Quantum Mechanics II

Content:

1) Time - independent perturbation:
   non - degenerate / generate quantum systems

   Brillouin-Wigner method
2) Time - dependent perturbation theory:
   absorption and emission of radiation
3) Scattering theory:
   scattering amplitude cross - section
   Rutherford scattering
4) Path integral description of quantum mechanics:
   free particle $\longleftrightarrow$ harmonic oscillator
5) Addition of angular momenta:
   Clebsch - Gordon coefficients, Landé factors
6) Relativistic wave equations:
   Klein - Gordon / Dirac equations: hydrogen atom

Chapter 1: Formal Principles of Quantum Mechanics

1.1 Classical Mechanics:

particle in 3 D, mass m, potential $V(\vec{x})$

$$A[\vec{x}(\cdot)] = \int_{t_a}^{t_b} dt \; L(\vec{x}(t), \dot{\vec{x}}(t)) \quad ; \quad L(\vec{x}, \dot{\vec{x}}) = \frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x})$$

action is functional of path $\vec{x}(t)$

Hamilton principle of Lagrange mechanics:

$$\frac{\delta A[\vec{x}(\cdot)]}{\delta \vec{x}(t)} = 0 \qquad \text{(functional derivative)}$$

$$\frac{dx}{dx} = 1 \qquad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad \frac{\delta x(t)}{\delta x(t')} = \delta(t-t')$$

total derivative, partial derivative, functional derivative

This is basic definition for a functional derivative, all basic rules of differentiation can be extended to functional derivatives.

Greiner, Reinhardt, Volume $\overline{\text{VII}}$ A

Einstein summation

$$\frac{\delta A[\vec{x}(\cdot)]}{\delta x_i(t)} = \frac{\delta}{\delta x_i(t)} \int_{t_a}^{t_b} dt' L(\vec{x}(t'), \dot{\vec{x}}(t'))$$

$$= \int_{t_a}^{t_b} dt' \left\{ \frac{\partial L}{\partial x_j(t')} \frac{\delta x_j(t')}{\delta x_i(t)} + \frac{\partial L}{\partial \dot{x}_j(t')} \frac{\delta \dot{x}_j(t')}{\delta x_i(t)} \right\}$$

$$= \delta_{ij}\,\delta(t-t') \qquad \frac{d}{dt'}\frac{\delta x_j(t')}{\delta x_i(t)} = \frac{\delta}{\delta x_i(t)}\frac{d}{dt'}x_j(t')$$

partial integration

$$= \underbrace{\left[\frac{\partial L}{\partial \dot{x}_j(t)}\frac{\delta x_j(t')}{\delta x_i(t)}\right]_{t_a=t'}^{t_b=t'}}_{=0 \quad t_a<t<t_b} + \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial x_j(t')} - \frac{d}{dt'}\frac{\partial L}{\partial \dot{x}_j(t')}\right\}\underbrace{\frac{\delta x_j(t')}{\delta x_i(t)}}_{=\delta_{ij}\delta(t-t')}$$

$= 0 \Rightarrow$ Euler - Lagrange equations

$$\underbrace{\frac{\partial L}{\partial \vec{x}(t)}}_{-\frac{\partial V(\vec{x}(t))}{\partial \vec{x}(t)}} - \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\vec{x}}(t)}}_{=\frac{m}{2}2\dot{\vec{x}}} = 0 \;;\quad L = \frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x})$$

Newton equation

$$\Rightarrow m\ddot{\vec{x}}(t) = -\underbrace{\frac{\partial V(\vec{x}(t))}{\partial \vec{x}(t)}}_{= +\vec{\nabla} V(\vec{x}(t))}$$

Lagrange mechanics $\rightarrow$ Hamilton mechanics

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} \Rightarrow \dot{\vec{x}} = \frac{\vec{p}}{m}$$

Legendre transformation:

$$H(\vec{p},\vec{x}) = \vec{p}\cdot\dot{\vec{x}} - L = \vec{p}\cdot\frac{\vec{p}}{m} - \left\{\frac{m}{2}\frac{\vec{p}^2}{2m} - V(\vec{x})\right\} = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

action in phase space

$$\mathcal{A} = \mathcal{A}[\vec{p}(\cdot), \vec{x}(\cdot)] \hat{=} \int_{t_a}^{t_b} dt \left\{ \vec{p}(t)\, \dot{\vec{x}}(t) - H(\vec{p}(t); \vec{x}(t)) \right\}$$

Hamilton principle in Hamilton mechanics:

$$\frac{\delta \mathcal{A}}{\delta \vec{x}(t)} = 0 \quad \underset{\text{equations}}{\overset{\text{Euler-Lagrange}}{\Longrightarrow}} \quad \dot{\vec{p}}(t) = -\frac{\partial H}{\partial \vec{x}(t)} = -\vec{\nabla} V(\vec{x})$$

$$\frac{\delta \mathcal{A}}{\delta \vec{p}(t)} = 0 \quad \Longrightarrow \quad \dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}(t)} = \frac{\vec{p}}{m}$$

Hamilton equations $\hat{=}$ Newton equation

1.2 Quantization:

classical observable $\longrightarrow$ hermitian operator

$$\vec{x} \longrightarrow \hat{\vec{x}}, \quad \vec{p} \longrightarrow \hat{\vec{p}}, \quad H(\vec{p}; \vec{x}) = \hat{H} = H(\hat{\vec{p}}; \hat{\vec{x}})$$

Heisenberg uncertainty relation enforces:

$$[\hat{x}_j, \hat{x}_k]_- = [\hat{p}_j, \hat{p}_k]_- = 0, \quad [\hat{p}_j, \hat{x}_k]_- = \frac{\hbar}{i} \delta_{jk}$$

$$[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Dirac: independent representation

state: $|\psi(t)\rangle$ state vector

time evolution: time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

coordinate representation: discrete basis $\Rightarrow$ eigenstates of $\hat{\vec{x}}$

$$\hat{\vec{x}}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle$$

orthonormality: $\langle \vec{x}\,|\,\vec{x}'\rangle = \delta(\vec{x}-\vec{x}')$

completeness: $\int d^3x\, |\vec{x}\rangle\langle\vec{x}| = \hat{1}$

coordination representation for $\hat{\vec{p}}$: Jordan rule

$$\langle\vec{x}|\,\hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla}\,\langle\vec{x}|$$

$$|\psi(t)\rangle = \hat{1}\,|\psi(t)\rangle = \int d^3x\,|\vec{x}\rangle\underbrace{\langle\vec{x}|\psi(t)\rangle}_{=\psi(\vec{x},t)}$$ wave function

$\langle\vec{x}|$

$$i\hbar\frac{\partial}{\partial t}\underbrace{\langle\vec{x}|\psi(t)\rangle}_{=\psi(\vec{x},t)} = \langle\vec{x}|\left\{\frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{x}})\right\}|\psi(t)\rangle$$

$$= \left\{-\frac{\hbar^2}{2m}\Delta\,\langle\vec{x}| + V(\vec{x})\,\langle\vec{x}|\right\}|\psi(t)\rangle$$

$$= \underbrace{\hat{H}}\,\psi(\vec{x},t)$$

$$= -\frac{\hbar^2}{2m}\underbrace{\Delta}_{=\vec{\nabla}^2} + V(\vec{x})$$

stationarity:

$$\psi(\vec{x},t) = e^{-\frac{i}{\hbar}Et}\,\psi_E(\vec{x})$$

time-dependent $\Longrightarrow$ Schrödinger eq.

$$\boxed{\hat{H}\,\psi_E(\vec{x}) = \underbrace{E}_{\text{energy eigenvalue}}\ \underbrace{\psi_E(\vec{x})}_{\text{energy eigenfunction}}}$$

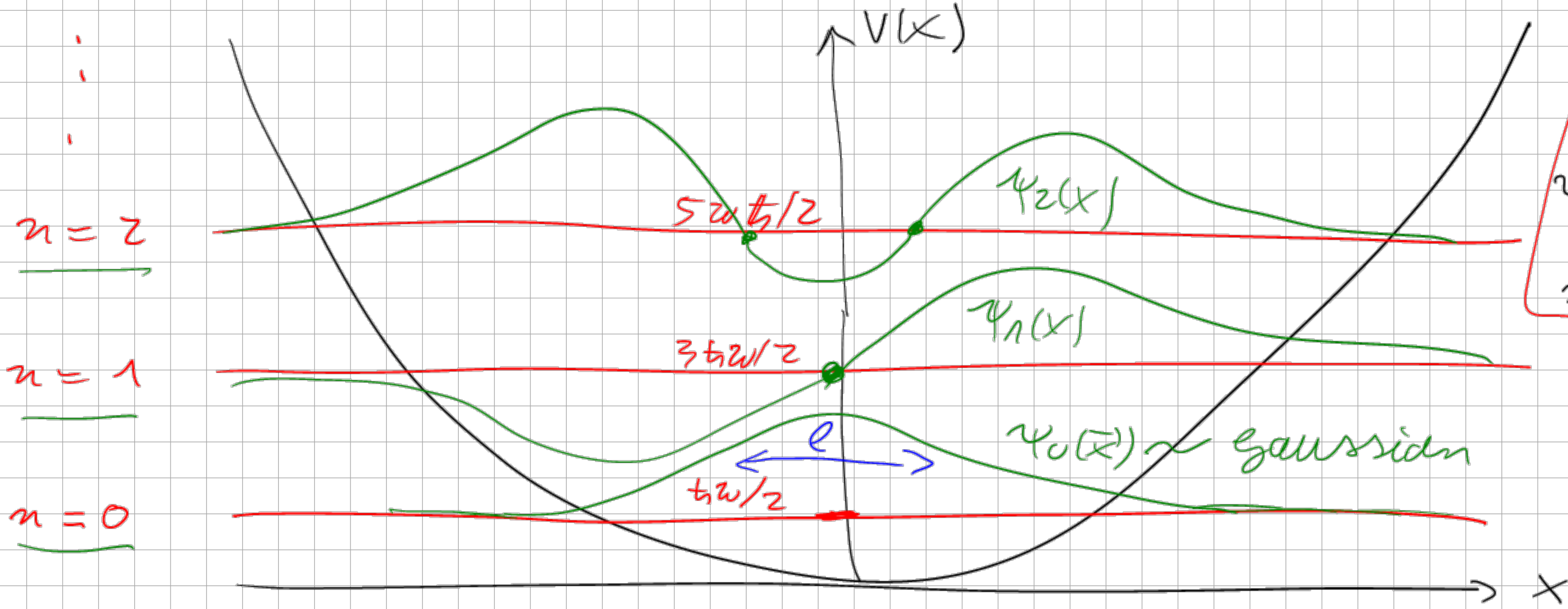
differential equation of Schrödinger + boundary conditions
$\Rightarrow$ quantization of energies

1.3 Harmonic Oscillator: 1D

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2$$

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2 \right\} \psi_E(x) = E\, \psi_E(x)$$

Dirichlet boundary conditions: $\lim_{x \to \pm\infty} \psi_E(x) = 0$



node rule:
$\psi_n(x)$ has n nodes

oscillator length: $l = \sqrt{\frac{\hbar}{m\omega}}$

energy eigenvalues:
$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$; $n = 0, 1, 2, \ldots$

^{87}Rb, $m = 87u$, $u = 1.66 \cdot 10^{-27}$ kg, $\omega = 2\pi \cdot 100$ Hz $\Rightarrow$ $l = 1.16\,\mu m$