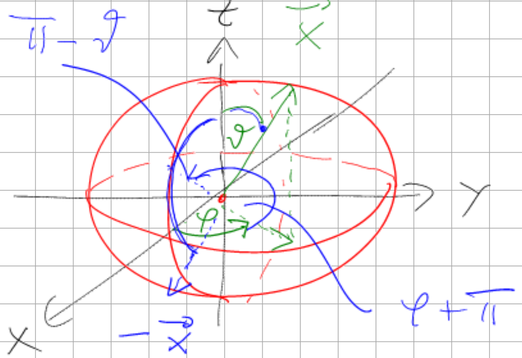
Indistinguishable Particles:

Scattering for bosons ($\varepsilon = +1$) and fermions ($\varepsilon = -1$):

$$\underbrace{\psi^{\varepsilon}(\vec{x})}_{\text{wave function for indistinguishable particles (normalised)}} = \frac{1}{\sqrt{2}} \Big\{ \underbrace{\psi(\vec{x})}_{\text{wave function for distinguishable particles (normalized)}} + \varepsilon \; \underbrace{\psi(-\vec{x})}_{\text{inversion is due to particle exchange}} \Big\} \quad ; \quad \underbrace{\vec{x} = \vec{x}_1 - \vec{x}_2}_{\text{relative coordinate}}$$

(symmetrization $\varepsilon = +1$ bosons
antisymmetrization $\varepsilon = -1$ fermions)

$$\underset{\text{scattering problem}}{=} \frac{1}{\sqrt{2}} \Big(\underbrace{e^{ikz}}_{\text{incoming plane wave}} + \varepsilon \, e^{-ikz} \Big) + \underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave}} \cdot \underbrace{f^{\varepsilon}(\vartheta)}_{\text{scattering amplitude}}$$



$$\vec{x} = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix} \iff -\vec{x} = r \begin{pmatrix} \sin(\pi-\vartheta)\cos(\varphi+\pi) \\ \sin(\pi-\vartheta)\sin(\varphi+\pi) \\ \cos(\pi-\vartheta) \end{pmatrix}$$

$$\boxed{\vartheta \iff \pi - \vartheta} \qquad \begin{cases} \sin(\pi-\vartheta) = +\sin\vartheta \\ \cos(\pi-\vartheta) = -\cos\vartheta \leftarrow \end{cases}$$

$$\varphi \iff \varphi + \pi \qquad \begin{cases} \sin(\varphi+\pi) = -\sin\varphi \\ \cos(\varphi+\pi) = -\cos\varphi \end{cases}$$

$$f^{\varepsilon}(\vartheta) \underset{\text{cylinder symmetry}}{=} \frac{1}{\sqrt{2}} \Big\{ f(\vartheta) + \varepsilon \, f(\pi - \vartheta) \Big\} \underset{\substack{\text{partial} \\ \text{wave} \\ \text{decomposition}}}{\longrightarrow} \frac{1}{\sqrt{2}} \Big\{ P_l(\cos\vartheta) + \varepsilon \, \underbrace{P_l(\underbrace{\cos(\pi-\vartheta)}_{= -\cos\vartheta})}_{= (-1)^l P_l(\cos\vartheta)} \Big\}$$

partial wave decomposition for scattering $f^{\varepsilon}(\vartheta)$:

$$P_\ell(\cos\vartheta) \longrightarrow P_\ell(\cos(\pi-\vartheta)) = P_\ell(-\cos\vartheta) = (-1)^\ell P_\ell(\cos\vartheta)$$

Legendre polynomial

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell \; ; \quad P_\ell(-z) = (-1)^\ell P_\ell(z)$$

odd $\ell \iff$ odd $P_\ell(z)$, even $\ell \iff$ even $P_\ell(z)$

$$\left(\frac{d\sigma}{d\Omega}\right)^{\varepsilon}(\vartheta) = \frac{1}{k^2}\sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}(2\ell+1)(2\ell'+1)e^{i(\delta_\ell-\delta_{\ell'})}\sin\delta_\ell \sin\delta_{\ell'}$$

$$\frac{1}{2}\left[1+\varepsilon(-1)^\ell\right]\left[1+\varepsilon(-1)^{\ell'}\right] P_\ell(\cos\vartheta)P_{\ell'}(\cos\vartheta) \quad \Big| \; 2\pi \cdot \int_0^\pi d\vartheta \sin\vartheta$$

$$\sigma^{\varepsilon} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)^{\varepsilon}(\vartheta) = \underbrace{\frac{4\pi}{\cancel{k^2}}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell}_{\text{distinguishable particle}} \; \frac{1}{2}\left[1+\varepsilon(-1)^\ell\right]^2$$

$$\int_{-1}^{+1} dz\, P_\ell(z) P_{\ell'}(z) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

ultracold $ka \ll 1$

$\sin^2\delta_\ell \approx (\cancel{k}a)^2$

s-wave scattering

$= 4\pi a^2 \cdot \underbrace{\frac{(1+\varepsilon)^2}{2}}_{\left\{\begin{array}{l}2\\0\end{array}\right.}$

$\ell = 0$

$\varepsilon = +1 \implies \sigma^{+1} = 8\pi a^2$

$\varepsilon = -1 \implies \sigma^{-1} = \boxed{0}$

There is NO scattering for fermions in same state

$\hat{=}$ free fermions

Dipole-dipole interaction:

two magnetic dipolar atoms

$$V_{DD}(\vec{x}) = -\frac{\mu_0}{4\pi}\,\frac{3(\vec{x}\cdot\vec{m})^2 - \vec{m}^2\vec{x}^2}{|\vec{x}|^5}$$

$\vec{m}$: magnetic dipole moment

two heteronuclear molecules

$$V_{DD}(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\,\frac{3(\vec{x}\cdot\vec{p})^2 - \vec{d}^2\vec{x}^2}{|\vec{x}|^5}$$

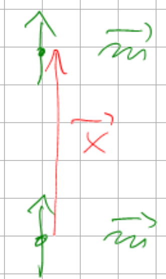
$\vec{p}$: electric dipole moment

repulsive

m: magic

no interaction

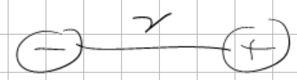
$$\vartheta_m = \arccos\frac{1}{\sqrt{3}} = 54.74°$$



attractive

- anisotropic interaction
- $1/|\vec{x}|^3$: "long-range" interaction

$$V_{DD}^{(magn.)} \sim \boxed{\frac{\mu_0}{4\pi}\,\frac{m^2}{r^3}} \longleftrightarrow V_{DD}^{(elec)} \sim \frac{1}{4\pi\varepsilon_0}\,\frac{p^2}{r^3}$$



$$m \sim \underbrace{m_B}_{\text{Bohr magneton}} = I\cdot F = \frac{e\hbar}{2M} \qquad \frac{\in}{a_B}$$

$p \sim e\,r$ (D: Debye)

$F = \pi a_B^2$, area

$I = \frac{e}{T} = \frac{e}{2\pi}\,\omega = \frac{e\hbar}{2\pi M \omega a_B^2}$, current

$$L = M v a_B = M\omega a_B^2 \overset{!}{=} \hbar \Rightarrow \omega = \frac{\hbar}{M\omega a_B^2}$$

Bohr-Sommerfeld quantization

comparison of magnetic / electric DDI:

$$\frac{V_{DD}^{(mag)}}{V_{DD}^{(el)}} = \frac{\frac{\mu_0}{\cancel{4\pi}} \cancel{\frac{1}{a_B^3}} \frac{e^2\hbar^2}{4m^2}}{\frac{1}{\cancel{4\pi}\varepsilon_0} e^2 a_B^2 \cancel{\frac{1}{a_B^3}}} = \dots = \boxed{\alpha^2 \sim 10^{-4}}$$

Bohr radius $a_B = \frac{4\pi\varepsilon_0 \hbar^2}{m e^2}$

Fourier transformed of DDI:

$$V_{DD}(\vec{x}) = -C_{DD} \cdot \frac{3(\vec{x}\cdot\vec{\mu})^2 - \vec{\mu}^2\vec{x}^2}{|\vec{x}|^5} = -C_{DD}\mu_i\mu_j \boxed{\frac{3x_ix_j - \delta_{ij}\vec{x}^2}{|\vec{x}|^5}}$$

$$\frac{\partial}{\partial x_i}\frac{1}{|\vec{x}|} = -\frac{x_i}{|\vec{x}|^3}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\vec{x}|} = \dots = \frac{3x_ix_j - \delta_{ij}|\vec{x}|^2}{|\vec{x}|^5} \quad ; \quad \vec{x} \neq \vec{0}$$

$$\Delta \frac{1}{|\vec{x}|} = -4\pi\,\delta(\vec{x}) \quad \text{(electrostatics)}$$

$$\frac{\partial^2}{\partial x_i^2}\frac{1}{|\vec{x}|} = \frac{1}{3}\Delta\frac{1}{|\vec{x}|} = -\frac{4\pi}{3}\delta(\vec{x})$$

$$\boxed{\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\vec{x}|} = \boxed{\frac{3x_ix_j - \delta_{ij}|\vec{x}|^2}{|\vec{x}|^5}} - \frac{4\pi}{3}\cancel{\delta(\vec{x})}\delta_{ij}}$$

distributional identity

this is missing in Jackson's first editions

only taken care of in more recent editions

$$-C_{DD}\frac{3x_ix_j - \delta_{ij}(\vec{x})}{|\vec{x}|^5}\mu_i\mu_j = V_{DD}(\vec{x}')$$

$$= -C_{DD}\,\mu_i \mu_j \left\{ \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\vec{x}|} + \frac{4\pi}{3}\,\delta(\vec{x})\,\delta_{ij} \right\}$$

convert this into Fourier space:

$$V_{DD}(\vec{k}) = \frac{4\pi}{3}\, C_{DD} \left\{ \frac{3(\vec{k}\cdot\vec{\mu})^2}{\vec{k}^2} - \vec{\mu}^2 \right\}$$

Theory of Angular Momenta:

Motivation:

Angular momenta important for atomic physics:

- orbital angular momentum L
- spin angular momentum S
- nuclear angular momentum I

L, S: electronic angular momentum $J = |L-S|, |L-S|+1, \ldots, L+S-1, L+S$

J, I: total angular momentum of atom $F = |J-I|, \ldots, J+I$

add 1 unit

Adding angular momenta leads to a shift of energies

LS coupling $\Rightarrow$ fine structure of energy levels

stems from Dirac equation for hydrogen atom by non-relativistic limit

JI - coupling $\Rightarrow$ hyperfine structure of energy levels

further examples: Zeeman effect $\rightsquigarrow$ energy splitting due to external magnetic fields, proportional to Landé factor

$$g = \frac{\dfrac{\text{magnetic moment}}{\text{Bohr magneton}}}{\dfrac{\text{angular momentum}}{\hbar}}$$

Another example: scattering theory:

atom with F_1

atom with F_2

compound

with: $F = |F_1 - F_2|, \ldots, F_1 + F_2$

F_1

F_2

$F_1 \neq F_2$: distinguishable particles

$F_1 = F_2$: indistinguishable "