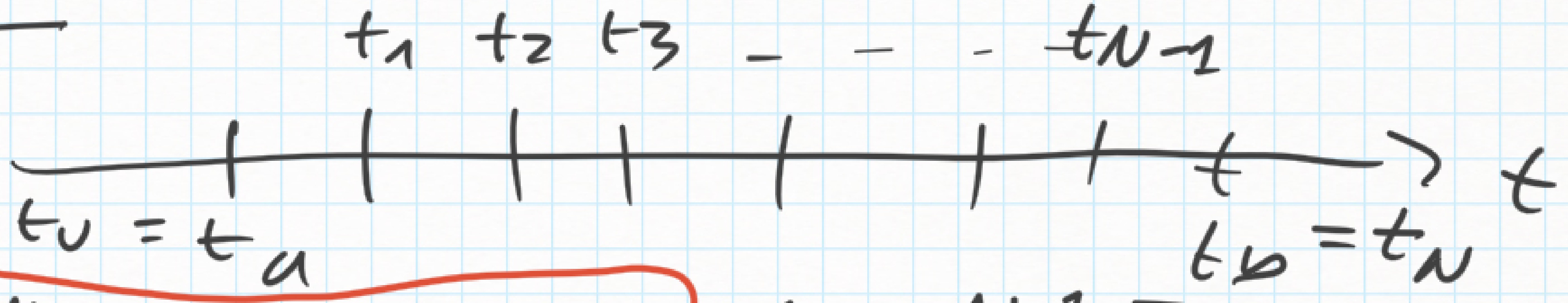
Path Integrals:

$$\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}$$

1) Discrete Version:



$$G(x_b, t_b; x_a, t_a) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon = t_b - t_a}} \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dx_j\right\} \exp\left\{\frac{i}{\hbar}\varepsilon\sum_{j=0}^{N-1}\left[\frac{M}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2 - V(x_j)\right]\right\}$$

true definition, example: • free particle → last lecture
• harmonic oscillator on finite lattice
→ problem set 4

2) continuous version:

$$G(x_b, t_b; x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\; \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left[\frac{M}{2}\dot{x}^2(t) - V(x(t))\right]\right\}$$

useful definition, example: harmonic oscillator → today

so far: Schrödinger ⟶ Feynman

now: Feynman ⟶ Schrödinger

8.7 Schrödinger Equation from Path Integral:

$$\Psi(x_b,t_b) = \int_{-\infty}^{+\infty} dx_a \, G(x_b, t_b; x_a, t_a)\, \Psi(x_a, t_a)$$

$$\frac{\partial \Psi(x_b,t_b)}{\partial t_b} = \lim_{\varepsilon \downarrow 0} \frac{\Psi(x_b, t_b+\varepsilon) - \Psi(x_b, t_b)}{\varepsilon}$$

$$\Psi(x_b, \underline{t_b+\varepsilon}) = \int_{-\infty}^{+\infty} dx_a \, \underbrace{G(x_b, \varepsilon; x_a, 0)} \, \Psi(x_a, \underline{t_b}) \quad \Leftarrow$$

$$= \sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \exp\left\{ \frac{i}{\hbar} \varepsilon \left[\frac{m}{2} \left(\frac{x_b - x_a}{\varepsilon} \right)^2 - V(x_a) \right] \right\}$$

$$\frac{\Psi(x_b,t_b) - \xi \Psi'(x_b,t_b) + \frac{1}{2}\xi^2 \Psi''(x_b,t_b) + \dots}{}$$

$$\xi(x_a) = x_b - x_a$$

$$\overbrace{\Psi(x_b - \xi, t_b)}$$

$$\Psi(x_b, t_b+\varepsilon) = \int_{-\infty}^{+\infty} d\xi \sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \, e^{-\frac{m}{2 i \hbar \varepsilon}\xi^2 - \frac{i}{\hbar}\varepsilon \underbrace{V(x_b - \xi)}}$$

$$\frac{V(x_b) - \cancel{V'(x_b)\xi} + \cancel{\frac{1}{2} V''(x_b)\xi^2} + \dots}{}$$

$$\Psi(x_b, t_b+\varepsilon) = \int_{-\infty}^{+\infty} d\xi \boxed{\sqrt{\frac{m}{2\pi i \hbar \varepsilon}} \, e^{-\frac{m}{2 i \hbar \varepsilon}\xi^2}} \left\{ 1 - \frac{i}{\hbar} \varepsilon V(x_b) + \dots \right\}$$

$$\cdot \left\{ \Psi(x_b, t_b) - \frac{\partial \Psi(x_b,t_b)}{\partial x_b} \xi + \frac{1}{2} \frac{\partial^2 \Psi(x_b,t_b)}{\partial x_b^2} \xi^2 + \dots \right\}$$

$$\int_{-\infty}^{+\infty} d\xi\, \xi^{2n+1} \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\, e^{-\frac{m}{2i\hbar\varepsilon}\xi^2} = 0$$

$$\int_{-\infty}^{+\infty} d\xi\, \xi^{2n} \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\, e^{-\frac{m}{2i\hbar\varepsilon}\xi^2} \sim \varepsilon^{n} \Rightarrow \boxed{\xi \mathrel{\hat{=}} \varepsilon^{\frac{1}{2}}}$$

basic rule for path integral manipulations at discrete version

$$\int_{-\infty}^{+\infty} d\xi\, e^{-\lambda\xi^2} = \sqrt{\frac{\pi}{\lambda}}, \quad \int_{-\infty}^{+\infty} d\xi\, \xi^2 e^{-\lambda\xi^2} = -\frac{\partial}{\partial\lambda}\sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\pi}{\lambda}}\,\frac{1}{2\lambda}$$

$$\int_{-\infty}^{+\infty} d\xi\, \xi^4 e^{-\lambda\xi^2} = \sqrt{\frac{\pi}{\lambda}}\,\frac{1}{2\lambda}\,\frac{3}{2\lambda} = \left(-\frac{\partial}{\partial\lambda}\right)$$

$$\int_{-\infty}^{+\infty} d\xi\, \xi^{2n} e^{-\lambda\xi^2} = \sqrt{\frac{\pi}{\lambda}}\,\frac{1}{2\lambda}\,\frac{3}{2\lambda}\,\frac{5}{2\lambda}\,\frac{7}{2\lambda}\cdots\frac{2n-1}{2\lambda} = \sqrt{\frac{\pi}{\lambda}}\,\frac{(2n-1)!!}{(2\lambda)^{n}}$$

$\Rightarrow$ evaluation only up to order ε necessary

$$\psi(x_0, t_0+\varepsilon) = \psi(x_0, t_0) - \frac{i}{\hbar}\varepsilon V(x_0)\psi(x_0, t_0) + \frac{1}{2}\frac{\partial^2\psi(x_0, t_0)}{\partial x_0^2}\,\frac{1}{2}\,\frac{2i\hbar\varepsilon}{m} + \cdots$$

$$\frac{\partial\psi(x_0, t_0)}{\partial t_0} i\hbar = i\hbar \lim_{\varepsilon\downarrow 0}\frac{\psi(x_0, t_0+\varepsilon) - \psi(x_0, t_0)}{\varepsilon} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_0^2} + V(x_0)\right\}\psi(x_0, t_0)$$

Schrödinger/Heisenberg

operators: operator ordering

$\vec{p} \cdot \vec{A}(\vec{x}) \rightarrow \hat{\vec{p}}\ \hat{\vec{A}}(\vec{x}),\ \hat{\vec{A}}(\vec{x})\,\hat{\vec{p}}$

$\frac{1}{2}\{\hat{\vec{p}}\,\hat{\vec{A}}(\vec{x}) + \hat{\vec{A}}(\vec{x})\,\hat{\vec{p}}\}$

Feynman

classical paths

$S[x(\cdot)]$ - - $V(x(t))$

discrete

postpoint $V(x_{j+1})$

prepoint $V(x_j)$

midpoint $V\left(\frac{x_j + x_{j+1}}{2}\right)$

direct access to Schrödinger equation

more efficient

9 Path Integral of Harmonic Oscillator:

9.1 Classical Path:

$$G(x_0, t_0; x_a, t_a) = \int_{x(t_0)=x_0}^{x(t_a)=x_a} \mathcal{D}x \; e^{\frac{i}{\hbar} S[x(\cdot)]}$$

$\rightarrow \infty$ in $\hbar \rightarrow 0$ classical limit

$\Rightarrow$ classical path has largest contribution to the path integral

$$\left.\frac{\delta A[x(\cdot)]}{\delta x(t)}\right|_{x(t)=x_{cl}(t)} = 0, \quad A[x(\cdot)] = \int_{t_a}^{t_b} dt \left[\frac{m}{2}\dot{x}^2(t) - V(x(t))\right]$$

$$\left.\frac{\partial L}{\partial x(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}(t)}\right|_{x(t)=x_{cl}(t)} = 0 \Rightarrow \ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) = 0$$

$$\Uparrow$$

$$V(x) = \frac{m}{2}\omega^2 x^2$$

$\Rightarrow$ general solution: $x_{cl}(t) = c_1 \cos\omega t + c_2 \sin\omega t$

two boundary conditions: $x_{cl}(t_a) = x_a, \quad x_{cl}(t_b) = x_b$

$$\begin{pmatrix} \cos\omega t_a & \sin\omega t_a \\ \cos\omega t_b & \sin\omega t_b \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

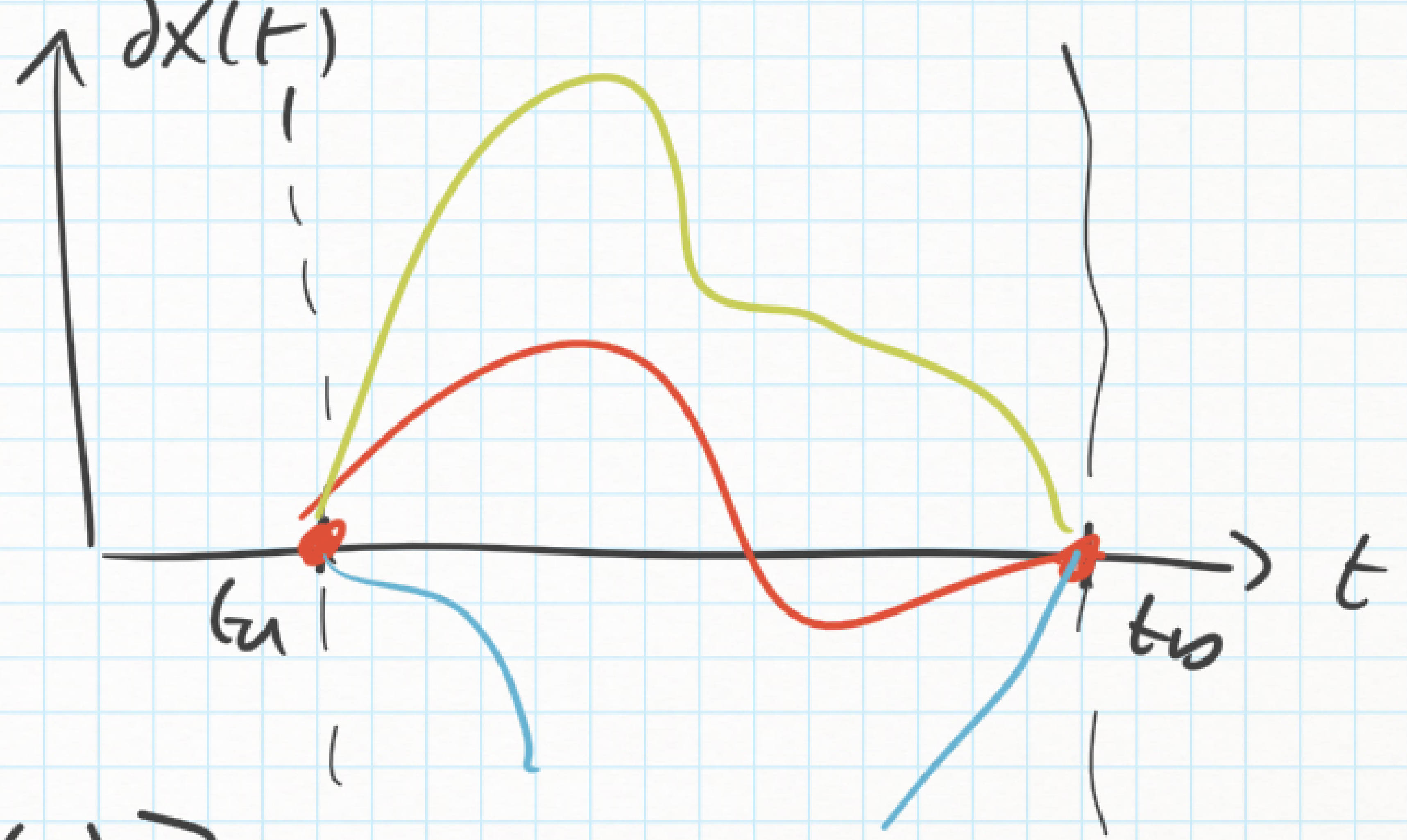
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sin\omega(t_b - t_a)}\begin{pmatrix} \sin\omega t_b & -\sin\omega t_a \\ -\cos\omega t_b & \cos\omega t_a \end{pmatrix}\begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$$\Rightarrow \cdots\cdots \Rightarrow x_{cl}(t) = \frac{x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)}{\sin\omega(t_b - t_a)}$$

9.2 Factorization of Propagator:

Idea: $\underbrace{x(t)}_{\text{sum over all paths}} = \underbrace{x_{cl}(t)}_{\text{fixed}} + \underbrace{\delta x(t)}_{\text{sum over all fluctuations}}$

$x(t_a)=x_a,\ x(t_b)=x_b \qquad x_{cl}(t_a)=x_a,\ x_{cl}(t_b)=x_b \qquad \delta x(t_a)=0,\ \delta x(t_b)=0$



$$\mathcal{D}x = \mathcal{D}\,\delta x$$

$$G(x_b,t_b;x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\; e^{\frac{i}{\hbar}\mathcal{A}[x(\cdot)]}$$

$$\underset{x(t)=x_{cl}(t)+\delta x(t)}{=} \int_{\delta x(t_a)=0}^{\delta x(t_b)=0} \mathcal{D}\delta x\; e^{\frac{i}{\hbar}\mathcal{A}[x_{cl}(\cdot)+\delta x(\cdot)]}$$

$$\mathcal{A}[x(\cdot)] = \int_{t_a}^{t_b} dt \left\{\frac{M}{2}\dot{x}^2(t) - \frac{M}{2}\omega^2 x^2(t)\right\} = \mathcal{A}[x_{cl}(\cdot)] + \mathcal{A}[\delta x(\cdot)] + \cancel{\Delta\mathcal{A}[x_{cl}(\cdot),\delta x(\cdot)]}$$

$$= \int_{t_a}^{t_b} dt \left(\frac{M}{2}\left[\underline{\dot{x}_{cl}^2(t)} + \underline{2\dot{x}_{cl}(t)\,\delta\dot{x}(t)} + \underline{\delta\dot{x}(t)^2}\right] - \frac{M}{2}\omega^2\left[\underline{x_{cl}^2(t)} + \underline{2x_{cl}(t)\,\delta x(t)} + \underline{\delta x^2(t)}\right]\right)$$

$$\Delta A[x_{cl}(\cdot), \delta x(\cdot)] = \int_{t_a}^{t_b} dt \left\{ m\, \dot{x}_{cl}(t) \underbrace{\delta \dot{x}(t)} - m \omega^2 x_{cl}(t)\, \delta x(t) \right\}$$

$$= \delta \frac{d}{dt} x(t) = \frac{d}{dt} \delta x(t) \longrightarrow \text{partial integration}$$

$$= \underbrace{\cancel{\left[m\, \dot{x}_{cl}(t)\, \delta x(t) \right]_{t_a}^{t_b}}}_{= 0 \text{ due to } \delta x(t_a) = \delta x(t_b) = 0} - m \int_{t_a}^{t_b} dt\, \underbrace{\cancel{\left\{ \ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) \right\}}}_{= 0 \text{ due to classical path equation of motion}} \delta x(t)$$

$$= 0$$

$$G(x_b, t_b; x_a, t_a) = \underbrace{e^{\frac{i}{\hbar} A[x_{cl}(\cdot)]}}_{= G_1(x_b, t_b;\, x_a, t_a)} \cdot \underbrace{\int_{\delta x(t_a) = 0}^{\delta x(t_b) = 0} \mathcal{D}\delta x\; e^{\frac{i}{\hbar} A[\delta x(\cdot)]}}_{= G_2(x_b, t_b;\, x_a, t_a)}$$

This is exact due to harmonic potential.

9.3 Classical Action:

$\dot{x}_{cl}(t) \frac{d}{dt} x_{cl}(t) \Rightarrow$ partial integration

$$A[x_{cl}(\cdot)] = \int_{t_a}^{t_b} dt \left\{ \frac{m}{2} \dot{x}_{cl}^2(t) - \frac{m}{2} \omega^2 x_{cl}^2(t) \right\}$$

$$= \left[\frac{m}{2} \dot{x}_{cl}(t)\, x_{cl}(t) \right]_{t_a}^{t_b} dt - \int_{t_a}^{t_b} dt \; \frac{m}{2} \underbrace{\left(\ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) \right)}_{\equiv 0} x_{cl}(t)$$

due to equation of motion

$$= \frac{m}{2} \Big[\dot{x}_{cl}(t_b) \underbrace{x_{cl}(t_b)}_{= x_b} - \dot{x}_{cl}(t_a) \underbrace{x_{cl}(t_a)}_{= x_a} \Big]$$

$$\Rightarrow S[x_{cl}(\cdot)] = \frac{m}{2} \left\{ \dot{x}_{cl}(t_b) x_b - \dot{x}_{cl}(t_a) x_a \right\}$$

$$x_{cl}(t) = \frac{x_b \sin\omega(t - t_a) + x_a \sin\omega(t_b - t)}{\sin\omega(t_b - t_a)}$$

$$\dot{x}_{cl}(t) = \frac{\omega}{\sin\omega(t_b - t_a)} \left[x_b \cos\omega(t - t_a) - x_a \cos\omega(t_b - t) \right]$$

$$\Rightarrow S[x_{cl}(\cdot)] = \dots = \frac{m\omega}{2 \sin\omega(t_b - t_a)} \left\{ (x_b^2 + x_a^2) \cos\omega(t_b - t_a) - 2 x_a x_b \right\}$$

$$\Rightarrow G_1(x_b, t_b; x_a, t_a) = e^{\frac{i}{\hbar} S[x_{cl}(\cdot)]}$$

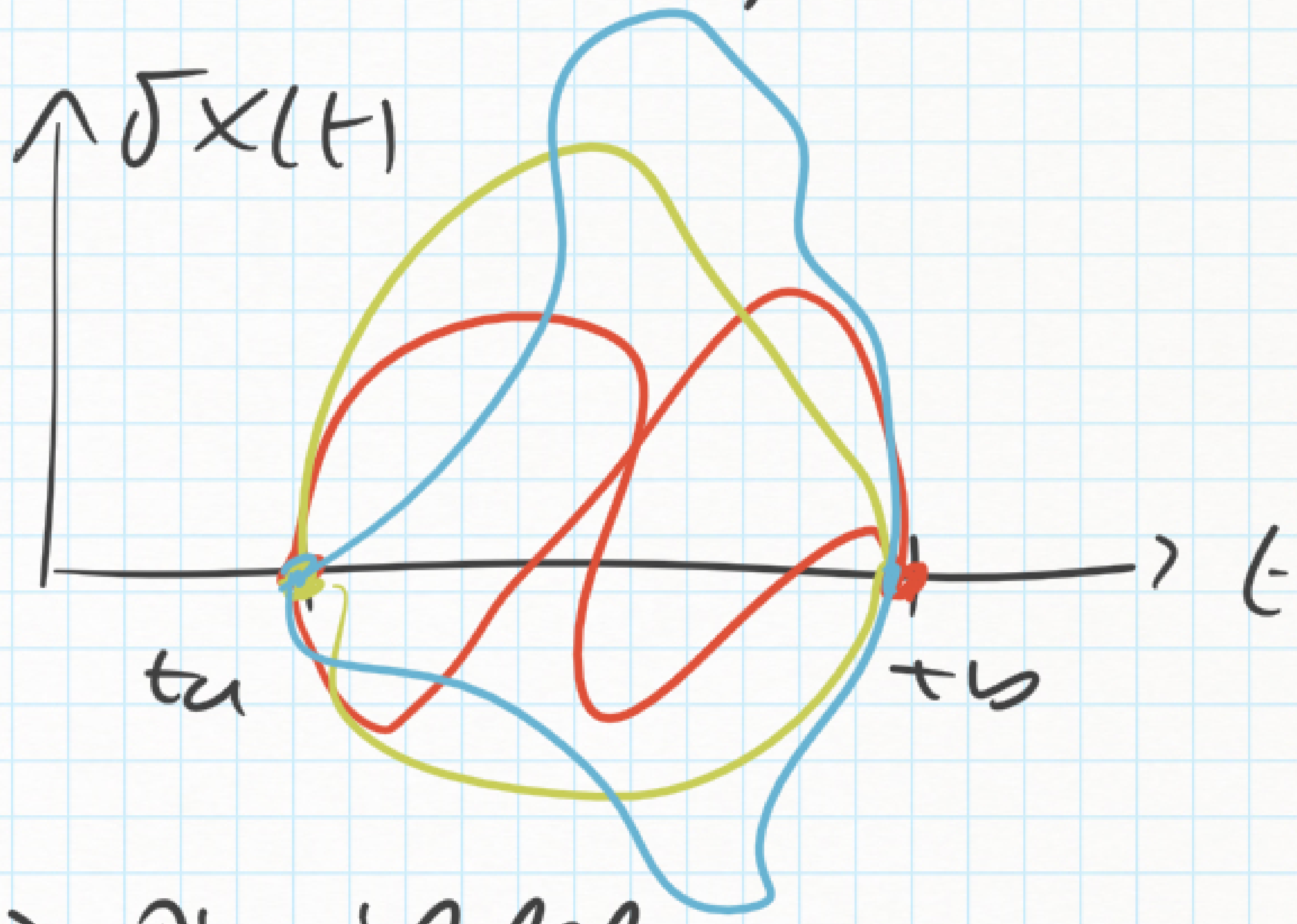
9.4 Fluctuation Propagator:

$$A[\delta x(\cdot)] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2} \underbrace{\delta\dot{x}^2(t)} - \frac{M}{2}\omega^2 \delta x^2(t) \right\}$$

$$= \delta\dot{x}(t) \frac{d}{dt} \delta x(t) \quad \Rightarrow \text{partial integration}$$

$$= \underbrace{\cancel{\left[\frac{M}{2} \delta\dot{x}(t)\, \delta x(t) \right]_{t_a}^{t_b}}}_{=0} - \frac{M}{2} \int_{t_a}^{t_b} dt \left\{ \delta\ddot{x}(t) + \omega^2 \delta x(t) \right\} \delta x(t)$$

$\delta x(t_a) = \delta x(t_b) = 0$



$$A[\delta x(\cdot)] = \int_{t_a}^{t_b} dt\; \delta x(t) \underbrace{\hat{O}(t)} \delta x(t)$$

$$= -\frac{M}{2} \left\{ \frac{d^2}{dt^2} + \omega^2 \right\}$$

boundary conditions: $\delta x(t_a) = 0 = \delta x(t_b) \Rightarrow$ Dirichlet eigenvalue problem of operator $\hat{O}(t)$

$$\underbrace{\hat{O}(t)}_{= -\frac{M}{2}\left\{\frac{d^2}{dt^2}+\omega^2\right\}} u_n(t) = \lambda_n u_n(t) \quad ; u_n(t_a) = 0 = u_n(t_b)$$

$\hat{=}$ Schrödinger equation of particle in a finite box of length L

$L \hat{=} t_b - t_a$

$$u_n(t) = N_n \sin\frac{n\pi(t-t_a)}{t_b-t_a} \quad ; n = 1, 2, 3, \ldots$$

$$\lambda_n = \frac{M}{2}\left[\left(\frac{n\pi}{t_b-t_a}\right)^2 - \omega^2\right]$$

normalization constants

$$\int_{t_a}^{t_b} dt\, u_n(t)\, u_{n'}(t) = \delta_{n,n'} \Rightarrow N_n = \sqrt{\frac{2}{t_b-t_a}}$$

conclusion: The eigenfunction $u_n(t)$ represent an orthonormal basis for all possible fluctuation $\delta x(t)$ contributing to the path integral $\int_{\delta x(t_a)=0}^{\delta x(t_b)=0} \mathcal{D}\delta x(t)$