Chapter 6: Different Pictures

Motivation:

- dynamics for both states and operators
- Irrespective of picture: always the same expectation value
- 3 pictures:

Schrödinger picture	Dirac (interaction) picture	Heisenberg picture
states time-dependent	both states and	states time-independent
operators time independent	operators are time-dependent	operators time-dependent

6.1 Schrödinger and Heisenberg Picture:

Note: Simplification → all operators in Schrödinger picture time independent

$$\begin{cases} i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle = \hat{H}_S |\Psi_S(t)\rangle \\ i\hbar \frac{\partial}{\partial t} \hat{O}_S = 0 \end{cases}$$

S: Schrödinger picture

time-independent

formal solution: $|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S t} \underbrace{|\Psi_S(0)\rangle}_{=:|\Psi_H\rangle}$

definition:

$$|\Psi_S(t)\rangle = \underbrace{e^{-\frac{i}{\hbar}\hat{H}_S t}}_{\text{time-evolution operator}} |\Psi_H\rangle \iff \boxed{|\Psi_H\rangle = e^{\frac{i}{\hbar}\hat{H}_S t} \; |\Psi_S(t)\rangle}$$

$$\underline{i\hbar \frac{\partial}{\partial t}|\Psi_H\rangle} = \left(-\hat{H}_S + \hat{H}_S\right) e^{\frac{i}{\hbar}\hat{H}_S t} |\Psi_S(t)\rangle = 0$$

Equation of motion of operator in H picture

$$\underbrace{\langle\Psi_S(t)|}_{=\langle\Psi_H| e^{\frac{i}{\hbar}\hat{H}_S t}} \hat{O}_S \underbrace{|\Psi_S(t)\rangle}_{= e^{-\frac{i}{\hbar}\hat{H}_S t}|\Psi_H\rangle} \stackrel{!}{=} \langle\Psi_H| \underline{\hat{O}_H(t)} |\Psi_H\rangle$$

$$\Rightarrow \quad \hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}_S t} \, \hat{O}_S \, e^{-\frac{i}{\hbar}\hat{H}_S t}$$

example: $\hat{H}_H(t) = e^{\frac{i}{\hbar}\hat{H}_S t} \hat{H}_S e^{-\frac{i}{\hbar}\hat{H}_S t} \equiv \hat{H}_S$

$$i\hbar \frac{\partial}{\partial t}\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}_S t}\left(-\hat{H}_S \hat{O}_S + \hat{O}_S \hat{H}_S\right) e^{-\frac{i}{\hbar}\hat{H}_S t}$$

$$i\hbar \frac{\partial}{\partial t}\hat{O}_S = 0$$

$$= \underbrace{e^{\frac{i}{\hbar}\hat{H}_S t}\hat{O}_S e^{-\frac{i}{\hbar}\hat{H}_S t}}_{=\hat{O}_H(t)} \hat{H}_S - \hat{H}_S \underbrace{e^{\frac{i}{\hbar}\hat{H}_S t}\hat{O}_S e^{-\frac{i}{\hbar}\hat{H}_S t}}_{=\hat{O}_H(t)}$$

$$= \left[\hat{O}_H(t), \hat{H}_S\right]_- = \left[\hat{O}_H(t), \hat{H}_H(t)\right]_-$$

Heisenberg equation

6.2 Dirac Picture:

$$\hat{H}_S(t) = \underbrace{\hat{H}_S^{(0)}}_{\text{time-independent, unperturbed}} + \underbrace{\hat{H}_S^{(int)}(t)}_{\text{time-dependent perturbed}}$$

$$i\hbar \frac{\partial}{\partial t} |\psi_S(t)\rangle = \hat H_S(t)|\psi_S(t)\rangle = \left\{ \hat H_S^{(0)} + \hat H_S^{(int)}(t) \right\} |\psi_S(t)\rangle$$

$$|\psi_D(t)\rangle = e^{\frac{i}{\hbar}\hat H_S^{(0)} t} |\psi_S(t)\rangle \iff |\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat H_S^{(0)} t} |\psi_D(t)\rangle$$

$\Rightarrow$ partially "redo" the dynamics in $|\psi_S(t)\rangle$

$$\langle\psi_D(t)| \hat O_D(t) |\psi_D(t)\rangle = \underbrace{\langle\psi_S(t)|}_{= \langle\psi_D(t)| e^{\frac{i}{\hbar}\hat H_S^{(0)} t}} \hat O_S \underbrace{|\psi_S(t)\rangle}_{= e^{-\frac{i}{\hbar}\hat H_S^{(0)} t} |\psi_D(t)\rangle}$$

$$\Rightarrow \quad \hat O_D(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)} t} \, \hat O_S \, e^{-\frac{i}{\hbar}\hat H_S^{(0)} t}$$

example: $\hat O_S = \hat H_S^{(0)}$

$$\hat H_D^{(0)}(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)} t} \, \hat H_S^{(0)} \, e^{-\frac{i}{\hbar}\hat H_S^{(0)} t} = \hat H_S^{(0)}$$

$$e^{-\frac{i}{\hbar}\hat H_S^{(0)} t} \; e^{\frac{i}{\hbar}\hat H_S^{(0)} t}$$

$$i\hbar \frac{\partial}{\partial t} |\psi_D(t)\rangle = \left\{ \cancel{- \hat H_S^{(0)} e^{\frac{i}{\hbar}\hat H_S^{(0)} t}} + e^{\frac{i}{\hbar}\hat H_S^{(0)} t} \left(\cancel{\hat H_S^{(0)}} + \hat H_S^{(int)}(t) \right) \right\} |\psi_S(t)\rangle$$

$$= \underbrace{e^{\frac{i}{\hbar}\hat H_S^{(0)} t} \, \hat H_S^{(int)}(t) \, e^{-\frac{i}{\hbar}\hat H_S^{(0)} t}}_{\hat H_D^{(int)}(t)} |\psi_D(t)\rangle$$

$$i\hbar \frac{\partial}{\partial t} \hat O_D(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)} t} \left\{ - \hat H_S^{(0)} \hat O_S + \hat O_S \hat H_S^{(0)} \right\} e^{-\frac{i}{\hbar}\hat H_S^{(0)} t}$$

$$= \left[\hat O_D(t), \hat H_S^{(0)} \right]_- = \left[\hat O_D(t), \hat H_D^{(0)}(t) \right]_-$$

6.3 Time evolution Operator:

$$|\psi_D(t_2)\rangle = \underline{\hat U_D(t_2,t_1)}\,|\psi_D(t_1)\rangle \qquad |\psi_S(t_2)\rangle = \underline{\hat U_S(t_2,t_1)}\,|\psi_S(t_1)\rangle$$

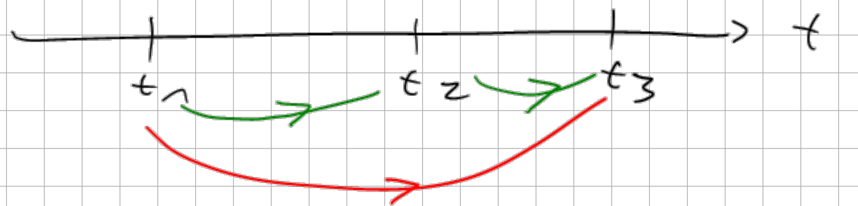
$$e^{\frac{i}{\hbar}\hat H_S^{(0)} t_2}|\psi_S(t_2)\rangle = e^{\frac{i}{\hbar}\hat H_S^{(0)} t_2}\,\hat U_S(t_2,t_1)\underbrace{|\psi_S(t_1)\rangle}_{= e^{-\frac{i}{\hbar}\hat H_S^{(0)} t_1}|\psi_D(t_1)\rangle}$$

$$\Rightarrow \hat U_D(t_2,t_1) = e^{\frac{i}{\hbar}\hat H_S^{(0)} t_2}\,\hat U_S(t_2,t_1)\,e^{-\frac{i}{\hbar}\hat H_S^{(0)} t_1} \qquad (*)$$

Properties of time evolution operators:

1) $\hat U_S(t_1,t_1) = 1 \overset{(*)}{\Longrightarrow} \hat U_D(t_1,t_1) = 1$

2) group property:



$$\hat U_S(t_3,t_1) = \hat U_S(t_3,t_2)\,\hat U_S(t_2,t_1) \qquad (**)$$

$$\hookrightarrow \hat U_D(t_3,t_1) \overset{(*)}{=} e^{\frac{i}{\hbar}\hat H_S^{(0)} t_3}\,\underbrace{\hat U_S(t_3,t_1)}_{(**)}\,e^{-\frac{i}{\hbar}\hat H_S^{(0)} t_1} = \hat U_D(t_3,t_2)\,\hat U_D(t_2,t_1)$$

$$= \hat U_S(t_3,t_2)\,\underbrace{\;}_{e^{-\frac{i}{\hbar}\hat H_S^{(0)} t_2}\; e^{\frac{i}{\hbar}\hat H_S^{(0)} t_2}}\,\hat U_S(t_2,t_1)$$

3) $t_3 = t_1$: $\quad \hat{U}_D(t_1,t_2)\,\hat{U}_D(t_2,t_1) \overset{2)}{=} \hat{U}_D(t_1,t_1) \overset{1)}{=} 1$

$$\Rightarrow \quad \hat{U}_D(t_1,t_2) = \hat{U}_D^{-1}(t_2,t_1)$$

4) $$\hat{U}_D^{+}(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}\, \underbrace{\hat{U}_S^{+}(t_2,t_1)}_{\text{assumption}}\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2} = \hat{U}_D(t_1,t_2) \overset{3)}{=} \hat{U}_D^{-1}(t_2,t_1)$$

unitary $= \hat{U}_S^{-1}(t_2,t_1) \overset{3)}{=} \hat{U}_S(t_1,t_2)$

$$i\hbar\frac{\partial}{\partial t_2}\hat{U}_D(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\left\{-\cancel{\hat{H}_S^{(0)}} + \cancel{\hat{H}_S^{(0)}} + \hat{H}_S^{(int)}(t_2)\right\}\, \overbrace{}^{e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\; e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}}\, \underline{\hat{U}_S(t_2,t_1)\, e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}}$$

$$= \hat{H}_D^{(int)}(t_2)\,\hat{U}_D(t_2,t_1)$$

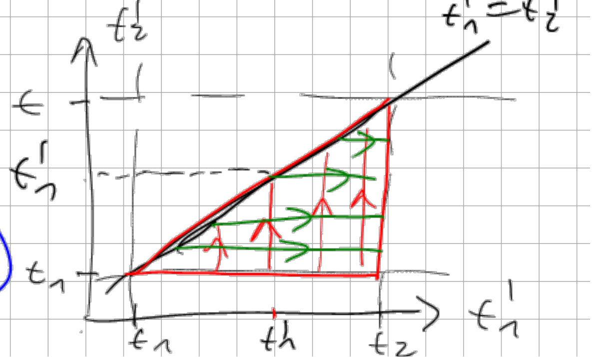
$\Rightarrow$ differential equation for $\hat{U}_D(t_2,t_1)$

$$\hat{U}_D^{(n+1)}(t_2,t_1) = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2} dt_1'\, \hat{H}_D^{(int)}(t_1')\, \hat{U}_D^{(n)}(t_1',t_1)$$ integral equation

$$\frac{d}{dx}\int_{a(x)}^{b(x)} dt\, f(t,x) \underset{\text{Leibniz}}{=} b'(x)\, f(b(x),x) - a'(x)\, f(a(x),x) + \int_{a(x)}^{b(x)} dt\, \frac{\partial f(t,x)}{\partial x}$$

Von Neumann series

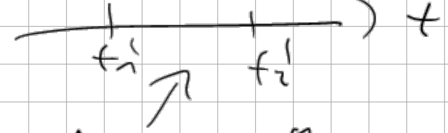
$$\underline{\hat{U}_D(t_2,t_1)} = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2} dt_1'\, H_D^{(int)}(t_1')$$
$$+ \left(-\frac{i}{\hbar}\right)^2 \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2'\, \hat{H}_D^{(int)}(t_1')\, \hat{H}_D^{(int)}(t_2')$$



$$+\dots+\left(\frac{-i}{\hbar}\right)^n\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\dots\int_{t_1}^{t_{n-1}'}dt_n'\,\hat{H}_D^{(int)}(t_1')\,\hat{H}_D^{(int)}(t_2')\dots\hat{H}_D^{(int)}(t_n')$$

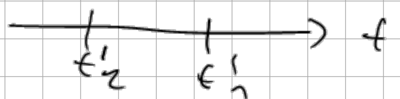
$$+\dots$$

$$\int_{t_1}^{t_2}dt_{\cancel{1}\,2}'\int_{t_{\cancel{1}}'}^{t_2}dt_{2\,1}'\,\hat{H}_D^{(int)}(t_2')\,\hat{H}_D^{(int)}(t_{\cancel{1}}')$$



$$=\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_2}dt_2'\left\{\underbrace{\Theta(t_1'-t_2')\,\hat{H}_D^{(int)}(t_1')\,\hat{H}_D^{(int)}(t_2')+\Theta(t_2'-t_1')\,\hat{H}_D^{(int)}(t_2')\,\hat{H}_D^{(int)}(t_1')}\right\}$$

$$=:\hat{T}\left(\hat{H}_D(t_1')\,\hat{H}_D^{(int)}(t_2')\right)$$



time-ordering operator

$$\hat{U}_D(t_2,t_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{t_1}^{t_2}dt_1'\dots\int_{t_1}^{t_2}dt_n'\,\hat{T}\left(\hat{H}_D^{(int)}(t_1')\dots\hat{H}_D^{(int)}(t_n')\right)$$

$$=\hat{T}\left(\exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t_2}dt\,\hat{H}_D^{(int)}(t)\right\}\right)\xleftarrow{\text{extension}}e^{-\frac{i}{\hbar}\hat{H}_D^{(int)}t}$$

solves $i\hbar\frac{\partial}{\partial t_2}\hat{U}_D(t_2,t_1)=\hat{H}_D^{(int)}(t_2)\,\hat{U}_D(t_2,t_1)$

$$i\hbar \frac{\partial}{\partial t_2} \hat{U}_D(t_2,t_1) = \sum_{n=1}^{\infty} \cancel{n}\, \frac{1}{\cancel{n!}\;(n-1)!} \cancel{i\hbar} \left(\frac{-i}{\hbar}\right)^{n-1} \int_{t_1}^{t_2} dt_1' \cdots \int_{t_1}^{t_2} dt_{n-1}' \; \hat{H}_D^{(int)}(t_2)\, \hat{T}\left(\hat{H}_D^{(int)}(t_1') \cdots \hat{H}_D^{(int)}(t_{n-1}')\right)$$

$$= \hat{H}_D^{(int)}(t_2)\, \underline{\hat{U}_D(t_2,t_1)}$$

Chapter 7: Scattering Theory

non-relativistic, scattering between atoms / ions

e.g. Rutherford scattering: α-particle + Au