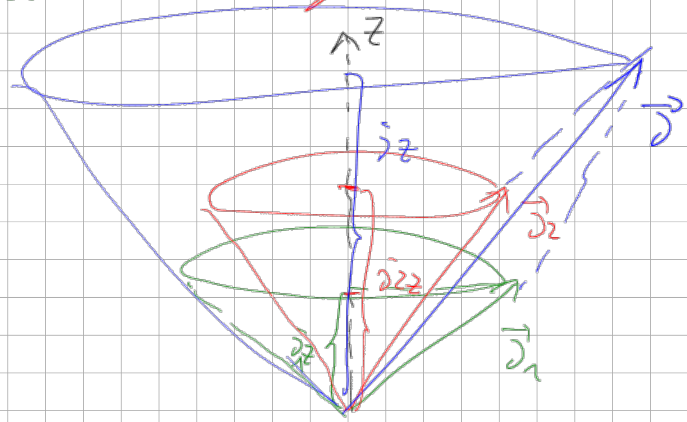
Summary: add angular momenta: $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$\underbrace{|j_1, j_2; j, m\rangle}_{\text{Option B}} = \sum_{m_1, m_2} \underbrace{\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}} \underbrace{|j_1, m_1; j_2, m_2\rangle}_{\text{Option A}}$$



$|\vec{J}_1| = \hbar\sqrt{j_1(j_1+1)}$

$j_{1z} = \hbar m_1$

Two major properties:

1) $m = m_1 + m_2$

2) triangular rule:

$j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}$

semiclassical vector model

Simplify notation: j_1, j_2 fixed $\to$ omitted

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \langle m_1, m_2; j, m\rangle \; |m_1, m_2\rangle$$

11.8 Proton / Neutron built from quarks:

quarks are spin 1/2 and Landé factor 2 described by Dirac equation

generation \ charge	$-\frac{e}{3}$	$+\frac{2}{3}e$
1	d (down)	u (up)
2	s (strange)	c (charm)
3	b (bottom)	t (top)

mass increases

$\longrightarrow$ $\boxed{M_d \approx M_u}$

$\Rightarrow$ proton mass $\approx$ neutron mass quark content of proton and neutron ($\hat{=}$ nucleon)

$$|p\rangle = |u\,u\,d\rangle \;\xleftrightarrow[\text{exchange}]{u \leftrightarrow d}\; |n\rangle = |d\,d\,u\rangle$$

Problem to solve: How to add three $\underbrace{\text{spin } 1/2\text{'s}}_{\text{quarks}}$ to a total $\underbrace{\text{spin } 1/2}_{\text{nucleon}}$

First attempt

$$\underbrace{\left.\begin{matrix}\left.\begin{matrix}\frac{1}{2}\\ \\ \frac{1}{2}\end{matrix}\right\} 0 \\ \\ \frac{1}{2}\end{matrix}\right\}}_{\text{quark}} \qquad \underbrace{\frac{1}{2}}_{\text{nucleon}}$$

Second attempt

$$\underbrace{\left.\begin{matrix}\left.\begin{matrix}\frac{1}{2}\\ \\ \frac{1}{2}\end{matrix}\right\} \boxed{1} \\ \\ \boxed{\frac{1}{2}}\end{matrix}\right\}}_{\text{quark}} \qquad \underbrace{\frac{1}{2} \quad \text{or} \quad \frac{3}{2}}_{\text{nucleon}} \nearrow \text{four } \Delta \text{ baryons}$$

$j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$: $j_{12} = 0$ or 1

$$|j_{12} = 1, m_{12} = 1\rangle = 1\,|m_1 = \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle$$

$$|j_{12} = 1, m_{12} = 0\rangle = \frac{1}{\sqrt{2}}\,|m_1 = \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|m_1 = -\tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle$$

Triplet

$$|j_{12} = 0, m_{12} = 0\rangle = \frac{1}{\sqrt{2}}\,|m_1 = +\tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}\,|m_1 = -\tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle$$

Singlet

$$|j_{12} = 1, m_{12} = -1\rangle = 1\,|m_1 = -\tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle$$

$$\hat{\vec{J}}^2 = (\hat{\vec{J}}_1 + \hat{\vec{J}}_2)^2 = \hat{\vec{J}}_1^{\,2} + \hat{\vec{J}}_2^{\,2} + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$\hat{\vec{J}}^2 \underbrace{|m_1 = \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle}\ \overset{\downarrow}{=}\ \hbar^2 \left\{2\,\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) + 2\,\tfrac{1}{2}\,\tfrac{1}{2}\right\} = \hbar^2\left(\tfrac{3}{2}+\tfrac{1}{2}\right) = \hbar^2 \underbrace{1}_{=j_{12}}(1+1)$$

$$\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$$

$$\hat{\vec{J}}^2 |\underbrace{\tfrac{1}{2}}_{m_1}, \underbrace{-\tfrac{1}{2}}_{m_2}\rangle = \hbar^2 \left\{\underbrace{2\,\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) + 2\cdot\tfrac{1}{2}\,\tfrac{-1}{2}}_{=1}\right\} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + \hbar^2 |-\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$\hat{\vec{J}}^2 |-\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \hbar^2 1 \cdot |-\tfrac{1}{2}, \tfrac{1}{2}\rangle + \hbar^2 |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\hat{\vec{J}}^2 \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}, +\tfrac{1}{2}\rangle\right) = \hbar^2 \underbrace{2}_{\substack{= \underbrace{1}_{j_{12}=1}\cdot(1+1)}} \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$$

$$j_{12} = 1,\ j_3 = \tfrac{1}{2}:\quad j = \tfrac{1}{2} \text{ or } j = \tfrac{3}{2}$$

$$|j = \tfrac{1}{2}, m = \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\ |m_{12} = 1, m_3 = -\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}\ |m_{12} = 0, m_3 = \tfrac{1}{2}\rangle$$

according to table

$$= \sqrt{\tfrac{2}{3}}\ |m_1 = \tfrac{1}{2}, m_2 = \tfrac{1}{2}, m_3 = -\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{6}}\ |m_1 = \tfrac{1}{2}, m_2 = -\tfrac{1}{2}, m_3 = \tfrac{1}{2}\rangle$$
$$- \tfrac{1}{\sqrt{6}}\ |m_1 = -\tfrac{1}{2}, m_2 = \tfrac{1}{2}, m_3 = \tfrac{1}{2}\rangle$$

proton wave function:

$$|p^\uparrow\rangle = \sqrt{\tfrac{2}{3}}\ |u^\uparrow u^\uparrow d^\downarrow\rangle - \tfrac{1}{\sqrt{6}}\ |u^\uparrow u^\downarrow d^\uparrow\rangle - \tfrac{1}{\sqrt{6}}\ |u^\downarrow u^\uparrow d^\uparrow\rangle$$

neutron wave function:

$$|n^\uparrow\rangle = \sqrt{\tfrac{2}{3}}\ |d^\uparrow d^\uparrow u^\downarrow\rangle - \tfrac{1}{\sqrt{6}}\ |d^\uparrow d^\downarrow u^\uparrow\rangle - \tfrac{1}{\sqrt{6}}\ |d^\downarrow d^\uparrow u^\uparrow\rangle$$

u – d exchange

Magnetic moment of point-like spin 1/2 according to Dirac theory like electron

$$\vec{\mu} = \underbrace{-}_{\text{electron charge}} \underbrace{\mu_B}_{\text{Bohr magneton}} \underbrace{g_S}_{\substack{=2 \\ \text{Landé factor}}} \frac{\vec{S}}{\hbar} \leftarrow \text{electron spin}$$

Bohr magneton

= magnetic moment of electron in 1s

$$\mu_B = \frac{e\hbar}{2m} \leftarrow \text{electron mass}$$

Magnetic moment for proton and neutron

$$\mu_p^{expected} = + 2 \frac{e\hbar}{2M_p} \qquad \mu_n^{expected} = 0 \cdot 2 \frac{e\hbar}{2M_n}$$

positive charge

Dirac ?

but experimentall you get:

$$\mu_p^{exp.} = 2.79 \frac{e\hbar}{2M_p} \qquad \mu_n^{exp} = -1.91 \frac{e\hbar}{2M_n}$$

proton and neutron are not point-like

i.e. they must have a sub-structure (here: 3 quarks)

$$\mu_{pz}^{theory} = \langle \underbrace{\Psi_p | \mu_z | \Psi_p}_{\text{built of quark wave functions}} \rangle, \quad \vec{\mu} = \vec{\mu}_{u_1} + \vec{\mu}_{u_2} + \vec{\mu}_d$$

built of quark wave functions

$$= \left(\sqrt{\frac{2}{3}} \langle u^\uparrow u^\uparrow d^\downarrow | - \frac{1}{\sqrt{6}} \langle u^\uparrow u^\downarrow d^\uparrow | - \frac{1}{\sqrt{6}} \langle u^\downarrow u^\uparrow d^\uparrow | \right)$$

$$(\mu_{u_1} + \mu_{u_2} + \mu_d) \left(\sqrt{\frac{2}{3}} | u^\uparrow u^\uparrow d^\downarrow \rangle - \frac{1}{\sqrt{6}} | u^\uparrow u^\downarrow d^\uparrow \rangle - \frac{1}{\sqrt{6}} | u^\downarrow u^\uparrow d^\uparrow \rangle \right)$$

$$= \frac{2}{3}(\mu_u + \mu_u - \mu_d) + \frac{1}{6}(\cancel{\mu_u} - \cancel{\mu_u} + \mu_d) + \frac{1}{6}(\cancel{\mu_u} - \cancel{\mu_u} + \mu_d)$$

$$= \frac{4}{3}\mu_u - \frac{1}{3}\mu_d \quad , \quad \mu^{theory}_{n,Z} = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u$$

doublett of first generation: $M_u \simeq M_d$

$$\mu_u = +\frac{2}{3}\, 2\, \frac{e\hbar}{2M_u} \quad , \quad \mu_d = -\frac{1}{3}\, 2\, \frac{e\hbar}{2M_d}$$

$\mu_u = -2\mu_d, \quad \mu_d = -\frac{1}{2}\mu_u$

$$\mu^{theory}_{p,Z} = \left(\frac{4}{3} - \frac{1}{3}\cdot -\frac{1}{2}\right)\mu_u = \left(\frac{4}{3} + \frac{1}{6}\right)\mu_u = \frac{3}{2}\mu_u$$

$$\mu^{theory}_{n,Z} = \left(\frac{4}{3}\cdot -\frac{1}{2} - \frac{1}{3}\right)\mu_u = \left(-\frac{2}{3} - \frac{1}{3}\right)\mu_u = -\mu_u$$

$$\frac{\mu^{theory}_{n,Z}}{\mu^{theory}_{p,Z}} = \frac{-1\mu_u}{\frac{3}{2}\mu_u} = -\frac{2}{3} \approx -0.667$$

$$\frac{\mu^{exp}_{n,Z}}{\mu^{exp}_{p,Z}} = \frac{-1.91\,\frac{e\hbar}{2M_n}}{2.79\,\frac{e\hbar}{2M_p}} \underset{\overline{\overline{}}}{\overset{M_n \simeq M_p}{=}} -\frac{1.91}{2.79} \approx -0.685$$

excellent agreement

11.9 Recursion relations for Clebsch-Gordan coefficients

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \underbrace{\langle m_1, m_2 | j, m\rangle}_{\text{unitary matrix of rank } (2j_1+1)\cdot(2j_2+1)} |m_1, m_2\rangle$$

fulfilling

- orthonormality of columns:

$$\sum_{m_1, m_2} \langle m_1, m_2 | j, m \rangle \langle m_1, m_2 | j', m' \rangle^* = \delta_{jj'} \delta_{mm'}$$

- orthonormality of rows:

$$\sum_{j, m} \langle m_1, m_2 | j, m \rangle \langle m_1', m_2' | j, m \rangle^* = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

Note: Without loss of generality phases of states $|j, m\rangle$ can be chosen such that Clebsch-Gordan coefficient are real

Step 1: j_1, j_2 fixed

maximal j: $j = j_1 + j_2$, maximal m: $m = m_1 + m_2 = j_1 + j_2$

only one term: $|\underbrace{j_1 + j_2}_{= j}, \underbrace{j_1 + j_2}_{= m}\rangle = \underbrace{\langle j_1, j_2 | j_1 + j_2, j_1 + j_2 \rangle}_{= 1} |\underbrace{j_1}_{m_1}, \underbrace{j_2}_{m_2}\rangle$

example: $|j = 1, m = 1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$

From here successive application of $\vec{J}_-$ allow to obtain all states belonging to $j = j_1 + j_2$ and $-j \leq m \leq +j$