13 Dirac Equation:

Motivation:

- Klein-Gordon equation: linear differential equation of second order in space and time, describing spin 0 relativistic quantum particles.
- Instead Dirac equation is a relativistic wave equation which is linear and of first order in space and time. Describes spin 1/2 particles.
- The Dirac equation describes correctly the fine structure of hydrogen atom.

13.1 Heuristic Derivation:

- Ansatz: linear differential equation of first order

$$(*)\left(i\gamma^\mu \partial_\mu - \underbrace{\frac{mc}{\hbar}}_{\text{"mass term"}} \right) \underbrace{\Psi}_{\text{spinor}} (x^\lambda) = 0$$

"mass term" $\hat{=}$ inverse Compton wave length

spinor

x^λ: space-time four vector

$(x^\lambda) = (ct, x, y, z)$ contravariant components

$\lambda = 0, 1, 2, 3$ (latin index)

$(x_\lambda) = (ct, -x, -y, -z)$ covariant components

γ^μ: coefficients not yet known

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \hat{=} \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$ four-dimensional gradient

$$\underbrace{\gamma^\mu \partial_\mu}_{} \underset{\substack{\text{Einstein} \\ \text{summation} \\ \text{convention}}}{=} \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu = \gamma^0 \underbrace{\frac{1}{c}\frac{\partial}{\partial t}}_{} + \gamma^1 \underbrace{\frac{\partial}{\partial x}}_{} + \gamma^2 \underbrace{\frac{\partial}{\partial y}}_{} + \gamma^3 \underbrace{\frac{\partial}{\partial z}}_{}$$

$\underline{\left(-i\gamma^\nu\partial_\nu - \frac{mc}{\hbar}\right)(*)}$: $\left(-i\gamma^\nu\partial_\nu - \frac{mc}{\hbar}\right)\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\Psi(x\lambda) = 0$

$$\Big(\underbrace{\gamma^\nu\gamma^\mu}\ \underbrace{\partial_\nu\partial_\mu}_{\partial_\mu\partial_\nu} - \cancel{\frac{mc}{\hbar}\, i\gamma^\mu\partial_\mu} + \cancel{i\gamma^\nu\partial_\nu\frac{mc}{\hbar}} + \frac{m^2c^2}{\hbar^2}\Big)\Psi(x\lambda) = 0$$

(second derivatives commute)

$$\Big\{\frac{1}{2}\left(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right)\partial_\mu\partial_\nu + \frac{m^2c^2}{\hbar^2}\Big\}\Psi(x\lambda) = 0$$

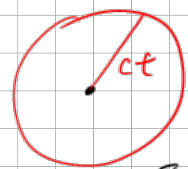
$\hat{=}$ Klein-Gordon equation provided that

$$\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = \boxed{\frac{1}{2}[\gamma^\mu,\gamma^\nu]_+ = \eta^{\mu\nu}}$$

→ Clifford algebra

Minkowski metric:

$$(\eta^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



light cone propagation

$x^\mu x^\nu \eta_{\mu\nu} = 0$

$c^2t^2 - \vec{x}^2 = 0$

$$\Big(\underbrace{\eta^{\mu\nu}\partial_\mu\partial_\nu} + \frac{m^2c^2}{\hbar^2}\Big)\Psi(x\lambda) = 0$$

$$\underbrace{\eta^{00}}_{=1}\partial_0^2 + \underbrace{\eta^{11}}_{=-1}\partial_1^2 + \underbrace{\eta^{22}}_{=-1}\partial_2^2 + \underbrace{\eta^{33}}_{=-1}\partial_3^2 = \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

Mathematical consequence: 4 elements of Clifford algebra $\gamma^0, \gamma^1, \gamma^2, \gamma^3$

→ They are not simply complex numbers!

$\mu = \nu = 0$: $\underbrace{(\gamma^0)^2 = 1}$, $\mu = \nu = \underbrace{i}_{1,2,3}$: $(\gamma^i)^2 = -1$ → possible implemented by complex numbers

$\mu \neq \nu$: $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ → not fulfilled for complex numbers

Next guess: γ^μ must be a matrix $\Rightarrow$ Ψ must have several components

Rewrite Dirac equation in terms of a Schrödinger-type equation

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\Psi(x^\lambda) = 0 \Rightarrow \left(i\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + i\gamma^k\partial_k - \frac{mc}{\hbar}\right)\Psi(\vec{x},t) = 0 \quad \Big| \, \hbar c\gamma^0 \cdot$$

$$i\hbar\underbrace{(\gamma^0)^2}_{=1}\frac{\partial}{\partial t}\Psi(\vec{x},t) = \Big\{c\underbrace{\gamma^0\gamma^k}_{=\alpha^k}\frac{\hbar}{i}\partial_k + mc^2\underbrace{\gamma^0}_{\beta}\Big\}\Psi(\vec{x},t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t) = \hat{H}\,\Psi(\vec{x},t) \,, \quad \hat{H} = c\,\frac{\hbar}{i}\vec{\nabla}\cdot\vec{\alpha} + mc^2\beta$$

<u>Algebra of α^k, β:</u>

$$[\beta,\beta]_+ = 2\beta^2 = 2(\gamma^0)^2 = 2$$

$$[\alpha^k,\beta]_+ = \ldots = 0$$

$$[\alpha^k,\alpha^l]_+ = \ldots = \boxed{2\delta^{kl}}$$

First idea: realize α^i, β with matrices; what about 2×2 matrices

Pauli matrices σ^k with $[\sigma^k,\sigma^l]_+ = \boxed{2\delta^{kl}}$ and $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

form a basis in space of 2×2 matrices with

$$[\sigma^k, \underbrace{I}_{=\beta}]_+ = 2\sigma^k \neq 0$$

Realization of α^k, β with 2×2 matrices fails

4×4-matrices?

$$\vec{\alpha} = \left(\begin{array}{c|c} 0 & \vec{\sigma} \\ \hline \vec{\sigma} & 0 \end{array}\right), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow [\beta, \beta]_+ = 2, \quad [\alpha^k, \beta]_+ = 0, \quad [\alpha^k, \alpha^l]_+ = 2\delta^{kl} \quad \checkmark$$

This means that Ψ must have 4 components. This turns out not to be vector but a spinor due to its special transformation properties concerning rotations.

13.2 Continuity:

$$(+) \quad i\hbar \frac{\partial}{\partial t}\Psi = -i\hbar c(\vec{\alpha}\vec{\nabla}\Psi) + \cancel{mc^2\beta\Psi} \quad \Big| \; \Psi^+ \cdot$$

$$(-) \quad -i\hbar \frac{\partial}{\partial t}\Psi^+ = +i\hbar c(\vec{\nabla}\Psi^+)(\underbrace{\vec{\alpha}^+}_{=\vec{\alpha}}) + \cancel{mc^2\Psi^+\underbrace{\beta^+}_{=\beta}} \quad \Big| \cdot \Psi$$

$=\beta$ self-adjoint

$$i\hbar \underbrace{\left(\Psi^+\frac{\partial}{\partial t}\Psi + \frac{\partial\Psi^+}{\partial t}\Psi\right)}_{=\frac{\partial}{\partial t}(\underbrace{\Psi^+\Psi}_{=\rho})} = -i\hbar c\{\underbrace{(\Psi^+\vec{\alpha}\vec{\nabla}\Psi + (\vec{\nabla}\Psi^+)\vec{\alpha}\Psi}_{=\vec{\nabla}(\Psi^+\vec{\alpha}\Psi)}\}$$

$$\Rightarrow \text{continuity eq.:} \quad \frac{\partial}{\partial t}\underbrace{(\underbrace{\Psi^+\Psi}_{=\rho})}_{\text{density is positive definite}} + \operatorname{div}\underbrace{(c\,\Psi^+\vec{\alpha}\,\Psi)}_{=\vec{j}} = 0$$

13.3 Nonrelativistic Limit:

interaction of spin 1/2 with charge q to electromagnetic field: minimal coupling

$$\hat{\vec{p}} \Rightarrow \hat{\vec{p}} - q\vec{A}, \quad \hat{E} \Rightarrow \hat{E} - q\varphi$$

$$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\,\Psi, \quad \hat{H} = c\,\vec{\alpha}\cdot(\hat{\vec{p}} - q\vec{A}) + \underline{mc^2}\,\boxed{\beta} + q\varphi \longrightarrow \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Note: local gauge transformation like in Schrödinger

structure of β-matrix: upper two spinor components are lifted by $+mc^2$, lower $-mc^2$

Non-relativistic ansatz:

$$\underbrace{\Psi}_{\text{Dirac spinor}} = \begin{pmatrix} u \\ v \end{pmatrix} \begin{matrix} \leftarrow \text{upper Weyl spinor (2 components)} \\ \leftarrow \text{lower Weyl spinor ''} \end{matrix}$$

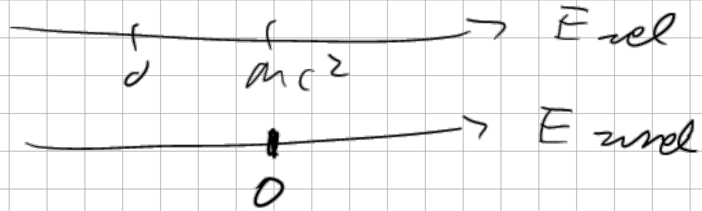
$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \left\{ c \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} (\hat{\vec{p}} - q\vec{A}) + mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + q\varphi \right\} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$i\hbar\frac{\partial u}{\partial t} = c\,\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})\,v + (+mc^2 + q\varphi)\,u$$

$$i\hbar\frac{\partial v}{\partial t} = c\,\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})\,u + (-mc^2 + q\varphi)\,v$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} e^{-\frac{i}{\hbar} mc^2 t}$$

non-relativistic limit



$$i\hbar\frac{\partial \tilde{u}}{\partial t} = c\,\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})\,\tilde{v} + q\varphi\,\tilde{u} \quad (1)$$

$$\underbrace{\cancel{i\hbar\frac{\partial \tilde{v}}{\partial t}}}_{\approx 0} = c\,\vec{\sigma}(\hat{\vec{p}} - q\vec{A})\,\tilde{u} + (\underbrace{-2mc^2}_{} + \underbrace{\cancel{q\varphi}}_{\text{negligible in nonrelativistic limit}})\,\tilde{v}$$

$\tilde{v}$ changes fast in time

Adiabatic elimination of lower Weyl spinor:

$\tilde{u}$ slowly varying
$\tilde{v}$ fast varying $\to$ eliminated

$$0 \approx c\,\vec{\sigma}(\hat{\vec{p}} - q\vec{A})\,\tilde{u} - 2mc^2\,\tilde{v}$$

$\tilde{v} = \frac{1}{2mc^2} c\vec{\sigma}\,(\vec{p} - q\vec{A})\,\tilde{u}$ (2): $\tilde{v}$ follows adiabatically $\tilde{u}$ dynamics

(2) in (1):
$$i\hbar \frac{\partial \tilde{u}}{\partial t} = \frac{1}{2m}\underbrace{\left[\vec{\sigma}(\vec{p} - q\vec{A})\right]\left[\vec{\sigma}\cdot(\vec{p} - q\vec{A})\right]}_{= ?}\tilde{u} + q\varphi\,\tilde{u}$$

Properties of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clifford algebra: $[\sigma^k, \sigma^l]_+ = \sigma^k\sigma^l + \sigma^l\sigma^k = 2\delta^{kl}\,I$

Lie algebra: $[\sigma^k, \sigma^l]_- = \sigma^k\sigma^l - \sigma^l\sigma^k = 2i\,\varepsilon_{klm}\sigma^m \leftarrow$

$$\sigma^k\sigma^l = \delta^{kl}\,I + i\,\varepsilon_{klm}\sigma^m \quad (*)$$

Physical consequence:

$$\underbrace{\boxed{\hat{S}^k = \frac{\hbar}{2}\sigma^k}}_{\text{fulfills angular momentum algebra}},\quad [\hat{S}^k, \hat{S}^l]_- = \dots = i\hbar\,\varepsilon_{klm}\hat{S}_m$$

Which quantum number s corresponds to this angular momentum

$$\hat{\vec{S}}^2 = \sum_k \hat{S}^k\hat{S}^k = \frac{\hbar^2}{4}\underbrace{\sum_k \sigma^k\sigma^k}_{\text{Clifford } = 2\delta^{kk} = 2} = \frac{\hbar^2}{4}\sum_k 2 = \frac{\hbar^2}{4}\cdot 2\cdot 3 = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) = \underbrace{\hbar^2 s(s+1)}$$

$$\Rightarrow \boxed{s = \frac{1}{2}}$$

Result: The Dirac equation describes spin 1/2 particles.

$$\boxed{} = \sigma^k\sigma^l\left(\frac{\hbar}{i}\partial_k - qA_k\right)\left(\frac{\hbar}{i}\partial_l - qA_l\right)$$

$$\overset{(*)}{=} (\hat{\vec{p}} - q\vec{A})^2 + i\vec{\sigma}\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)\times\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)$$

$$= \underbrace{\frac{\hbar}{i}\vec{\nabla}\times\frac{\hbar}{i}\vec{\nabla}}_{=0} - \cancel{q\vec{A}\times\frac{\hbar}{i}\vec{\nabla}} - \underbrace{\frac{\hbar}{i}\vec{\nabla}\times q\vec{A}}_{\text{product rule}} + \cancel{q\vec{A}\times q\vec{A}}$$

$$= q\frac{\hbar}{i}\underbrace{\mathrm{rot}\,\vec{A}}_{=\vec{B}} - \cancel{q\frac{\hbar}{i}\vec{A}\times\vec{\nabla}}$$

Pauli equation for $\tilde{u}$:

$$i\hbar\frac{\partial}{\partial t}\tilde{u} = \Big[\underbrace{\frac{1}{2m}(\vec{p}-q\vec{A})^2}_{} + \underbrace{\frac{-q\hbar}{2m}\vec{\sigma}\cdot\vec{B}}_{=\frac{-q}{2m}\vec{S}\cdot\vec{B}\cdot\underbrace{2}_{=g_S}} + q\varphi\Big]\tilde{u}$$

Landé factor

$$\overset{\text{see above}}{=} \frac{\vec{p}^{\,2}}{2m} - \frac{1}{2m}2q\vec{A}\cdot\vec{p} - \frac{q}{2m}\frac{\hbar}{i}\underbrace{\mathrm{div}\,\vec{A}}_{=0} + \underbrace{\cancel{\frac{q^2\vec{A}^2}{2m}}}_{=\text{quadratic Zeeman}}$$

const. magn. field

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{x}$$

$$\mathrm{rot}\,\vec{A} = \vec{B}$$

$$\frac{-q}{m}\frac{1}{2}(\vec{B}\times\vec{x})\cdot\vec{p} = \frac{-q}{2m}\underbrace{(\vec{x}\times\vec{p})}_{=\vec{L}}\cdot\vec{B}\cdot\underbrace{1}_{=g_\ell}$$

Conclusion: Heuristic derivation of Dirac equation with the aim to get a relativistic wave equation led us to insight that spin 1/2 particles with Landé factor $g_S = 2$ are described.