{(spin vector operator)}
$$

analogous procedure for spin 1 yields

$$
\hat J_x = \frac{\hbar}{\sqrt2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat J_y = \frac{i\hbar}{\sqrt2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

1.1.6 Formal Theory of Angular Momentum Addition:

Consider two angular momentum operators $\vec J_1$, $\vec J_2$ acting on different Hilbert spaces $\mathcal H_1$, $\mathcal H_2$:

- $[\hat J_{1j}, \hat J_{1k}]_- = i\hbar \varepsilon_{jkl} \hat J_{1l}$, $\quad [\hat J_{2j}, \hat J_{2k}]_- = i\hbar \varepsilon_{jkl} \hat J_{2l}$ $\quad (1)$

- independence: $[\hat{J}_{1j}, \hat{J}_{2k}]_- = 0$ (2)

total angular momentum operator: $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$

(1) + (2): $[\hat{J}_j, \hat{J}_k]_- = i\hbar\, \varepsilon_{jkl}\, \hat{J}_l$

Eigenvalue discussion with ladder operators applies also here to $\hat{\vec{J}}_1^2, \hat{J}_{1z}$; $\hat{\vec{J}}_2^2, \hat{J}_{2z}$; $\hat{\vec{J}}^2, \hat{J}_z$. This leads to two sets of basis states:

commute among themselves

Option A: simultaneous eigenstates of $\hat{\vec{J}}_1^2, \hat{\vec{J}}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$ denoted by

$$|j_1, j_2; m_1, m_2\rangle = \underbrace{|j_1, m_1\rangle}_{\in \mathcal{H}_1} \underbrace{|j_2, m_2\rangle}_{\in \mathcal{H}_2}$$

fixed j_1, j_2: $\dim(\mathcal{H}_1) = 2j_1 + 1$, $\dim(\mathcal{H}_2) = 2j_2 + 1$

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1$: $\dim \mathcal{H} = (2j_1 + 1)\cdot(2j_2 + 1)$

$\hat{\vec{J}}_k^2 |j_1, j_2; m_1, m_2\rangle = \hbar^2 j_k(j_k+1) |j_1, j_2; m_1, m_2\rangle$; $\hat{J}_{kz} |j_1, j_2; m_1, m_2\rangle = \hbar m_k |j_1, j_2; m_1, m_2\rangle$

for $k = 1, 2$

commute among themselves

Option B: simultaneous eigenstates of $\hat{\vec{J}}_1^2, \hat{\vec{J}}_2^2, \hat{\vec{J}}^2, \hat{J}_z$ denoted by

$$|j_1, j_2; j, m\rangle$$

$\hat{\vec{J}}_k^2 |j_1, j_2; j, m\rangle = \hbar^2 j_k(j_k+1) |j_1, j_2; j, m\rangle$, $k = 1, 2$

$\hat{\vec{J}}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) |j_1, j_2; j, m\rangle$; $\hat{J}_z |j_1, j_2; j, m\rangle = \hbar m |j_1, j_2; j, m\rangle$

Note: $\hat{\vec{J}}^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + \underbrace{\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2 + \hat{\vec{J}}_2 \cdot \hat{\vec{J}}_1}$

$= 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1x}\hat{J}_{2x} + \hat{J}_{1y}\hat{J}_{2y} + \hat{J}_{2x}\hat{J}_{1x} + \hat{J}_{2y}\hat{J}_{1y}$

$$= \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$[\hat{\vec{J}}^2, \hat{J}_{1z}]_- = \ldots = \hbar\{-\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}\} \neq 0$$

$$[\hat{\vec{J}}^2, \hat{J}_{2z}]_- = \ldots = \hbar[+\hat{J}_{1+}\hat{J}_{2-} - \hat{J}_{1-}\hat{J}_{2+}]$$

$$[\hat{\vec{J}}^2, \hat{J}_z]_- = 0$$

$\Rightarrow$ We can not add $\hat{\vec{J}}^2$ to option A

$\Rightarrow$ We can not add $\hat{J}_{1z}$ or $\hat{J}_{2z}$ to option B

Result: Two basis sets of eigenstates to two maximal sets of mutually compatible observables.

$|j_1, j_2; m_1, m_2\rangle$ and $|j_1, j_2; j, m\rangle$ represent two orthonormal basis states of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$\Rightarrow \quad |j_1, j_2; j, m\rangle = \sum_{m_1, m_2} \underbrace{\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}} |j_1, m_1; j_2, m_2\rangle$$

Clebsch – Gordan coefficients represent a unitary transformation between both bases

$$\sum_{m_1, m_2} |j_1, m_1; j_2, m_2\rangle\langle j_1, m_1; j_2, m_2| = 1$$

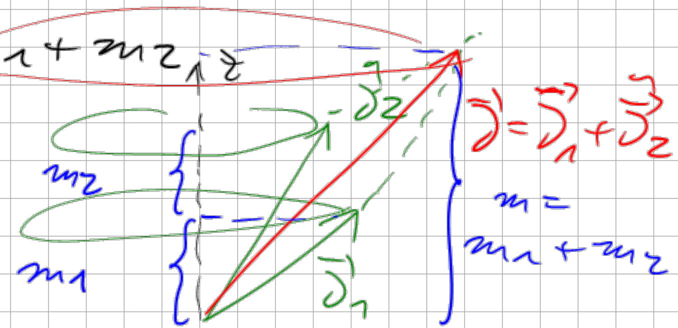
11.7 Properties of Clebsch-Gordan Coefficients:

1) $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} \quad \Rightarrow \quad (\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z})|j_1, j_2; j, m\rangle = 0 \quad \Big| \langle j_1, m_1; j_2, m_2| \cdot$

$$\Rightarrow \quad (m - m_1 - m_2)\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle = 0$$

$\neq 0$ $\qquad\qquad\qquad\qquad$ $= 0$

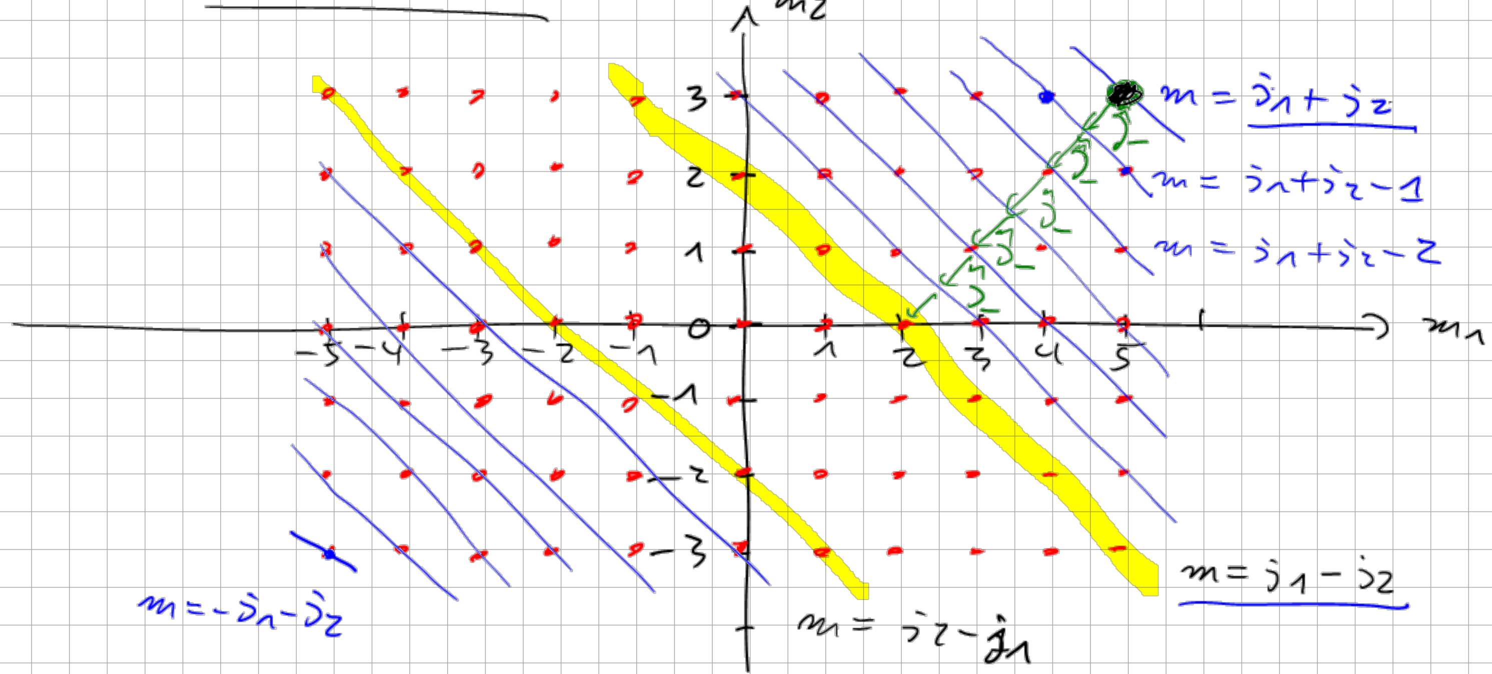
Clebsch-Gordan coefficients vanish unless $m = m_1 + m_2$



2) degeneracy of m is determined by all pairs (m_1, m_2) with $m = m_1 + m_2$

Assume without loss of generality $j_1 \geq j_2$

Here: $\underline{j_1 = 5,\ j_2 = 3}$



$j_1 = 5,\quad j_2 = 3$

j	$2j+1$
8	17
7	15
6	13
5	11
4	9
3	7
2	5

$$77 = \underbrace{11}_{=2\cdot 5+1} \cdot \underbrace{7}_{=2\cdot 3+1}$$

$j = |j_1 - j_2|, \ldots, j_1 + j_2$ triangle rule

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

1) maximal value: $m_{max} = j_1 + j_2$ $\quad |j_1, j_2; j, \underbrace{j_1+j_2}_{= m_{max}}\rangle = |j_1, j_2; \underbrace{j_1}_{m_1}\, \underbrace{j_2}_{m_2}\rangle$

$$\hat J_{k+} |j_1, j_2; j, j_1+j_2\rangle = 0 \qquad k = 1, 2$$

$$\hat{\vec J}^2 = \hat{\vec J}_1^2 + \hat{\vec J}_2^2 + 2\hat J_{1z}\hat J_{2z} + \cancel{\hat J_{1+}\hat J_{2-}} + \cancel{\hat J_{1-}\hat J_{2+}}$$

$$\hat{\vec J}^2 |j_1, j_2; j, j_1+j_2\rangle = \hbar^2 j(j+1) |j_1, j_2; j, j_1+j_2\rangle$$

$$= \hbar^2 \underbrace{\{ j_1(j_1-1) + j_2(j_2-1) + 2 j_1 j_2 \}}_{= (j_1+j_2)(j_1+j_2-1)} |j_1, j_2; j, j_1+j_2\rangle \qquad \Rightarrow \quad j = j_1 + j_2$$

$\Rightarrow \quad j = j_1 + j_2, \quad m = j_1 + j_2 = j \; : \; |j_1, j_2; j_1+j_2, j_1+j_2\rangle = |j_1, j_2; j_1, j_2\rangle$

2) successive applications of $\hat J_-$ to that state yield $2j+1$ eigenstates

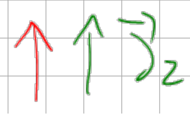
$$|j_1, j_2; \underbrace{j_1+j_2}_{= j}, m\rangle \sim \hat J_-^{\,j-m} |j_1, j_2; \underbrace{j_1+j_2}_{= j}, \underbrace{j_1+j_2}_{= m}\rangle \quad ; \quad -j \leq m \leq +j$$

$\Rightarrow$ multiplet with angular momentum $j = j_1 + j_2$

3) which states are left? The largest magnetic quantum number left is $m_{max} = j_1 + j_2 - 1 \Rightarrow$ corresponding multiplet with $j = j_1 + j_2 - 1$

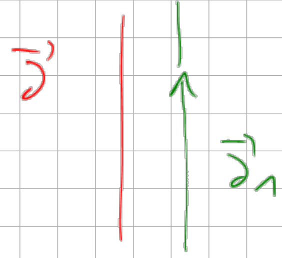
4) $j = j_1 + j_2, \; j_1 + j_2 - 1, \; j_1 + j_2 - 2, \; \text{-----}, \; j_{min} = |j_1 - j_2|$

check for degeneracy $(j_1 \supset j_2)$

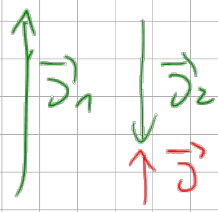


$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = \left(\sum_{j=1}^{j_1+j_2} - \sum_{j=1}^{j_1-j_2-1} \right) (2j+1)$$

$$= \dots = (2j_1+1)(2j_2+1) \checkmark$$



$j_1 + j_2$



$j_1 - j_2$