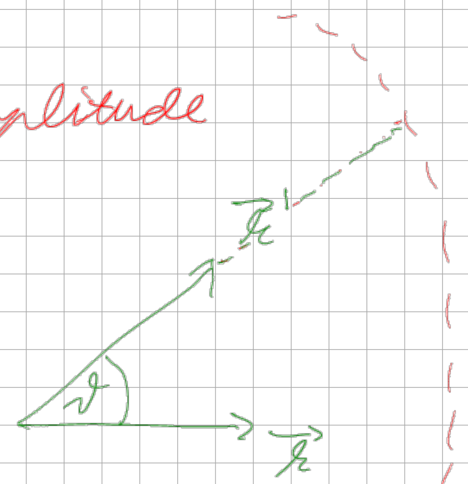
Summary:

wave function: scattering theory

$$\psi(\vec{x}) = \underbrace{e^{i\vec{k}\cdot\vec{x}}}_{\text{incoming plane wave}} + \underbrace{\frac{e^{ikr}}{r} f(\vartheta,\varphi)}_{\text{outgoing spherical wave}}$$

scattering amplitude

$[f] = 1\,m$



elastic scattering

$|\vec{k}| = |\vec{k}'|$

$$\vec{k}' = k \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

Determine scattering amplitude:

- solve exact Lippmann-Schwinger equation: only possible for special case
- Born approximation: 1st order perturbation theory

$$f(\vartheta,\varphi) = -\frac{m}{2\pi\hbar^2} V^{(int)}(\vec{k}-\vec{k}')$$

Prominent example: Rutherford scattering

$$V^{(int)}(\vec{x}) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |\vec{x}|} \Rightarrow V^{(int)}(\vec{k}) = \frac{Q_1 Q_2}{\varepsilon_0} \frac{1}{\vec{k}^2}$$

① ②

- Yukawa potential:

$$V_Y^q(\vec{x}) = C \frac{1}{|\vec{x}|} e^{-q|\vec{x}|}$$

screening parameter

- isotropic interaction potential

Gamma function

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} \quad ; \; x > 0$$

$\downarrow t = a\tau$

$$\frac{1}{a^x} = \frac{1}{\Gamma(x)} \int_0^\infty d\tau\, \tau^{x-1} e^{-a\tau}$$

$\uparrow$

Schwinger method

$$V^{(int)}(\vec k) = \int d^3x\, V^{(int)}(\vec x)\, e^{i\vec k\cdot\vec x} = C\int d^3x\, \underbrace{\frac{1}{|\vec x|}}_{=\frac{1}{\sqrt{\vec x^2}} = \frac{1}{(\vec x^2)^{1/2}}} e^{i\vec k\cdot\vec x}$$

Schwinger-method: $a = \vec x^2$, $x = \frac{1}{2}$

$$= C\int d^3x\, \frac{1}{\Gamma(\frac12)} \int_0^\infty d\tau\, \tau^{\frac12 - 1}\, e^{-\vec x^2\tau}\, e^{i\vec k\cdot\vec x}$$

$$\Gamma\left(\tfrac12\right) = \int_0^\infty dt\, t^{-\frac12} e^{-t} \quad \underset{dt = 2s\,ds}{\overset{t=s^2}{=}} \quad \int_0^\infty ds\, 2s\, \frac{1}{s} e^{-s^2} = 2\int_0^\infty ds\, e^{-s^2} = 2\,\frac{1}{2}\sqrt{\pi} = \sqrt{\pi}$$

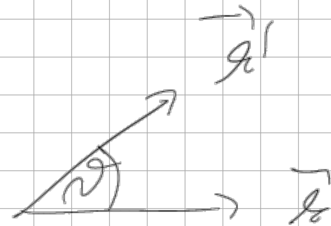
$$V^{(int)}(\vec k) = \frac{C}{\sqrt\pi}\int_0^\infty d\tau\, \tau^{-\frac12} \underbrace{\int d^3x\, e^{-\tau\vec x^2 + i\vec k\cdot\vec x}}_{=\left(\frac{\pi}{\tau}\right)^{3/2} e^{-\frac14\frac{\vec k^2}{\tau}}}$$

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\mu x^2 - \nu x} = \sqrt{\frac{\pi}{\mu}}\, e^{\frac14\frac{\nu^2}{\mu}}}$$

$$= C\pi\int_0^\infty d\tau\, \frac{1}{\tau^2}\, e^{-\frac{\vec k^2}{4\tau}}$$

$u = \frac{1}{\tau}$, $du = -\frac{1}{\tau^2}\,du$

$$= C\pi\int_0^\infty du\, \cancel{\tau^2}\, \cancel{\frac{1}{\tau^2}}\, e^{-\frac{\vec k^2}{4}u} = C\pi\,\frac{4}{\vec k^2} = \frac{4\pi C}{\vec k^2}$$



$$f(\vartheta,\varphi) = -\frac{\mu}{2\pi\hbar^2}\cdot\frac{Q_1Q_2}{\varepsilon_0}\,\frac{1}{(\vec k - \vec k')^2}$$

$$(\vec k - \vec k')^2 = k^2 - 2\underbrace{\vec k\cdot\vec k'}_{|\vec k||\vec k'|\cos\vartheta} + \underbrace{\vec k'^2}_{=\vec k^2}, \quad |\vec k'| = |\vec k|$$

$$= 2(1-\cos\vartheta)\,k^2$$

$$= 4(\sin^2\vartheta/2)\cdot k^2$$

$$= -\frac{\mu}{2\pi\hbar^2}\,\frac{Q_1Q_2}{\varepsilon_0}\,\frac{1}{4k^2(\sin^2\frac{\vartheta}{2})}$$

cylinder symmetry
≙ independent of φ

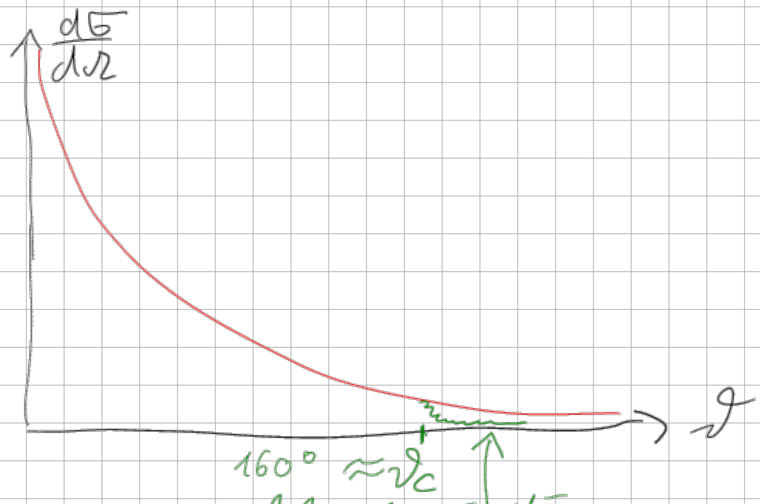
energy dependence

$$E = \frac{\hbar^2k^2}{2\mu} \Rightarrow k^2 = \frac{2\mu E}{\hbar^2}$$

$$f(\vartheta;E) = -\frac{\cancel{m}}{2\pi\cancel{\hbar^2}}\,\frac{Q_1Q_2}{\varepsilon_0}\,\frac{1}{4\cdot\frac{2\cancel{m}E}{\cancel{\hbar^2}}\,\sin^2\frac{\vartheta}{2}}$$

observation: $\hbar$ gone $\longrightarrow$ classical result

$$\left(\frac{d\sigma}{d\Omega}\right) = |f(\vartheta,E)|^2 = \left(\frac{Q_1Q_2}{16\pi\varepsilon_0 E}\right)^2\frac{1}{\sin^4(\frac{\vartheta}{2})}$$



measurable deviation from prediction of Rutherford formula: "anomalous Rutherford scattering"

This coincides with classical result.

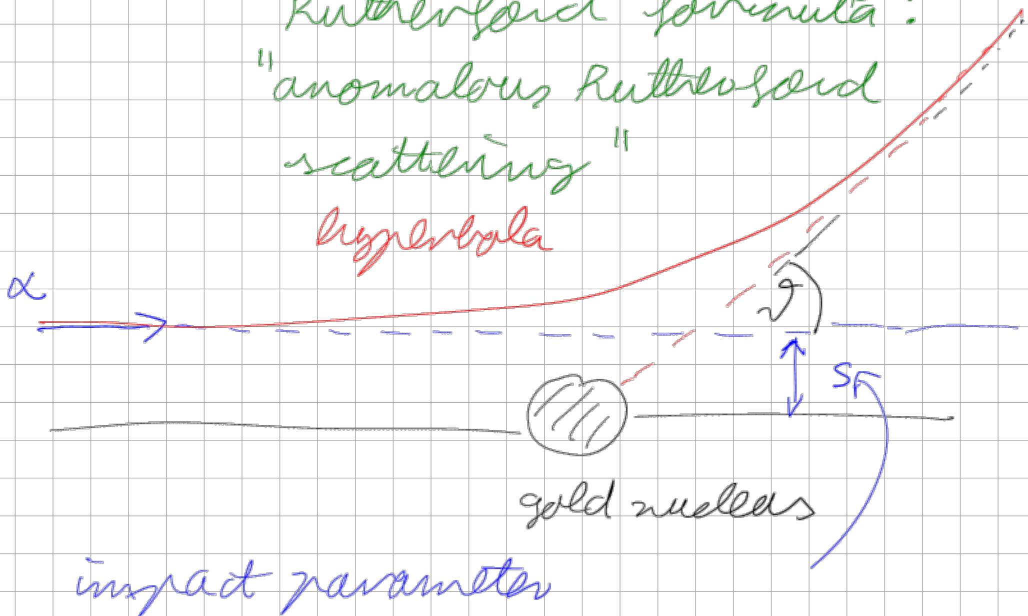
BUT: This is only Born approximation!

Higher order results in a quantum treatment of this Rutherford scattering might lead to deviations.

Rutherford scattering:

α particles ($^4He^{2+}$) scattered with gold

$Q_1 = 2e > 0$, $\quad Q_2 = \underbrace{Z}_{=79}\, e > 0$



combine energy and angular conservation:

$$S_c = \frac{Q_1 Q_2 e^2}{8\pi\varepsilon_0 E}\operatorname{ctg}\frac{\vartheta_c}{2}$$

~~for some energy~~ $= 5\cdot\underbrace{10^{-15}\,m}_{=1\,fm}$

impact parameter $\quad$ scattering angle

explanation for anomalous Rutherford scattering: strong interaction

Note: $\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right) \longrightarrow$ diverges $\hat{=}$ long-range character of Coulomb interaction

σ: total cross-section, $\frac{d\sigma}{d\Omega}$: differential cross-section

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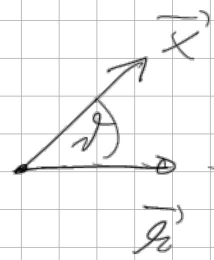
Student Talks:

- you can pick any topic you want in realm of QM
- List of 14 suggestions: see email

Partial Waves:

Specialize: $V^{(int)}(\vec{x}) = v^{(int)}(|\vec{x}|) \longrightarrow$ cylinder symmetry

Plane Wave:



$$e^{i\vec{k}\vec{x}} = e^{ikr\cos\vartheta}$$

divergency at origin

$$= \sum_{l=0}^{\infty} \cancel{\sum_{m=-l}^{+l}} \underbrace{\left\{ A_{lm}\, j_l(kr) + B_{lm}\, \cancel{n_l(kr)} \right\}}_{\text{radial dependence}} \underbrace{Y_{lm}(\vartheta,\varphi)}_{\text{spherical harmonics}}$$

$\hat{=}$ free solution of Schrödinger equation

$$\Rightarrow e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} A_l\, j_l(kr) \underbrace{Y_{l0}(\vartheta,\varphi)}_{\sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\vartheta)}$$

Legendre polynomial

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2-1)^\ell$$

$$\int_{-1}^{+1} dz\, P_\ell(z) P_{\ell'}(z) = \frac{2}{2\ell+1}\delta_{\ell\ell'} \leftarrow$$

$$e^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty} \boxed{\underbrace{A_\ell}_{=?}} j_\ell(kr) \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\vartheta) \qquad \Big| \cdot \int_0^\pi d\vartheta \sin\vartheta\, P_{\ell'}(\cos\vartheta)$$

partial wave decomposition:

$\ell = 0,\quad 1,\quad 2,\quad \ldots$

$s,\quad p,\quad d,\quad \ldots$

$$\int_{-1}^{+1} dz\, P_{\ell'}(z)\, e^{ikrz} = \sum_{\ell=0}^{\infty} A_\ell \sqrt{\frac{2\ell+1}{4\pi}} j_\ell(kr) \underbrace{\int_{-1}^{+1} dz\, P_{\ell'}(z) P_\ell(z)}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$z = \cos\vartheta$

$$= A_{\ell'} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2\ell'+1}} j_{\ell'}(kr)$$

$$A_\ell\, j_\ell(kr) = \sqrt{\pi(2\ell+1)} \int_{-1}^{+1} dz\, P_\ell(z)\, e^{ikrz}$$

$r \downarrow 0$

$$A_\ell \boxed{\frac{(kr)^\ell}{(2\ell+1)!!}} = \sqrt{\pi(2\ell+1)} \int_{-1}^{+1} dz\, P_\ell(z) \left\{ \ldots + \cancel{\frac{(ikrz)^{\ell-1}}{(\ell-1)!}} + \boxed{\frac{(ikrz)^\ell}{\ell!}} + \cancel{\frac{(ikrz)^{\ell+1}}{(\ell+1)!}} + \ldots \right.$$

do not contribute in limit $r \downarrow 0$

go away due to orthogonality relation with Legendre polynomial

$$z^\ell = \frac{2^\ell (\ell!)^2}{(2\ell)!} P_\ell(z) + \sum_{m=0}^{\ell-1} c_m P_m(z)$$

power basis | Legendre basis

$$A_\ell \frac{(k \cancel{r})^\ell}{(2\ell+1)!!} = \sqrt{\pi(2\ell+1)}\,\frac{2}{2\ell+1}\,\frac{(i k \cancel{r})^\ell}{\ell!}\,\frac{2^\ell(\ell!)^2}{(2\ell)!}$$

$$\Rightarrow A_\ell = i^\ell \sqrt{4\pi(2\ell+1)}$$