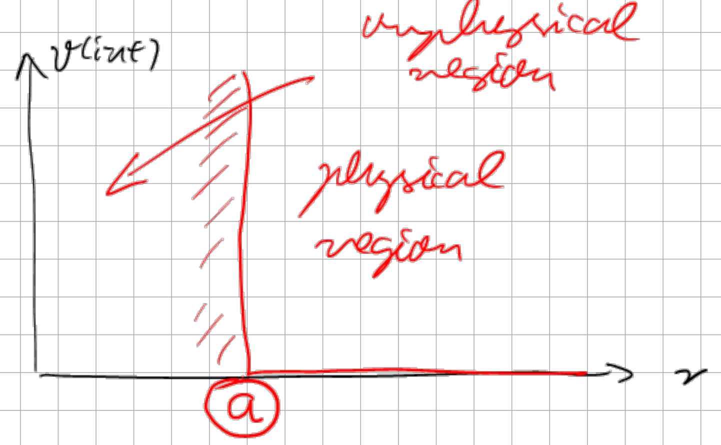
Summary:

hard spheres



$$R_\ell(r) = \begin{cases} 0 & ;\ 0 \le r \le a \\ A_\ell\, j_\ell(kr) + B_\ell\, n_\ell(kr) & ;\ r > a \end{cases}$$

phase shift

$$\delta_\ell = -\frac{(2\ell+1)(ka)^{2\ell+1}}{[(2\ell+1)!!]^2}$$

repulsive

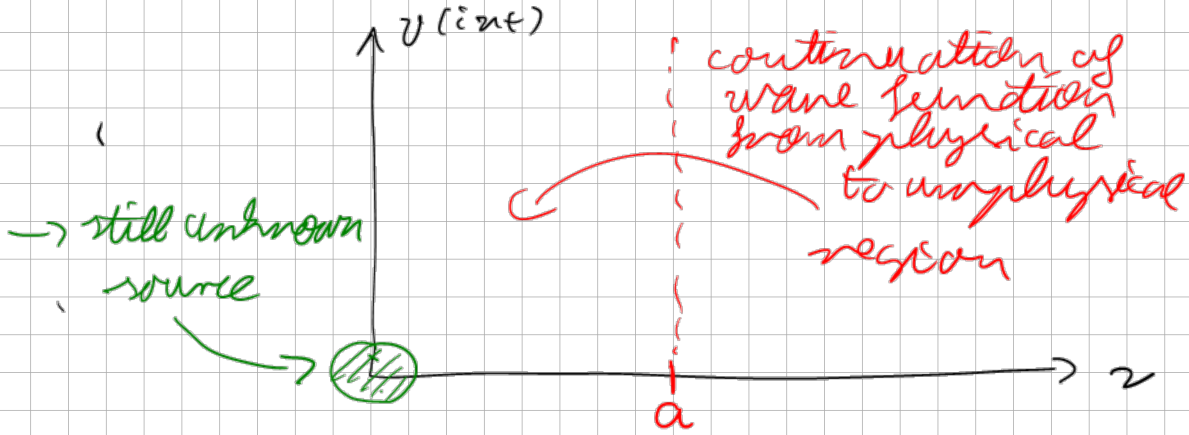
$$k = \sqrt{\frac{2E\mu}{\hbar^2}}$$

low energies, i.e. small k : $ka \ll 1$

$\Rightarrow$ $\ell = 0$ dominant contribution

s-wave scattering: $\delta_0 = -ka$

7.6 pseudopotential method



$$\tilde{R}_0(r) = A_0\, j_0(kr) + B_0\, n_0(kr) ;\ 0 \le r \le a$$

$$j_0(\rho) = \frac{\sin\rho}{\rho}, \quad n_0(\rho) = -\frac{\cos\rho}{\rho}$$

$$\boxed{\tilde{R}_0(r) = \frac{A_0 \sin(kr) - B_0 \cos(kr)}{kr}} = \frac{C_0 \sin(kr + \delta_0)}{kr}$$

$$\left.\begin{aligned} A_0 &= C_0 \cos\delta_0 \\ B_0 &= -C_0 \sin\delta_0 \end{aligned}\right\} \quad \delta_0 = -\arctan\frac{B_0}{A_0}$$

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{2\mu}{\hbar^2} v^{(int)}(r) \right\} \tilde{R}_0(r) = 0 \quad ;\ r \ge 0$$

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2\right\}\tilde{R}_0(r) = 0 \quad ; \quad r > 0$$ Helmholtz equation

$v^{(int)}(r)$ must be a "destributional source"

$$\int_{C(R)} d^3x \left\{ \underbrace{\Delta_{\vec{x}}}_{=\vec{D}_{\vec{x}}\cdot\vec{D}_{\vec{x}}} + k^2 \right\} \tilde{R}_0(r) \overset{\text{Gauß}}{=} \cancel{\underbrace{\oint_{\partial C(R)} \underbrace{d\vec{f}}_{R^2 \sin\vartheta\, d\vartheta\, d\varphi\, \vec{e}_r} \cdot \underbrace{\vec{D}_{\vec{x}} \tilde{R}_0(r)}_{\vec{e}_r\, \tilde{R}_0'(r)\,,\ r=R}}_{4\pi R^2 \frac{\partial \tilde{R}_0(R)}{\partial R}}} + \underbrace{k^2 \int_{C(R)} d^3x\, \tilde{R}_0(r)}_{=4\pi \cancel{k^2} \int_0^R dr\, r^2 \underbrace{k^2}_{= -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}} \tilde{R}_0(r)}$$

$$= -4\pi \left\{ \int_0^R dr\, r^2 \frac{\partial^2}{\partial r^2} \tilde{R}_0(r) + 2\int_0^R dr\, r \frac{\partial}{\partial r} \tilde{R}_0(r) \right\}$$

$$= -4\pi \left\{ \left[r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right]_0^R \cancel{- \int_0^R dr\, 2r \frac{\partial \tilde{R}_0(r)}{\partial r} + 2\int_0^R dr\, r \frac{\partial}{\partial r} \tilde{R}_0(r)} \right\}$$

$$= -4\pi \left\{ \cancel{R^2 \frac{\partial \tilde{R}_0(R)}{\partial R}} - \lim_{r \downarrow 0} \left(r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right) \right\}$$

$$\int_{C(R)} d^3x \left\{ \Delta_{\vec{x}} + k^2 \right\} \tilde{R}_0(r) = 4\pi \underbrace{\lim_{r \downarrow 0} \left\{ r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right\}}_{= \frac{B_0}{k}}$$ valid for all radii R

$$\tilde{R}_0(r) = \frac{A_0 \sin(kr) - B_0 \cos(kr)}{kr}$$

$$\frac{\partial \tilde{R}_0(r)}{\partial r} = -\frac{1}{k}\frac{1}{r^2}\left\{ A_0 \sin(kr) - B_0 \cos(kr) \right\} + \frac{\cancel{k}}{\cancel{k}\, r}\left\{ A_0 \cos(kr) + B_0 \sin(kr) \right\}$$

$$\lim_{r \downarrow 0} \left\{ r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right\} = \frac{B_0}{k}$$

$$\Rightarrow \left(\Delta_{\vec{x}} + k^2\right) \tilde{R}_0(r) = \frac{4\pi B_0}{k} \delta(\vec{x}) = \frac{4\pi \tan(ka)}{k} \lim_{r \downarrow 0} \frac{\partial}{\partial r}\left(r \tilde{R}_0(r)\right) \delta(\vec{x})$$

$$\tan \delta_0 = -\frac{B_0}{A_0} \qquad \delta_0 = -ka \quad (ka \ll 1) \qquad \Rightarrow \tan(ka) = \frac{B_0}{A_0}, \quad B_0 = A_0 \tan(ka)$$

$$r \tilde{R}_0(r) = \frac{1}{k}\left\{ A_0 \sin(kr) - B_0 \cos(kr) \right\}$$

$$\frac{\partial}{\partial r}\left(r \tilde{R}_0(r)\right) = A_0 \cos(kr) + B_0 \sin(kr) \Bigg\} \Rightarrow \lim_{r \downarrow 0} \frac{\partial}{\partial r}\left(r \tilde{R}_0(r)\right) = A_0$$

$$\left(\Delta_{\vec{x}} + k^2\right) \tilde{R}_0(r) = \frac{4\pi \tan(ka)}{k} \delta(\vec{x}) \frac{\partial}{\partial r}\left(r \tilde{R}_0(r)\right)$$

$$= \frac{2\mu}{\hbar^2} v^{(int)}(r) \tilde{R}_0(r)$$

you are tempted to ignore it

pseudopotential

$$\Rightarrow v^{(int)}(r) \bullet = \frac{\hbar^2}{2\mu} 4\pi \underbrace{\frac{\tan(ka)}{k}}_{ka \ll 1 \;= a} \boxed{\delta(\vec{x}) \frac{\partial}{\partial r}\left(r \;\bullet\right)}$$

contact interaction: $\rightarrow$ short-ranged, $\rightarrow$ isotropic

$$\frac{2\pi \hbar^2 a}{\mu} \qquad \frac{\mu = \frac{M}{2}}{M_1 = M_2 = M} \qquad \frac{4\pi \hbar^2 a_s}{M} = g \leftarrow$$

strength of contact interaction

s-wave scattering length

example: $a_s(^{87}Rb) = +100\, a_B$

$\uparrow$ repulsive

Remarks: $$\hat{O} \bullet = \frac{\partial}{\partial r}\left(r \;\bullet\right)\Big|_{r=0}$$

- $f(r)$ regular: $\hat{O} f(r) = \frac{\partial}{\partial r}\left(r f(r)\right)\Big|_{r=0} = \left\{ f(r) + r f'(r) \right\}\Big|_{r=0} = f(0)$

$\Rightarrow \hat{O} = 1$ for regular functions

- $f(r) = \frac{k}{r}$, $\hat{O} f(r) = \frac{\partial}{\partial r}\left(r \cdot \frac{k}{r}\right)\Big|_{r=0} = 0$

$\Rightarrow$ singular function like $\frac{k}{r}$ does not contribute

---

Possible Topics for Student Talks (suggestions)

- Bender-Wu recursion relation } combined
- Variational perturbation theory
- Wentzel-Kramers-Brillouin (WKB) approximation
- EPR paradox, Bell inequality, Physics Nobel Prize 2022 (2 people)
- Bohm quantum mechanics
- Wigner function: $W(\vec{x}, \vec{p})$
- Path integral of harmonic oscillator on a (time) lattice
- Semiclassical path integral
- Perturbation theory with path integrals
- Scattering theory in 2D
- Scattering theory of dipole-dipole interaction

Rules:

- deadline: 31.12.22, first come - first served

- dates of talks: ??? ⇒ at the beginning of 2023
  ⇒ roughly: second half of January
- 10 minutes talk + 10 minutes discussion

Announcement:

1) No lectures in week 12. – 16.12.

2) December 1, Thursday: 13.45 – 15.15
   + 15.15 – 15.30 break
   + 15.30 – 17.00 → 1 substitute

7.7 Scattering Amplitude:

Schrödinger equation of scattering problem: $\Big\{ \Delta_{\vec{x}} + \overbrace{k^2}^{=\frac{2\mu E}{\hbar^2}} - \frac{2\mu}{\hbar^2} V^{(int)}(\vec{x}) \Big\} \psi(\vec{x}) = 0$ stationary scattering problem

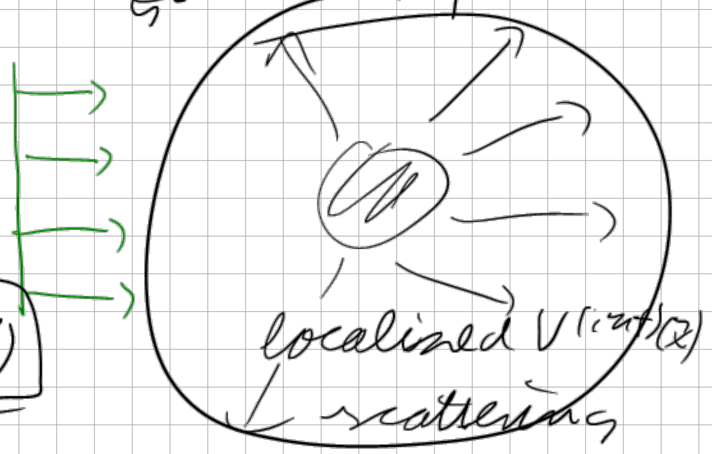
→ differential equation

Equivalent integral equation:

$$\left(\Delta_{\vec{x}} + k^2\right) \psi(\vec{x}) = \frac{2\mu}{\hbar^2} V^{(int)}(\vec{x}) \, \psi(\vec{x})$$

$$\psi(\vec{x}) = \underbrace{e^{i \vec{k} \cdot \vec{x}}}_{\text{homogeneous solution}} + \underbrace{\boxed{\frac{2\mu}{\hbar^2} \int d^3x' \, G(\vec{x}, \vec{x}') \, V^{(int)}(\vec{x}') \, \psi(\vec{x}')}}_{\text{particular solution}}$$

Green's function of Helmholtz equation



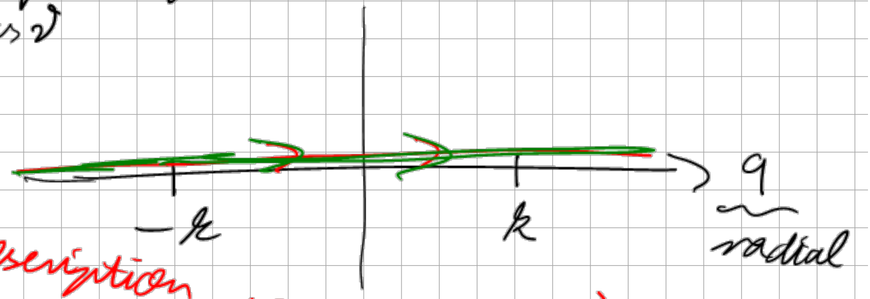
$$(\Delta_{\vec x} + k^2)\, \underbrace{G(\vec x, \vec x')}_{= G(\vec x - \vec x')} = \delta(\vec x - \vec x')$$

$$= G(\vec x - \vec x') = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec q(\vec x - \vec x')}\, G(\vec q)$$

$$\Rightarrow \quad (q^2 - k^2)\, G(\vec q) = -1 \quad \Rightarrow \quad G(\vec q) = \frac{-1}{q^2 - k^2}$$

$$G(\vec x - \vec x') = \frac{-1}{(2\pi)^3} \lim_{\varepsilon \downarrow 0} \underbrace{\int d^3q}_{= \int_0^\infty dq\, q^2 \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi} \frac{e^{i\vec q\cdot(\vec x - \vec x')}}{q^2 - k^2 - i\varepsilon}$$

$$\vec q\cdot(\vec x - \vec x') \to q|\vec x - \vec x'|\cos\vartheta$$



Feynman prescription

$\hat{=}$ causality (justification in 2023)

$$= \frac{-2\pi}{8\pi^3} \lim_{\varepsilon \downarrow 0} \int_0^\infty dq\, \frac{q^2}{q^2 - k^2 - i\varepsilon} \cdot \underbrace{\int_0^\pi d\vartheta \sin\vartheta\, e^{iq|\vec x - \vec x'|\cos\vartheta}}_{u = \cos\vartheta}$$

$= \ldots$ next time