So far: total amount of scattering amplitude

Now: partial wave decomposition

$\hat{=}$ decompose into $l = 0\,(s)$, $l = 1\,(p)$, $l = 2\,(d)$, ...

wave

special assumption: $V^{(int)}(\vec{x}) = v^{(int)}(|\vec{x}|)$

plane wave: partial wave decomposition

$$e^{i\underbrace{kr\cos\vartheta}_{=z}} = \sum_{l=0}^{\infty} i^l \, (2l+1) \, j_l(kr) \, P_l(\cos\vartheta)$$

outgoing spherical wave: partial wave decomposition

for convention

$$\underbrace{f(\vartheta,\varphi)}_{\text{scattering amplitude}} = f(\vartheta) = \sum_{l=0}^{\infty} (2l+1) \, f_l \, \underbrace{P_l(\cos\vartheta)}_{} \Leftarrow (*)$$

cylinder symmetry

f_l: partial wave amplitude of scattering amplitude

Legendre polynomials represent a basis for any function $f(\vartheta)$

Scattering wave function:

$$\psi(\vec{x}) = e^{ikr\cos\vartheta} + \frac{e^{ikr}}{r} f(\vartheta) = \sum_{l=0}^{\infty} (2l+1) \, P_l(\cos\vartheta) \left\{ i^l \, j_l(kr) + \boxed{f_l} \, \frac{e^{ikr}}{r} \right\}$$

partial wave decomposition

large distances: $r \to \infty$

Euler

$$j_l(s) \xrightarrow{s\to\infty} \frac{1}{s}\sin\left(s - \frac{l\pi}{2}\right) = \frac{1}{2is}\left\{ (-i)^l e^{is} - e^{-is}\, i^l \right\} ; \left[e^{il\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^l = i^l \right]$$

see beginning of scattering chapter

$$\psi(\vec{x}) \overset{|\vec{x}|\to\infty}{=} \sum_{l=0}^{\infty} \frac{\boxed{\boxed{(2l+1)}} P_l(\cos\vartheta)}{2ikr} \left\{ e^{ikr} \boxed{1 + 2ik \boxed{f_l}} - \boxed{(-1)^l} e^{-ikr} \right\}$$

outgoing spherical wave

scattering effect

incoming spherical wave

plane wave

far field solution of Schrödinger equation ($V^{(int)} = 0$)

$$\psi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-\infty}^{+\infty} c_{lm} R_l(r) Y_{lm}(\vartheta, \varphi)$$

cylinder symmetry

$$= \sum_{l=0}^{\infty} c_l \underbrace{R_l(r)}_{\overset{r\to\infty}{\simeq} \frac{\sin(kr - l\pi/2 + \delta_l)}{kr}} P_l(\cos\vartheta)$$

phase shift

see beginning of scattering chapter

fixes f_l

fixes c_l

$$= \sum_{l=0}^{\infty} \frac{\boxed{\boxed{c_l}} P_l(\cos\vartheta)}{2ikr} \left\{ \boxed{(-i)^l e^{i\delta_l}} e^{ikr} - \boxed{i^l e^{-i\delta_l}} e^{-ikr} \right\}$$

$e^{i\delta_l}$

$$\cancel{(2l+1)} \left[1 + 2ik f_l\right] = c_l (-i)^l$$

$$c_l\, i^l e^{-i\delta_l} = (-1)^l (2l+1)$$

$$\Rightarrow c_l = (2l+1)\, i^l\, e^{i\delta_l}$$

$$= \cancel{(2l+1)} \cancel{i^l} e^{i\delta_l} \cancel{(-i)^l} e^{i\delta_l}$$

$$= \cancel{(2l+1)}\, e^{2i\delta_l}$$

$$\boxed{f_l = \frac{1}{2ik} \left\{ e^{2i\delta_l} - 1 \right\} = \frac{e^{i\delta_l}}{\cancel{2i}k} \underbrace{\left(e^{i\delta_l} - e^{-i\delta_l} \right)}_{\cancel{2i} \sin\delta_l} = \frac{e^{i\delta_l} \sin\delta_l}{k}} \quad (**)$$

Partial wave contribution to the scattering amplitude f_ℓ is determined by the partial wave phase shift δ_ℓ.

Result: $f(\vartheta) \overset{(*)}{\underset{(**)}{=}} \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\vartheta)$

differential cross-section:

$$\left(\frac{d\sigma}{d\Omega}\right)(\vartheta) = |f(\vartheta)|^2 = \frac{1}{k^2} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'} P_\ell(\cos\vartheta) P_{\ell'}(\cos\vartheta)$$

total cross-section: integration over Ω

$$\boxed{\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)(\vartheta) = \frac{2\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)^2 \sin^2\delta_\ell \frac{2}{\cancel{2\ell+1}} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell}$$

orthogonality of Legendre polynomials

$$\int_0^\pi d\vartheta \sin\vartheta \, P_\ell(\cos\vartheta) P_{\ell'}(\cos\vartheta) = \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

$\hat{=}$ partial wave decomposition of the total cross-section

Example: hard spheres

$$\delta_\ell = - \frac{(2\ell+1)}{[(2\ell+1)!!]^2} (ka)^{2\ell+1}$$

total cross-section is affected by all plane waves

^{87}Rb: $\underset{\sim}{a} = 100\, a_B$, $a_B = 0.529 \cdot 10^{-10}\, m$, $M = 87 \cdot \underset{\sim}{u}$

s-wave scattering length

$= 1.67 \cdot 10^{-27} kg$

ultracold temperature: $T = 100\, nK = \underline{10^{-7} K}$

$$E = \frac{\hbar^2 k^2}{2\mu} \overset{\mu = \frac{M}{2}}{=} \frac{\hbar^2 k^2}{M} = k_B T \Rightarrow k = \sqrt{\frac{M k_B T}{\hbar^2}}$$

$\Rightarrow \quad ka = 0.097 \ll 1$

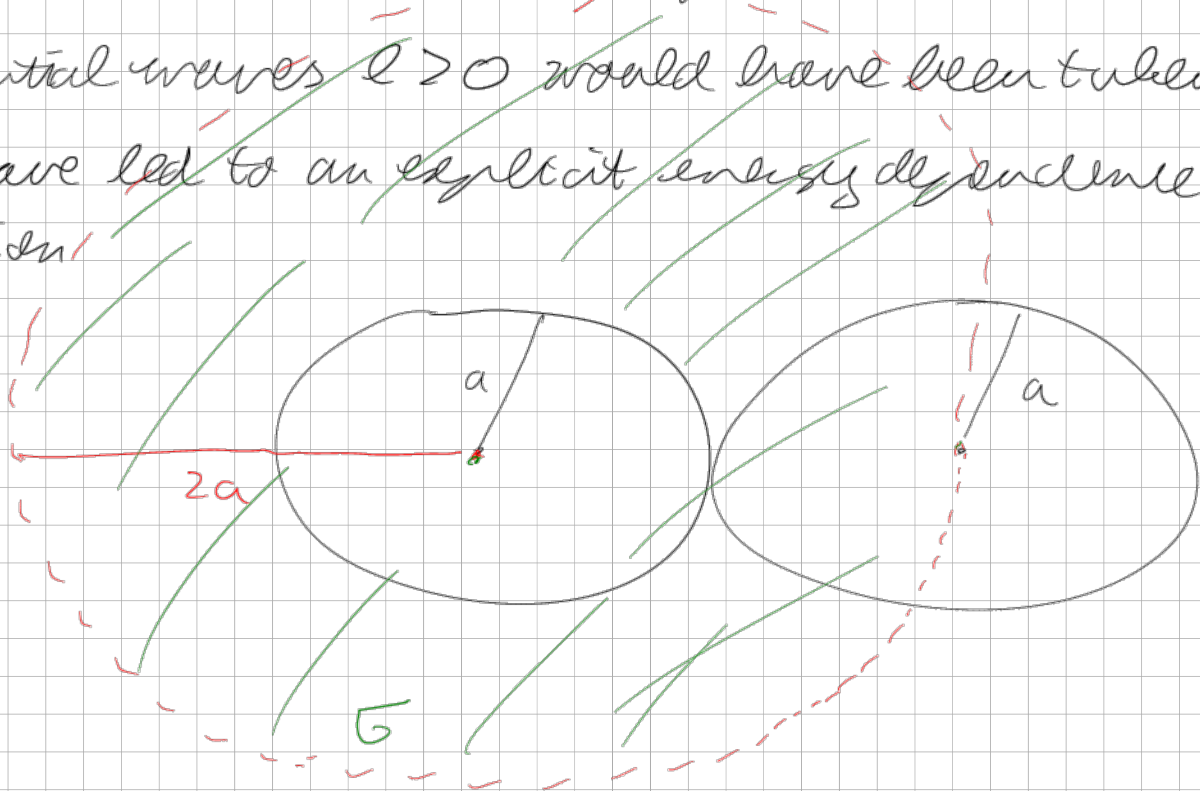
$\Rightarrow$ s-wave scattering with $l=0$ is the most relevant contribution

$$\delta_l = \delta_{l0}\,(-1)\,ka \Rightarrow \boxed{\sigma = \frac{4\pi}{\cancel{k^2}} \sin^2 \underbrace{\delta_0}_{-ka} = 4\pi a^2 = \pi (2a)^2}$$

$$ka \ll 1 \Rightarrow \cancel{k^2} a^2$$

independent of the scattering energy $\hat{=}$ low-energy limit

Note: If higher partial waves $l > 0$ would have been taken into account this would have led to an explicit energy dependence of the total cross-section



geometric interpretation of s-wave scattering

Indistinguishable Particles:

So far: particles which scatter, were assumed to be distinguishable
i.e. they can be enumerated

now: indistinguishable particles $\Rightarrow$ particles with the same properties i.e. the same mass $m_1 = m = m_2 \Rightarrow \mu = \frac{m}{2}$, and same quantum numbers

$$\underbrace{g}_{} = \frac{2\pi\hbar^2}{\mu} a \;\hat{=}\; \frac{4\pi\hbar^2}{m} a$$

strength of s-wave scattering

indistinguishable in 3D: bosons and fermions

symmetric wave function

antisymmetric wave function

Pauli spin-statistic theorem

integer spin

half-integer spin