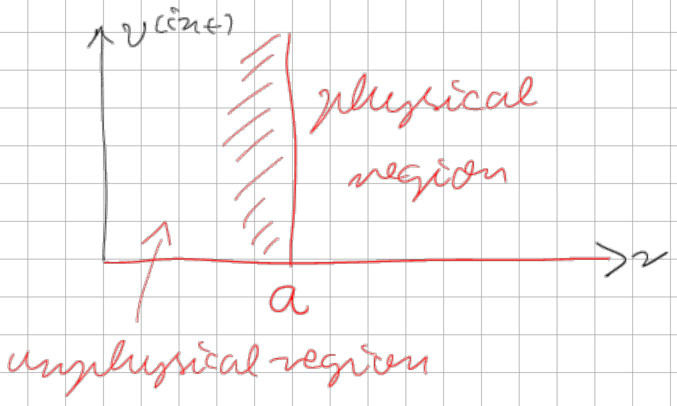
Summary:

hard spheres:



$$R_0(r) = \begin{cases} 0 & ; 0 \le r \le a \\ A_0\, j_0(kr) + B_0\, n_0(ka) & ; r > a \end{cases}$$

$\ell = 0$

analytic contin.

$$k = \sqrt{\frac{2E\mu}{\hbar^2}} \; ; \; \mu: \text{reduced mass}$$

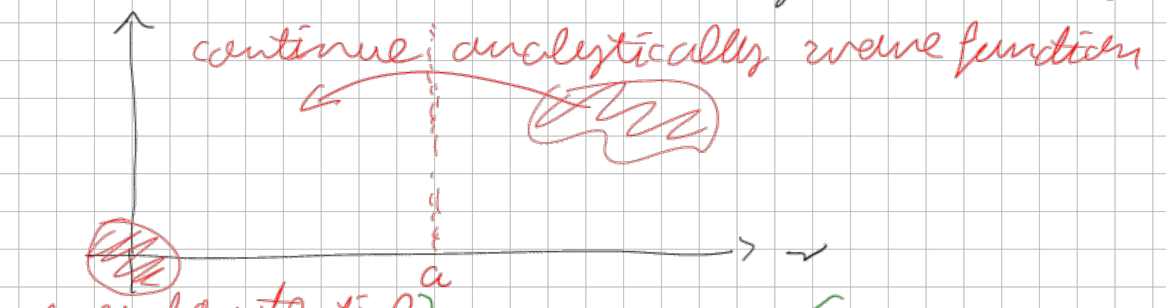
small energies: only $\ell = 0$ relevant (s-wave scattering)

$$\delta_0 = - ka$$

phase shift

repulsive

pseudopotential method: (goes back to Enrico Fermi; regularized by Kerson Huang)



pseudopotential at origin $r = 0$ (distribution) } WANTED

$$\tilde{R}_0(r) = A_0\, j_0(kr) + B_0\, n_0(kr) \; ; \; r > 0$$

$$j_0(\rho) = \frac{\sin\rho}{\rho}, \quad n_0(\rho) = -\frac{\cos\rho}{\rho}$$

$$\tilde{R}_0(r) = \frac{A_0 \sin(kr) - B_0 \cos(kr)}{kr} = \frac{C\sin(kr+\delta_0)}{r}$$

$A_0 = C\cos\delta_0 \qquad C = \sqrt{A_0^2 + B_0^2}$

$B_0 = -C\sin\delta_0 \qquad \delta_0 = -\arctan\frac{B_0}{A_0}$

$\delta_0 = -ka$: GIVEN

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key idea:

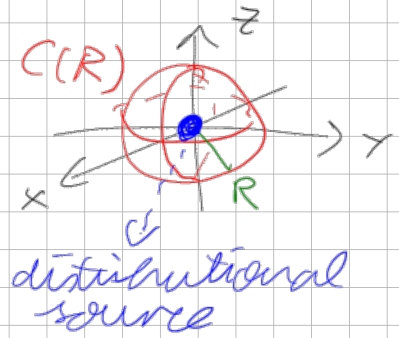
$$\Big\{ \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}}_{\Delta_{\vec{x}}} + k^2 - \frac{2\mu}{\hbar^2} \underbrace{v^{(int)}(r)}_{\text{WANTED}} \Big\} \underbrace{\tilde{R}_0(r)}_{\text{known}} = 0$$

$r > 0$: $\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 \right\} \tilde{R}_0(r) = 0 , \quad r > 0$ Helmholtz equation (*)

The wanted pseudopotential $v^{(int)}(r)$ vanishes for all $r > 0$

$\Rightarrow$ $v^{(int)}(r)$ must be a "distributional source" at the origin $r = 0$

Choose sphere $C(R)$ of radius R:



$$\int_{C(R)} d^3x \left\{ \underbrace{\Delta_{\vec{x}}}_{= \vec{\nabla}_{\vec{x}} \cdot \vec{\nabla}_{\vec{x}}} + k^2 \right\} \tilde{R}_0(r)$$

$$\overset{\text{Gauß}}{=} \cancel{\oint_{\partial C(R)} \underbrace{d\vec{S}}_{R^2 \sin\vartheta\, \vec{e}_r\, d\vartheta\, d\varphi} \cdot \underbrace{\vec{\nabla}_{\vec{x}} \tilde{R}_0(r)}_{\vec{e}_r \tilde{R}_0'(r)}} + k^2 \int_{C(R)} d^3x\, \tilde{R}_0(r)$$

$\vec{e}_r \cdot \vec{e}_r = 1$, $\quad \tilde{R}_0'(r) \to R$

$$= \cancel{4\pi R^2 \frac{\partial \tilde{R}_0(R)}{\partial R}}$$

$$k^2 \int_{C(R)} d^3x\, \tilde{R}_0(r) = 4\pi \int_0^R dr\, r^2\, \tilde{R}_0(r)$$

$$= 4\pi \int_0^R dr\, r^2 \left[k^2 \tilde{R}_0(r) \right]$$

$$k^2 \tilde{R}_0(r) \overset{(*)}{=} \left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} \right) \tilde{R}_0(r)$$

$$= -4\pi \left\{ \int_0^R dr\, r^2 \frac{\partial^2 \tilde{R}_0(r)}{\partial r^2} + 2 \int_0^R dr\, r \frac{\partial \tilde{R}_0(r)}{\partial r} \right\}$$

$u = r^2$, $\quad v' = \frac{\partial^2 \tilde{R}_0(r)}{\partial r^2}$

$$= -4\pi \left\{ \left[r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right]_0^R - \cancel{\int_0^R dr\, 2r \frac{\partial \tilde{R}_0(r)}{\partial r}} + \cancel{2 \int_0^R dr\, r \frac{\partial \tilde{R}_0(r)}{\partial r}} \right\}$$

$$\left[r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right]_0^R = \cancel{R^2 \frac{\partial \tilde{R}_0(R)}{\partial r}} - \lim_{r \downarrow 0} \left\{ r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right\}$$

$$\int_{C(R)} d^3x \left\{ \Delta_{\vec{x}} + k^2 \right\} \tilde{R}_0(r) = 4\pi \lim_{r \downarrow 0} \left\{ r^2 \frac{\partial \tilde{R}_0(r)}{\partial r} \right\}$$ valid for all R

$$\tilde{R}_0(r) = \frac{A_0 \sin(kr) - B_0 \cos(kr)}{r\, k} \Rightarrow \lim_{r \downarrow 0} \frac{\partial}{\partial r}\left(\underline{r\, \tilde{R}_0(r)} \right) = A_0$$

$$\frac{\partial R_0(r)}{\partial r} = -\frac{1}{k r^2} \left\{ A_0 \sin(kr) - B_0 \cos(kr) \right\} + \cancel{\frac{k}{kr}} \left\{ A_0 \cos(kr) + B_0 \sin(kr) \right\}$$

$$\lim_{r \downarrow 0} \left\{ r^2 \frac{\tilde{R}_0(r)}{r} \right\} = \frac{B_0}{k}$$

$$\int_{C(R)} d^3 r \left\{ \Delta_{\vec{x}} + k^2 \right\} \tilde{R}_0(r) = \frac{4\pi B_0}{k}$$ is constant for all R

$$\left\{ \Delta_{\vec{x}} + k^2 - \frac{2\mu}{\hbar^2} v^{(int)}(r) \right\} \tilde{R}_0(r) = 0$$

$$\int_{C(R)} d^3x\, v^{(int)}(r)\, \tilde{R}_0(r) = -\frac{\hbar^2}{2\mu} \frac{4\pi B_0}{k}$$ for all R $$= -\frac{\hbar^2}{2\mu} \frac{\tan(ka)}{k} \underline{A_0}$$

$$\delta_0 = -ka, \quad \tan \delta_0 = -\frac{B_0}{A_0} \Rightarrow \tan ka = \frac{B_0}{A_0}$$

see last lecture

$$B_0 = A_0 \cdot \tan(ka)$$

$$\int_{C(R)} d^3x\, v^{(int)}(r)\, \tilde{R}_0(r) = -\frac{\hbar^2}{2\mu} 4\pi \frac{\tan(ka)}{k} \lim_{r \downarrow 0} \frac{\partial}{\partial r} \left[r\, \tilde{R}_0(r) \right]$$

$$\hookrightarrow v^{(int)}(\vec{x}) \bullet = -\underbrace{\frac{\hbar^2}{2\mu}4\pi\frac{\tan ka}{k}}\boxed{\delta(\vec{x})\ \frac{\partial}{\partial r}\left(r\ \hat{R}_0(r)\right)}$$

contact interaction

for small k

weight of pseudopotential $\longrightarrow \frac{\hbar^2 4\pi a}{2\mu} \overset{\mu=\frac{M}{2}}{=} \frac{4\pi\hbar^2 a_s}{M} = g$

s-wave scattering length

example: $a_s(^{87}Rb) = 100\ a_B$

Remark: $\hat{O}\ \bullet = \frac{\partial}{\partial r}\left(r\ \bullet\right)\Big|_{r=0}$

1) $f(r)$ regular: $\hat{O}\ f(r) = \frac{\partial}{\partial r}\left(r\ f(r)\right)\Big|_{r=0}$

$$= \left\{ f(r) + r\ f'(r) \right\}\Big|_{r=0} = f(0)$$

$$\delta(\vec{x})\ f(r) = f(0)\ \delta(\vec{x})$$

2) $f(r) = \frac{K}{r}$, $\hat{O}\ f(r) = \frac{\partial}{\partial r}\left(r\ \frac{K}{r}\right)\Big|_{r=0} \equiv 0$

operator $\hat{O}$ removes divergency of the $1/r$-type

$\longrightarrow 1/r$ functions do not contribute

Scattering Amplitude:

Scattering problem: Schrödinger equation

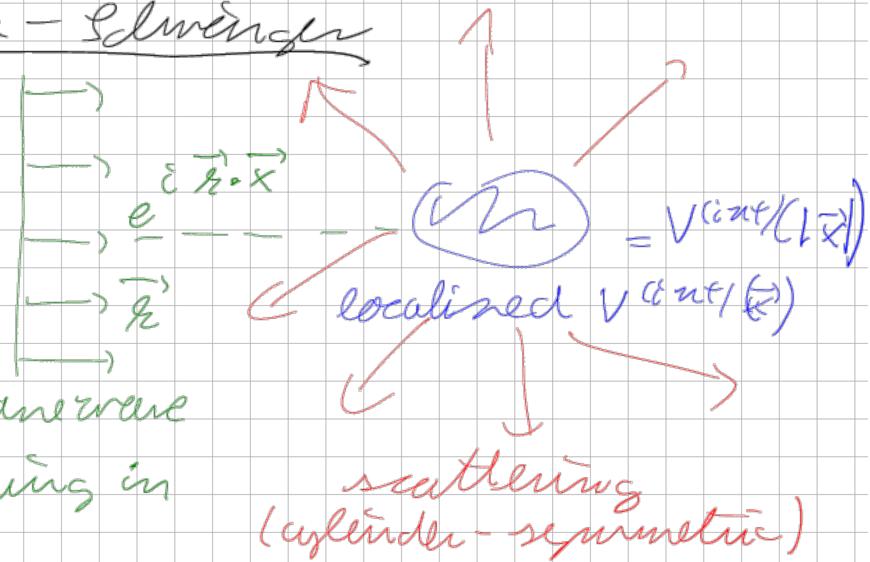
$$\left\{ \Delta_{\vec{x}} + k^2 - \frac{2\mu}{\hbar} V^{(int)}(\vec{x}) \right\} \psi(\vec{x}) = 0$$ stationary scattering problem

$\tilde{} = \frac{2\mu E}{\hbar^2}$, $E > 0$

Equivalent integral equation: Lippmann – Schwinger

$$(\Delta_{\vec x} + k^2)\,\psi(\vec x) = \frac{2\mu}{\hbar^2}\, V^{(int)}(\vec x)\,\psi(\vec x)$$

$$\psi(\vec x) = \underbrace{e^{i\vec k\cdot\vec x}}_{\text{homogeneous solution of Helmholtz equation}} + \underbrace{\frac{2\mu}{\hbar^2}\int d^3x'\, \underbrace{G(\vec x, \vec x')}_{\text{Green function}}\, V^{(int)}(\vec x')\,\psi(\vec x')}_{\text{particular solution of inhomogeneous Helmholtz equation}}$$



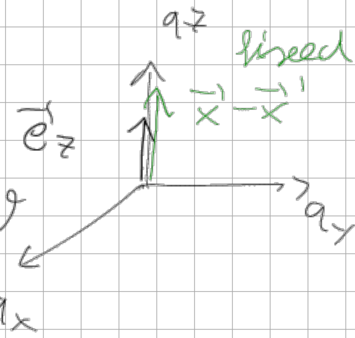
$$(\Delta_{\vec x} + k^2)\, \underbrace{G(\vec x, \vec x')}_{= G(\vec x - \vec x')} = \delta(\vec x - \vec x')$$

homogeneity in space

$$= G(\vec x - \vec x') = \int \frac{d^3q}{(2\pi)^3}\, G(\vec q)\, e^{i\vec q(\vec x - \vec x')}$$

$$\to (-\vec q^{\,2} + k^2)\, G(\vec q) = 1 \Rightarrow G(\vec q) = \frac{-1}{\vec q^{\,2} - k^2}$$

algebraic equation



$$G(\vec x - \vec x') = \frac{-1}{8\pi^3}\int_0^\infty dq\, q^2 \int_0^\pi d\vartheta \sin\vartheta \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \frac{1}{q^2 - k^2}\, e^{iq|\vec x - \vec x'|\cos\vartheta}$$

cylinder symmetry

$$= -\frac{1}{4\pi^2}\int_0^\infty dq\, q^2 \frac{1}{q^2 - k^2} \int_0^\pi d\vartheta \sin\vartheta\, e^{iq|\vec x - \vec x'|\cos\vartheta}$$

$du = -\sin\vartheta\, d\vartheta$ $\quad$ $u(\vartheta) = \cos\vartheta$

$$\int_{-1}^{+1} du\, e^{iq|\vec x - \vec x'|u} = \frac{e^{iq|\vec x - \vec x'|} - e^{-iq|\vec x - \vec x'|}}{iq|\vec x - \vec x'|}$$

$$G(\vec{x}-\vec{x}') = \underbrace{\frac{i}{4\pi}}_{\frac{1}{|\vec{x}-\vec{x}'|}}\left\{\int_0^\infty dq\, \frac{q}{q^2-k^2}\, e^{iq|\vec{x}-\vec{x}'|} + \underbrace{\int_0^\infty dq\, \frac{q}{q^2-k^2}\, e^{-iq|\vec{x}-\vec{x}'|}}_{q\to -q}\right\}$$

$$= \frac{i}{4\pi|\vec{x}-\vec{x}'|}\boxed{\lim_{\varepsilon\downarrow 0}}\int_{-\infty}^{+\infty} dq\, \frac{q}{q^2-k^2\boxed{-i\varepsilon}}\, e^{iq|\vec{x}-\vec{x}'|}$$

$$\int_{-\infty}^{0} dq\, \frac{(+q)}{q^2-k^2}\, e^{iq|\vec{x}-\vec{x}'|}$$

Feynman $i\varepsilon$ prescription

was derived with adiabatic switching on scattering potential within time dependent perturbation theory