Chapter 7: Scattering Theory (only for isotropic potentials)

7.1 Interaction Potentials:

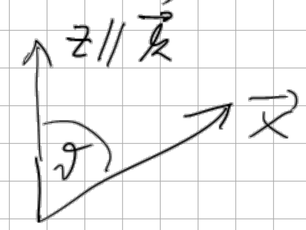
$$V^{(int)}(\vec{x},\vec{x}') \underset{\uparrow}{=} V^{(int)}(\vec{x}-\vec{x}') \underset{\uparrow}{=} \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{x}')}\, V^{(int)}(\vec{k})$$

Newton axiom: actio = - reactio Fourier transformation

$$V^{(int)}(\vec{k}) = \int d^3x\; e^{-i\vec{k}\vec{x}}\; V^{(int)}(\vec{x})$$

general statement: symmetry in $V^{(int)}(\vec{x})$ gets inherited by $V^{(int)}(\vec{k})$

example: $V^{(int)}(\vec{x}) = v(|\vec{x}|)$ rotational symmetry



$$V^{(int)}(\vec{k}) \underset{\uparrow}{=} \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{\infty} dr\, r^2\, e^{-ikr\cos\vartheta}\, v^{(int)}(r)$$

spherical coordinates

$$\vec{k} = k\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= 2\pi \int_0^{\infty} dr\, r^2 v^{(int)}(r) \underbrace{\int_0^{\pi} d\vartheta \sin\vartheta\, e^{-ikr\cos\vartheta}}_{}$$

$$u(\vartheta) = \cos\vartheta \quad = \int_{-1}^{+1} du\, e^{-ikru} = \frac{e^{-ikr}-e^{ikr}}{-ikr} = \frac{+2i\sin(kr)}{+ikr}$$

$$\Rightarrow \boxed{V^{(int)}(\vec{k}) = v^{(int)}(|\vec{k}|) = \frac{4\pi}{k}\int_0^{\infty} dr\, r\, \sin(kr)\, v^{(int)}(r)} \quad (*)$$

7.1.1 Yukawa Potential:

$$V_Y^q(\vec{x}) = C\,\frac{e^{-q|\vec{x}|}}{|\vec{x}|}$$

screened Coulomb-potential

q : screening parameter

$\frac{1}{q}$: screening length

1935 Hideki Yukawa: heuristic model for neutron-proton interaction

$$V_C(\vec{x}) = \lim_{q \downarrow 0} V_Y^q(\vec{x}) = \frac{C}{|\vec{x}|}, \quad C = \frac{Q_1 Q_2}{4\pi \varepsilon_0}$$

$V_Y^q(\vec{x})$ is rotationally symmetric:

$$V_Y^q(\vec{k}) = \frac{4\pi}{k}\int_0^\infty dr\, \cancel{r} \sin(kr)\, \frac{C}{\cancel{r}}\, e^{-qr} = \frac{4\pi C}{k}\, \mathrm{Im} \underbrace{\int_0^\infty dr\, e^{(-q+ik)r}}$$

$$\frac{q+ik}{q^2+k^2} = \frac{0+1}{+q - ik} = \frac{e^{(-q+ik)r}}{-q+ik}\Big|_0^\infty$$

$$= \frac{4\pi C}{k} \frac{k}{q^2+k^2} = \frac{4\pi C}{q^2+k^2}$$

$$V_C(\vec{k}) = \lim_{q \downarrow 0} V_Y^q(\vec{k}) = \frac{4\pi C}{k^2}$$

$\boxed{\text{Note: } e^{-qr} \text{ represents a convergence factor}}$

Alternative method:

$$V^{(int)}(\vec{k}) = \int d^3x\, e^{-i\vec{k}\vec{x}} \underbrace{V^{(int)}(\vec{x})}_{= C \frac{1}{|\vec{x}|} = C \frac{1}{\sqrt{\vec{x}^2}}}$$

Schwinger trick: based on Gamma function

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} \quad ; \ x > 0$$

$$\Gamma(x+1) = \int_0^\infty dt\, \underset{v}{t^x}\, \underset{u'}{e^{-t}} = \cancel{\left[t^x(-e^{-t})\right]_0^\infty} - \int_0^\infty dt\,(-e^{-t}) \times t^{x-1} = x\,\Gamma(x)$$

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n!\,\underbrace{\Gamma(1)}_{=1}$$

$$\Gamma(x) = a^x \int_0^\infty d\tau\, \tau^{x-1} e^{-a\tau} \quad \Rightarrow \quad \boxed{\frac{1}{a^x} = \frac{1}{\Gamma(x)} \int_0^\infty d\tau\, \tau^{x-1} e^{-a\tau}}$$

$t = a\tau$ $\qquad\qquad$ $x = 1/2$

$$V^{(int)}(\vec{k}) = c \int d^3x\, e^{-i\vec{k}\vec{x}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty d\tau\, \tau^{-\frac{1}{2}} e^{-\tau \vec{x}^2}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty dt\, t^{-\frac{1}{2}} e^{-t} \overset{t=s^2}{=} \int_0^\infty ds\, 2s\, \frac{1}{s} e^{-s^2} = \sqrt{\pi}$$

$dt = 2s\,ds$

$$= \frac{c}{\sqrt{\pi}} \int_0^\infty d\tau\, \tau^{-\frac{1}{2}} \underbrace{\int d^3x\, e^{-\tau\vec{x}^2 - i\vec{k}\vec{x}}}_{=\left(\frac{\pi}{\tau}\right)^{3/2} e^{-\frac{1}{4}\frac{\vec{k}^2}{\tau}}} = c\pi \int_0^\infty d\tau\, \tau^{-2} e^{-\frac{\vec{k}^2}{4\tau}}$$

$u = \frac{1}{\tau}$, $du = -\frac{1}{\tau^2} d\tau$

$$= c\pi \int_0^\infty du\, \tau^2 \frac{1}{\tau^2} e^{-\frac{\vec{k}^2}{4}u} = \frac{4\pi c}{\vec{k}^2}$$

Coulomb potential = long-range potential

7.1.2 Dipole-Dipole Interaction:

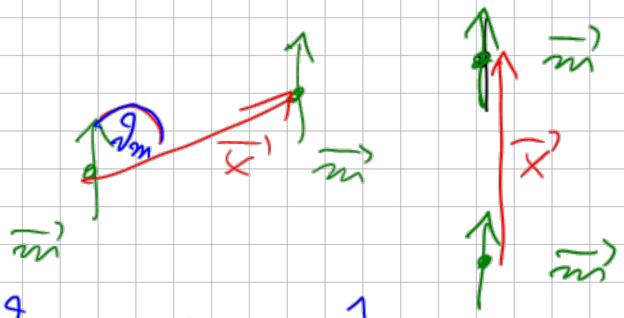
two magnetic dipoles (atoms) $\qquad\qquad$ two heteronuclear molecules

$$V_{DD}(\vec{x}) = -\frac{\mu_0}{4\pi}\,\frac{3(\vec{x}\cdot\vec{m})^2 - \vec{m}^2\vec{x}^2}{|\vec{x}|^5}$$

$$V_{DD}(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\,\frac{3(\vec{x}\cdot\vec{p})^2 - \vec{x}^2\vec{p}^2}{|\vec{x}|^5}$$

repulsion

magnetic dipole moment



attraction

$\vartheta_m = \arccos\frac{1}{\sqrt{3}} = 54.74°$

magic

$\vec{p}$: electric dipole moment

- anisotropic
- $\sim \frac{1}{r^3}$: "long"-range

$$V^{(magn.)}_{DD} \sim \frac{\mu_0}{4\pi}\,\frac{m^2}{r^3}$$

$$V^{(elect)}_{DD} \sim \frac{1}{4\pi\varepsilon_0}\,\frac{p^2}{r^3}$$

$$m \sim \underbrace{m_B}_{\text{Bohr magneton}} = I\,F = \frac{e\hbar}{2M}$$

$$p \sim e\,a_B$$

$$F = \pi a_B^2\,,\quad I = \frac{e}{T} = \frac{e}{2\pi}\,\omega = \frac{e}{2\pi}\,\frac{\hbar}{M a_B^2}$$

$$L = M v a_B = M\omega a_B^2 = \hbar \Rightarrow \omega = \frac{\hbar}{M a_B^2}$$

semiclassical Bohr-Sommerfeld quantization

$$\frac{V^{(magn)}_{DD}}{V^{(elect)}_{DD}} \sim \frac{\frac{\mu_0}{4\pi}\left(\frac{e\hbar}{2M}\right)^2}{\frac{1}{4\pi\varepsilon_0}\,e^2 a_B^2} = \text{- - - -} = \alpha^2 \sim 10^{-4}$$

$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m e^2}$

α: Sommerfeld fine structure constant generic form: $V_{DD}(\vec{x}) = -C_{DD} \frac{3(\vec{x}\vec{\mu})^2 - \vec{\mu}^2 \vec{x}^2}{|\vec{x}|^5}$

$$V_{DD}(\vec{k}) = \int d^3x \; e^{-i\vec{k}\vec{x}} \; V_{DD}(\vec{x})$$

$\vec{k} = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \vec{x} = r \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$ $\quad \vec{\mu} = \mu \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix}$

$\longrightarrow$ see later notes

7.1.3 Distributional Identities:

$$\frac{\partial}{\partial x_i} \frac{1}{|\vec{x}|} = - \frac{x_i}{|\vec{x}|^3}, \quad \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\vec{x}|} = \dots = \frac{3x_i x_j - \delta_{ij}\vec{x}^2}{|\vec{x}|^5} \qquad \vec{x} \neq \vec{0}$$

$$\Delta \frac{1}{|\vec{x}|} = -4\pi \, \delta(\vec{x})$$

$$\frac{\partial^2}{\partial x_i^2} \frac{1}{|\vec{x}|} = \frac{1}{3} \Delta \frac{1}{|\vec{x}|} = -\frac{4\pi}{3} \delta(\vec{x})$$

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\vec{x}|} = \frac{3x_i x_j - \delta_{ij}\vec{x}^2}{|\vec{x}|^5} - \frac{4\pi}{3} \delta_{ij} \, \delta(\vec{x})$$

$$V_{DD}(\vec{x}) = -C_{DD} \, \mu_i \mu_j \underbrace{\frac{3x_i x_j - \delta_{ij}\vec{x}^2}{|\vec{x}|^5}}_{\frac{\partial^2}{\partial x_i \partial x_j} \underbrace{\frac{1}{|\vec{x}|}}_{\int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} e^{-i\vec{k}\vec{x}}} + \frac{4\pi}{3} \delta_{ij} \underbrace{\delta(\vec{x})}_{\int \frac{d^3k}{(2\pi)^3} 1 \, e^{-i\vec{k}\vec{x}}}}$$

$$= \int \frac{d^3k}{(2\pi)} e^{-i\vec{k}\vec{x}} \; V_{DD}(\vec{k}), \quad V_{DD}(\vec{k}) = \frac{4\pi}{3} C_{DD} \left\{ \frac{3(\vec{k}\vec{\mu})^2}{\vec{k}^2} - \vec{\mu}^2 \right\}$$

7.2 Center-of-mass system (distinguishable particles)

$$\left\{ -\frac{\hbar^2}{2m_1}\Delta_{\vec{x}_1} - \frac{\hbar^2}{2m_2}\Delta_{\vec{x}_2} + V^{(int)}(\overbrace{\vec{x}_1, \vec{x}_2}^{\vec{x}_1-\vec{x}_2}) \right\} \Psi(\vec{x}_1, \vec{x}_2) = E\, \Psi(\vec{x}_1, \vec{x}_2)$$

collective coordinates: $\vec{X} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$, $\vec{x} = \vec{x}_1 - \vec{x}_2$

$\Rightarrow \vec{x}_1 = \vec{X} + \frac{m_2}{m_1+m_2}\vec{x}$, $\vec{x}_2 = \vec{X} - \frac{m_1}{m_1+m_2}\vec{x}$

$$\left\{ -\frac{\hbar^2}{2m_{tot}}\Delta_{\vec{X}} - \frac{\hbar^2}{2\mu}\Delta_{\vec{x}} + V^{(int)}(\vec{x}) \right\} \Psi(\vec{X}, \vec{x}) = E\, \Psi(\vec{X}, \vec{x})$$

$m_{tot} = m_1 + m_2$

$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$ reduced mass

$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

separation ansatz: $\Psi(\vec{X}, \vec{x}) = \frac{e^{i\vec{K}\vec{X}}}{(2\pi)^3} \cdot \psi(\vec{x})$, $E = \frac{\hbar^2 \vec{K}^2}{2m} + \varepsilon$

scattering: $\varepsilon = \frac{\hbar^2 \vec{k}^2}{2m}$

$$\left\{ -\frac{\hbar^2}{2\mu}\Delta_{\vec{x}} + V^{(int)}(\vec{x}) \right\} \psi(\vec{x}) = \varepsilon\, \psi(\vec{x})$$

Effective 1-particle problem: μ and $V^{(int)}(\vec{x})$