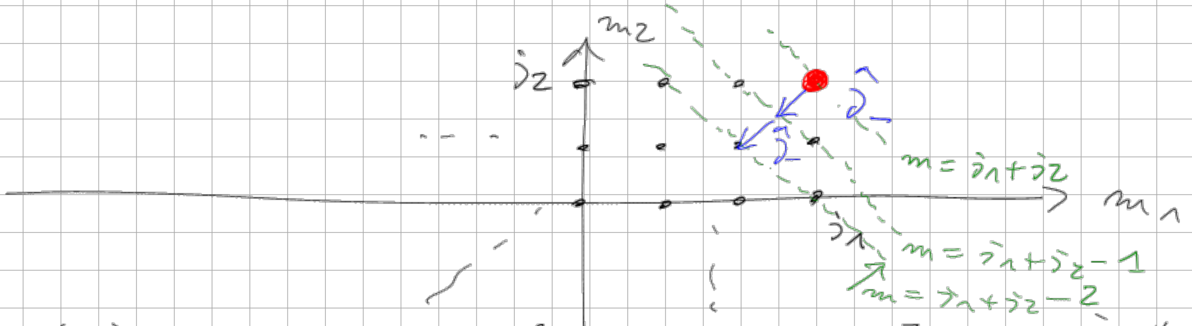
11. Recursion Relations for Clebsch-Gordan Coefficients

$$|j,m\rangle = \sum_{\substack{m_1,m_2 \\ \boxed{m_1+m_2=m}}} \underbrace{\langle m_1,m_2|\, j,m\rangle}_{C.\ G.\ C.} \quad |m_1,m_2\rangle \qquad (*)$$

Idea:



step 1: $j = j_1+j_2$, m maximal: $m = m_1+m_2 = j_1+j_2$

decomposition (*) can only consist of one term:

$$\underbrace{|j_a+j_2, j_1+j_2\rangle}_{\text{normalised}} = \underbrace{\langle j_1,j_2|\ \ j_1+j_2, j_1+j_2\rangle}_{=\ 1} \underbrace{|\overset{m_1}{\overbrace{j_1}}, \overset{m_2}{\overbrace{j_2}}\rangle}_{\text{normalised}}$$

example:

$j_1 = j_2 = 1/2$

$$|\underbrace{1}_{=j},\underbrace{1}_{m}\rangle = |\underbrace{\tfrac{1}{2}}_{m_1},\underbrace{\tfrac{1}{2}}_{m_2}\rangle$$

successively applying $\hat{J}_-$ operators states belonging to $j_1+j_2 = j$ and $-j \leq m \leq +j$.

1. Application:

$$\hat{J}_-|\underbrace{j_1+j_2}_{=j}, \underbrace{j_1+j_2}_{=m}\rangle = \hbar\sqrt{\big(\underbrace{(j_1+j_2)}_{=j} + \underbrace{(j_1+j_2)}_{=m}\big)\big(\underbrace{(j_1+j_2)}_{j} - \underbrace{(j_1+j_2)}_{m}+1\big)}\ \boxed{|j_1+j_2, j_1+j_2-1\rangle}$$

$$= (\hat{J}_{1-}+\hat{J}_{2-})\,|\underbrace{j_1}_{m_1}, \underbrace{j_2}_{m_2}\rangle = |j_1\rangle|j_2\rangle$$

$$= \hbar\sqrt{(j_1+j_1)\,(j_1-j_1+1)} \qquad |j_1-1, j_2\rangle$$

$$+ \hbar\sqrt{(j_2+j_2)\,(j_2-j_2+1)} \qquad |j_1, j_2-1\rangle$$

$$|j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1-1, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1, j_2-1\rangle$$

$$\langle m_1, m_2 | j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,\delta_{m_1, j_1-1}\,\delta_{m_2, j_2} + \sqrt{\frac{j_2}{j_1+j_2}}\,\delta_{m_1, j_1}\,\delta_{m_2, j_2-1}$$

example: $j_1 = 1/2 = j_2$

$$|\underbrace{1}_{=j}, \underbrace{0}_{m=}\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$$

	wave function	=	spatial part	·	spin part	
triplett	−		−		+	excited
singlett	−	=	+		−	ground

2. Application: another $\hat{J}_-$ application

$$|j_1+j_2, j_1+j_2-2\rangle = \sqrt{\frac{j_1(2j_1-1)}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1-2, j_2\rangle$$

$$+2\sqrt{\frac{j_1 j_2}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1-1, j_2-1\rangle + \sqrt{\frac{j_2(2j_2-1)}{(j_1+j_2)(2j_1+2j_2-1)}}\,|j_1, j_2-2\rangle$$

example: $j_1 = 1/2 = j_2$

$$|1, -1\rangle = 1 \quad |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Step 2:

Now we look for states with $j = j_1 + j_2 - 1$

We start with maximal m, i.e. $m = j_1 + j_2 - 1 = m_1 + m_2$

$$|\underbrace{j_1+j_2-1}_{=j}, \underbrace{j_1+j_2-1}_{=m}\rangle = \sum_{m_1 \cancel{m_2}} \langle m_1, j_1+j_2-1-m_1 | j_1+j_2-1, j_1+j_2-1\rangle\, |m_1, \underbrace{j_1+j_2-1-m_1}_{=m_2}\rangle$$

$m = m_1 + m_2 = j_1 + j_2 - 1$

$\Rightarrow m_2 = j_1 + j_2 - 1 - m_1$

$$= \alpha \; |j_1-1, j_2\rangle$$
$$+ \beta \; |j_1, j_2-1\rangle$$

to determine Clebsch-Gordan coefficients we use the fact that the state $\underbrace{|j_1+j_2, j_1+j_2-1\rangle}$ is orthogonal to $|j_1+j_2-1, j_1+j_2-1\rangle$

$$= \sqrt{\frac{j_1}{j_1+j_2}} \; |j_1-1, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}} \; |j_1, j_2-1\rangle$$

$$\Rightarrow \alpha \cdot \sqrt{\frac{j_1}{j_1+j_2}} + \beta \sqrt{\frac{j_2}{j_1+j_2}} = 0$$

$$\alpha = - \sqrt{j_2}\,\gamma, \quad \beta = + \sqrt{j_1}\,\gamma$$

and $\gamma = \frac{1}{\sqrt{j_1+j_2}}$ due to normalization

$$|j_1+j_2-1, j_1+j_2-1\rangle = -\sqrt{\frac{j_2}{j_1+j_2}} \; |j_1-1, j_2\rangle + \sqrt{\frac{j_1}{j_1+j_2}} \; |j_1, j_2-1\rangle$$

1. example: $j_1 = 1/2 = j_2$

$$|j_{12} = 0, m_{12} = 0\rangle = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}, \tfrac{1}{2}\rangle \right)$$ $\Rightarrow$ singlet state

2 example: $j_1 = 1$, $j_2 = 1/2$

$$|j_{12} = \tfrac{1}{2}, \tfrac{1}{2}\rangle = \underbrace{\sqrt{\frac{1}{1+\frac{1}{2}}}}_{= \sqrt{\frac{2}{3}}} \; |1, -\tfrac{1}{2}\rangle - \underbrace{\sqrt{\frac{1/2}{1+1/2}}}_{= \sqrt{\frac{1}{3}}} \; |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

11.10 LS Coupling:

important example: adding orbital angular momentum and spin 1/2

$j_1 = l$ (integer), $m_1 = m_l$
$j_2 = s = 1/2$, $m_2 = m_s = \pm 1/2$

allowed values of j:
1) $l = 0$: $j = 1/2$
2) $l > 0$: $j = l \pm 1/2$

example: p state $\hat{=}\ l = 1 \Rightarrow j = 1/2, 3/2$

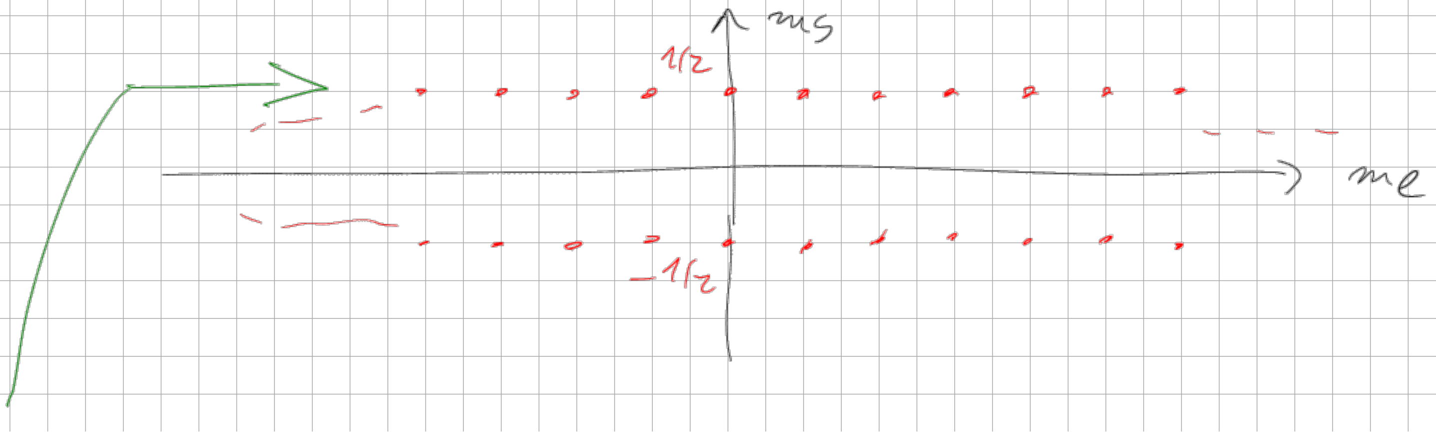
$$\underline{\ p\ } \longrightarrow \left\{ \begin{array}{l} p_{1/2} \\ p_{3/2} \end{array} \right.$$ with different energies

degeneracy ⟶ lifted degeneracy due to LS-coupling

In this lecture we will study this LS-coupling from 3 points of view

- Dirac equation contains this
- LS coupling is only one of three special relativistic corrections to Schrödinger eigenvalue values: a systematic non-relativistic limit of Dirac equation is implemented by a Foldy-Wouthessen transformation
- heuristic discussion of all 3 non-relativistic corrections

m_l, m_s - plane is particularly simple: two rows corresponding to $m_s = \pm 1/2$



concentrate on $j = l + 1/2 \quad (l > 0)$

$$\hat{J}_- |\underbrace{l+\tfrac{1}{2}}_{=j}, \underbrace{m+1}_{=m_j}\rangle = \hbar\sqrt{(\underbrace{l+\tfrac{1}{2}}_{j} + \underbrace{m+1}_{m_j})(\underbrace{l+\tfrac{1}{2}}_{j} \underbrace{-m-1}_{-m_j} + 1)}\,|l+\tfrac{1}{2}, \underbrace{m}_{m_j-1}\rangle$$

$$= (\hat{L}_- + \hat{S}_-) \sum_{m_l', m_s'} |m_l', m_s'\rangle\langle m_l', m_s'|l+\tfrac{1}{2}, m+1\rangle$$

$$m_j = m+1 = m_l' + m_s'$$

①

$$= \sum_{m_l', m_s'} \Big\{ \hbar\sqrt{(l+m_l')(l-m_l'+1)}\, \boxed{|m_l'-1, m_s'\rangle} \langle m_l', m_s'|l+\tfrac{1}{2}, m+1\rangle$$

$$\hbar\sqrt{(\tfrac{1}{2}+m_s')(\tfrac{1}{2}-m_s'+1)}\, \boxed{|m_l', m_s'-1\rangle} \langle m_l', m_s'|l+\tfrac{1}{2}, m+1\rangle \Big\}$$

②

Apply now $\langle m_l, m_s|$:

① $m_l = m_l' - 1$, $m_s = m_s' \Rightarrow \underline{m_l' = m_l + 1}$, $m_s' = \underline{m_s}$; $m+1 = m_l' + m_s' = m_l + m_s + 1 = m_j$

$\hookrightarrow \boxed{m = m_l + m_s}$ ✓

② $m_l = m_l'$, $m_s = m_s' - 1 \Rightarrow m_l' = m_l$, $m_s' = m_s + 1$; $m+1 = m_l' + m_s' = m_l + m_s + 1 = m_j$

$\Rightarrow \boxed{m = m_l + m_s}$

Choice : $\boxed{m_l = m - \tfrac{1}{2}}$, $m_s = \tfrac{1}{2} \Rightarrow m_l + m_s = m$ ✓

① $m_l' = m_l + 1 = m + \tfrac{1}{2}$, $m_s' = m_s = 1/2$

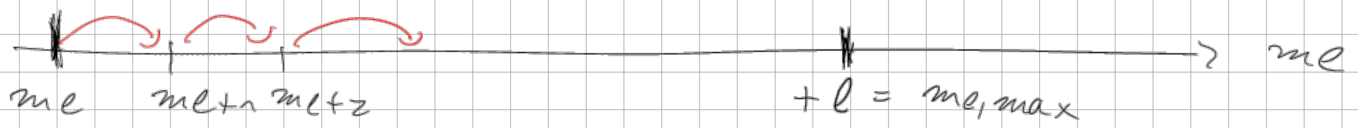
② $m_l' = m_l = m - 1/2$, $m_s' = m_s + 1 = 3/2$

$$(l+m+\tfrac{3}{2})(l-m+\tfrac{1}{2}) \boxed{\langle \overset{m_l}{m-\tfrac{1}{2}}, \overset{m_s}{\tfrac{1}{2}} | l+\tfrac{1}{2}, m\rangle} \qquad (*)$$

$$= \sqrt{(l+m+\tfrac{1}{2})(l-m-\tfrac{1}{2}+1)}\, \overbrace{\boxed{\langle m-\tfrac{1}{2}, \tfrac{1}{2} | m-\tfrac{1}{2}, \tfrac{1}{2}\rangle}}^{=1} \boxed{\langle \underbrace{m+\tfrac{1}{2}}_{m_l+1}, \underbrace{\tfrac{1}{2}}_{m_s} | l+\tfrac{1}{2}, m+1\rangle}$$

$$+\sqrt{(\tfrac{1}{2}+\tfrac{3}{2})(\cancel{\tfrac{1}{2}-\tfrac{3}{2}+1})}\, \boxed{\text{— — -}}$$

$m_s = +\frac{1}{2}$



iterate (*) $l - m_l$ times

$$\hat{=}\; l - m_l = \left(l+\frac{1}{2}\right) - \underbrace{\left(m_l + \frac{1}{2}\right)}_{= m} = l + \frac{1}{2} - m \text{ iterations}$$

$$\boxed{\left\langle m_l, \frac{1}{2} \middle| l+\frac{1}{2}, m \right\rangle}$$

$$\overset{(*)}{=} \sqrt{\frac{l+m+1/2}{l+m+3/2}} \left\langle \overbrace{m_l+1}^{m}+\frac{1}{2}, \frac{1}{2} \middle| l+\frac{1}{2}, m+1 \right\rangle$$

$$\overset{(*)}{=} \left(\sqrt{\frac{l+m+1/2}{\cancel{l+m+3/2}}} \cdot \sqrt{\frac{\cancel{l+m+3/2}}{l+m+5/2}}\right) \left\langle m+\frac{3}{2}, \frac{1}{2} \middle| l+\frac{1}{2}, m+2 \right\rangle$$

apply (*) $l - m_l = l + 1/2 - m$ times

$$= \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \; \boxed{\left\langle \underbrace{l}_{m_{max}}, \frac{1}{2} \middle| \underbrace{l+\frac{1}{2}}_{=j}, \underbrace{l+\frac{1}{2}}_{m_{j,max}} \right\rangle} \equiv 1$$

$$\left| \underbrace{l}_{m_{l,max}}, \underbrace{\frac{1}{2}}_{m_{s,max}} \right\rangle = \left| l+\frac{1}{2}, l+\frac{1}{2} \right\rangle$$

$$\Rightarrow \left\langle m_l, \frac{1}{2} \middle| l+\frac{1}{2}, m \right\rangle = \sqrt{\frac{l+m+1/2}{2l+1}}$$

This is only 1/4 of the whole story

$$\left| j = l+\frac{1}{2}, m \right\rangle = \boxed{\sqrt{\frac{l+m+1/2}{2l+1}}} \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle$$

$$+ \boxed{?} \; |m_l \mp m + \tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle$$

$$|j = l - \tfrac{1}{2}, m\rangle = \boxed{?} \; |m_l = m - \tfrac{1}{2}, m_s = \tfrac{1}{2}\rangle$$

$$+ \boxed{?} \; |m_l = m + \tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle$$

$$\begin{pmatrix} |l + \frac{1}{2}, m\rangle \\ |l - \frac{1}{2}, m\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{\text{structure fixed due to normalization + orthogonality}} \begin{pmatrix} |m - 1/2, \frac{1}{2}\rangle \\ |m + \frac{1}{2}, -1/2\rangle \end{pmatrix}$$

$$\cos\alpha = \sqrt{\frac{l + m + 1/2}{2l+1}} \Rightarrow \sin\alpha = \sqrt{\frac{l - m - 1/2}{2l+1}}$$

spinorial spherical harmonics:

$$\underline{\left|Y_{l,\frac{1}{2}}^{j = l \pm \frac{1}{2}, m}\right\rangle} = \pm\sqrt{\frac{l \pm m + 1/2}{2l+1}}\, |m - \tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{\frac{l \mp m + 1/2}{2l+1}}\, |m + \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|m_l, \tfrac{1}{2}\rangle \hat{=} Y_l^m(\vartheta, \varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |m_l, -\tfrac{1}{2}\rangle = Y_l^m(\vartheta, \varphi) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Y_{l,\frac{1}{2}}^{j = l \pm \frac{1}{2}, m}(\vartheta, \varphi) = \pm\sqrt{\frac{l \pm m + 1/2}{2l+1}}\, Y_l^{m - 1/2}(\vartheta, \varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ \sqrt{\frac{l \mp m + 1/2}{2l+1}}\, Y_l^{m + 1/2}(\vartheta, \varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$