addition of angular momenta: $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} \underbrace{\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}} |j_1, m_1; j_2, m_2\rangle$$

Major properties: 1) $m = m_1 + m_2$

2) $\underset{\text{triangular rule}}{j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2\}}$

From now on: j_1, j_2 omitted in notation as they are fixed

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \langle m_1, m_2 | j, m\rangle \, |m_1, m_2\rangle$$

11.8 Proton/Neutron Built from Quarks:

quarks are spin 1/2 particles

generation \ charge	$-\frac{e}{3}$	$+\frac{2}{3}e$
1	d (down)	u (up)
2	s (strange)	c (charm)
3	b (bottom)	t (top)

$\boxed{M_d \approx M_u}$

Quarks are constituents of proton / neutron (= nucleon):

$$|p\rangle = |u\,u\,d\rangle \;\xleftrightarrow[\text{exchange}]{u \leftrightarrow d}\; |n\rangle = |d\,d\,u\rangle$$

Problem to solve: how to add three spin 1/2 particles?

First attempt:

$$\left.\begin{matrix} \left.\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix}\right\} 0 \\ \frac{1}{2} \end{matrix}\right\} \frac{1}{2} \qquad \text{or} \qquad \left.\begin{matrix} \left.\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix}\right\} 1 \\ \frac{1}{2} \end{matrix}\right\} \frac{1}{2} \text{ or } \frac{3}{2}$$

$\underline{j_1 = \frac{1}{2},\ j_2 = \frac{1}{2}:}$ $\quad j_{12} = 0$ or 1

$$\left.\begin{aligned} &|j_{12}=1,\ m_{12}=1\rangle = 1\cdot\left|m_1=\tfrac{1}{2},\ m_2=\tfrac{1}{2}\right\rangle \quad (1)\\ &|j_{12}=1,\ m_{12}=-1\rangle = 1\cdot\left|m_1=-\tfrac{1}{2},\ m_2=-\tfrac{1}{2}\right\rangle \\ &|j_{12}=1,\ m_{12}=0\rangle = \tfrac{1}{\sqrt{2}}\left\{\left|m_1=\tfrac{1}{2},\ m_2=-\tfrac{1}{2}\right\rangle + \left|m_1=-\tfrac{1}{2},\ m_2=\tfrac{1}{2}\right\rangle\right\} \quad (2) \end{aligned}\right\} \text{Triplett}$$

$$|j_{12}=0,\ m_{12}=0\rangle = \tfrac{1}{\sqrt{2}}\left\{\left|m_1=\tfrac{1}{2},\ m_2=-\tfrac{1}{2}\right\rangle - \left|m_1=-\tfrac{1}{2},\ m_2=\tfrac{1}{2}\right\rangle\right\} \quad \Big\}\ \text{Singulett}$$

$\underline{j_{12} = 1,\ j_3 = \frac{1}{2}:}$ $\quad j = \frac{1}{2}$ or $j = \frac{3}{2}$

$$\left|j=\tfrac{1}{2},\ m=\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,\Big|\underbrace{m_{12}=1}_{(1)},\ m_3=-\tfrac{1}{2}\Big\rangle - \tfrac{1}{\sqrt{3}}\,\Big|\underbrace{m_{12}=0}_{(2)},\ m_3=\tfrac{1}{2}\Big\rangle$$

$= \sqrt{\frac{2}{3}} \,|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}, m_3 = -\frac{1}{2}\rangle - \frac{1}{\sqrt{6}} \,|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}, m_3 = \frac{1}{2}\rangle$

$- \frac{1}{\sqrt{6}} \,|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}, m_3 = \frac{1}{2}\rangle$

proton wave function:

$$|p^\uparrow\rangle = \boxed{\sqrt{\frac{2}{3}}}\,|u^\uparrow u^\uparrow d^\downarrow\rangle - \boxed{\frac{1}{\sqrt{6}}}\,|u^\uparrow u^\downarrow d^\uparrow\rangle - \boxed{\frac{1}{\sqrt{6}}}\,|u^\downarrow u^\uparrow d^\uparrow\rangle$$

$$|n^\uparrow\rangle = \boxed{\sqrt{\frac{2}{3}}}\,|d^\uparrow d^\uparrow u^\downarrow\rangle - \boxed{\frac{1}{\sqrt{6}}}\,|d^\uparrow d^\downarrow u^\uparrow\rangle - \boxed{\frac{1}{\sqrt{6}}}\,|d^\downarrow d^\uparrow u^\uparrow\rangle$$

u – d exchange

Magnetic moment of point – like spin 1/2 – particles according to Dirac theory:

$$\vec{\mu} = -\mu_B \, g_s \frac{\vec{S}}{\hbar}$$

electron charge

electron spin

Bohr magnetic moment: $\mu_B = \frac{e\hbar}{2M}$

electron mass

Landé factor: $\boxed{g_s = 2}$

example: electron in hydrogen atom

Expected result for magnetic moment of proton and neutron, which are spin 1/2 particles:

$$\mu_p^{\text{expected}} = 1 \cdot \frac{e\hbar}{2M_p}, \qquad \mu_n^{\text{expected}} = 0 \cdot \frac{e\hbar}{2M_n} = 0$$

But experimental results read:

$$\mu_p^{\text{experiment}} = \boxed{2.79}\,\mu_p^{\text{expected}}, \qquad \mu_n^{\text{experiment}} = \boxed{-1.91} \cdot \frac{e\hbar}{2M_n}$$

Result: Proton and neutron can not be point – like spin 1/2 particles,

i. e. they must have a substructure (here: 3 quarks)

$$\boxed{\mu_{pz}^{theory}} = \langle \Psi_p | \mu_z | \Psi_p \rangle \quad , \quad \vec{\mu} = \vec{\mu}_{u1} + \vec{\mu}_{u2} + \vec{\mu}_d$$

built out of quarks

$$= \left(\sqrt{\tfrac{2}{3}} \langle u\uparrow u\uparrow d\downarrow | - \tfrac{1}{\sqrt{6}} \langle u\uparrow u\downarrow d\uparrow | - \tfrac{1}{\sqrt{6}} \langle u\downarrow u\uparrow d\uparrow | \right)$$

$$(\mu_{u1z} + \mu_{u2z} + \mu_{dz}) \left(\sqrt{\tfrac{2}{3}} | u\uparrow u\uparrow d\downarrow \rangle - \tfrac{1}{\sqrt{6}} | u\uparrow u\downarrow d\uparrow \rangle - \tfrac{1}{\sqrt{6}} | u\downarrow u\uparrow d\uparrow \rangle \right)$$

$$= \tfrac{2}{3} (\mu_u + \mu_u - \mu_d) + \tfrac{1}{6} (\cancel{\mu_u} - \cancel{\mu_u} + \mu_d) - \tfrac{1}{6} (-\cancel{\mu_u} + \cancel{\mu_u} + \mu_d)$$

$$= \tfrac{4}{3} \mu_u - \tfrac{1}{3} \mu_d \quad , \quad \mu_{nz}^{theory} = \tfrac{4}{3} \mu_d - \tfrac{1}{3} \mu_u$$

$$\mu_u = + \tfrac{2}{3} \frac{e\hbar}{2M_u} \quad , \quad \mu_d = - \tfrac{1}{3} \frac{e\hbar}{2M_d}$$

dublett of 1. generation: $M_u \approx M_d \Rightarrow \mu_u = -2\mu_d, \; \mu_d = -\tfrac{1}{2} \mu_u$

$$\mu_{p,z}^{theory} = \left(\tfrac{4}{3} - \tfrac{1}{3} \tfrac{-1}{2} \right) \mu_u = \left(\tfrac{4}{3} + \tfrac{1}{6} \right) \mu_u = \frac{8+1}{6} \mu_u = \tfrac{3}{2} \mu_u$$

$$\mu_{n,z}^{theory} = \left(\tfrac{4}{3} \cdot \tfrac{-1}{2} - \tfrac{1}{3} \right) \mu_u = \left(- \tfrac{2}{3} - \tfrac{1}{3} \right) \mu_u = - \mu_u$$

$$\frac{\mu_{n,z}^{theory}}{\mu_{p,z}^{theory}} = \frac{-1 \cdot \mu_u}{\tfrac{3}{2} \mu_u} = - \tfrac{2}{3} \approx -0.667$$

$$\frac{\mu_{n,z}^{experiment}}{\mu_{p,z}^{experiment}} = \frac{-1.91 \, \frac{e\hbar}{2M_n}}{2.79 \, \frac{e\hbar}{2M_p}} \quad \underset{\approx}{M_n \approx M_p} \quad \frac{-1.91}{2.79} = -0.685$$

excellent agreement

$\Rightarrow$ Application of Clebsch-Gordan coefficients

11.9 Recursion Relations for Clebsch-Gordan Coefficients:

$$|j,m\rangle = \sum_{\substack{m_1,m_2\\ m=m_1+m_2}} \underbrace{\langle m_1,m_2|j,m\rangle}_{\text{unitary matrix of rank }(2j_1+1)\cdot(2j_2+1)\text{ fulfilling}} |m_1,m_2\rangle \qquad (*)$$

- orthonormality of columns:

$$\sum_{m_1,m_2} \langle m_1,m_2|j,m\rangle\langle m_1,m_2|j',m'\rangle^* = \delta_{jj'}\delta_{mm'}$$

- orthonormality of rows:

$$\sum_{j,m} \langle m_1,m_2|j,m\rangle\langle m_1',m_2'|j,m\rangle^* = \delta_{m_1m_1'}\delta_{m_2m_2'}$$

Note: without loss of generality phases of states $|j,m\rangle$ can be chosen such that Clebsch-Gordan coefficients are real.

Step 1:

maximal j: $j=j_1+j_2$, maximal m: $m=m_1+m_2=j_1+j_2$

(*) consists of only one term: $|\underbrace{j_1+j_2}_{=j},\underbrace{j_1+j_2}_{=m}\rangle = \underbrace{\langle j_1,j_2|j_1+j_2,j_1+j_2\rangle}_{=1} |\underbrace{j_1}_{m_1},\underbrace{j_2}_{m_2}\rangle$

normalization, special choice of phase

From here successive applications of $\hat{J}_-$ allow to obtain all states belonging to $j=j_1+j_2$ and $-j\le m\le +j$

1. Application:

$$\hat{J}_- |\underbrace{j_1+j_2}_{j}, \underbrace{j_1+j_2}_{m}\rangle = \hbar\sqrt{(\underbrace{j_1+j_2}_{j} + \underbrace{j_1+j_2}_{m})(\underbrace{j_1+j_2}_{j} - (\underbrace{j_1+j_2}_{m})+1)}\ \boxed{|j_1+j_2, j_1+j_2-1\rangle}$$

$$= (\underline{\hat{J}_{1-} + \hat{J}_{2-}})\,|\underbrace{j_1}_{m_1}, \underbrace{j_2}_{m_2}\rangle = \hbar\sqrt{(\underbrace{j_1+j_1}_{2j_1})(\underbrace{j_1-j_1+1}_{=1})}\,|j_1-1, j_2\rangle$$

$$+\hbar\sqrt{(\underbrace{j_2+j_2}_{2j_2})(\underbrace{j_2-j_2+1}_{=1})}\,|j_1, j_2-1\rangle$$

$$\boxed{|j_1+j_2, j_1+j_2-1\rangle} = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1-1, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_2-1\rangle \qquad (**)$$

$$\langle m_1, m_2|j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,\delta_{m_1, j_1-1}\,\delta_{m_2, j_2}$$

$$+\sqrt{\frac{j_2}{j_1+j_2}}\,\delta_{m_1, j_2}\,\delta_{m_2, j_2-1}$$

2. Application: another $\hat{J}_-$ application to $(**)$

$$|j_1+j_2, j_1+j_2-2\rangle = \sqrt{\frac{j_1(2j_1-1)}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1-2, j_2\rangle$$

$$+2\sqrt{\frac{j_1\ j_2}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1-1, j_2-1\rangle + \sqrt{\frac{j_2(2j_2-1)}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1, j_2-2\rangle$$

This procedure can be continued to all $2(j_1+j_2)-1$ states $|\underbrace{j_1+j_2}_{j}, m\rangle$ with $-(j_1+j_2) \le m \le j_1+j_2$.

Step 2: Now states with $j = j_1+j_2-1$

start considerations with maximal $\boxed{m = j_1+j_2-1 = m_1+m_2}$

$$\boxed{|\underbrace{j_1+j_2-1}_{j}, \underbrace{j_1+j_2-1}_{m}\rangle} = \sum_{m_1} \langle m_1, j_1+j_2-1 | j_1+j_2-1, j_1+j_2-1-m_1\rangle \, |m_1, j_1+j_2-1-m_1\rangle$$

$$= \underbrace{\langle j_1-1, j_1+j_2-1 | j_1+j_2-1, j_2\rangle}_{\alpha} \, |j_1-1, j_2\rangle$$

$$+ \underbrace{\langle j_1, j_1+j_2-1 | j_1+j_2-1, j_2-1\rangle}_{=\beta} \, |j_1, j_2-1\rangle$$

must be orthogonal to $|j_1+j_2, j_1+j_2-1\rangle$

$$\alpha \sqrt{\frac{j_1}{j_1+j_2}} + \beta \sqrt{\frac{j_2}{j_1+j_2}} = 0 \quad \Rightarrow \quad \alpha = \sqrt{j_2}\,\gamma, \quad \beta = -\sqrt{j_1}\,\gamma$$

$$|j_1+j_2-1, j_1+j_2-1\rangle = \gamma \left\{ \sqrt{j_2}\, |j_1-1, j_2\rangle - \sqrt{j_1}\, |j_1, j_2-1\rangle \right\}$$

normalization: $\gamma^2 (j_1+j_2) = 1 \quad \Rightarrow \quad \gamma = \frac{1}{\sqrt{j_1+j_2}}$

$$|\underbrace{j_1+j_2-1}_{j}, j_1+j_2-1\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1-1, j_2\rangle - \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1, j_2-1\rangle$$

successive application of $\hat{J}_-$ leads to all states $|\underbrace{j_1+j_2-1}_{=j}, m\rangle$ with $-j \leq m \leq +j$.

Outlook: Clebsch-Gordan-Coefficients for LS-Coupling