Theory of Angular Momenta

11.1 Commutation Relations:

angular momentum operators $\hat{J}_i$; $i = x, y, z$

- hermitian
- canonical commutation relations: $[\hat{J}_i, \hat{J}_j]_- = \hat{J}_i \hat{J}_j - \hat{J}_j \hat{J}_i = i\hbar \underbrace{\varepsilon_{ijk}} \hat{J}_k$

$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}$; $\varepsilon_{123} := 1$ Levi-Civita symbol

1. example: orbital angular momentum

$$\vec{\hat{L}} = \vec{r} \times \vec{\hat{p}} = \begin{pmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{pmatrix} ; [\hat{p}_j, \hat{x}_k]_- = \frac{\hbar}{i}\delta_{jk} \Big\} \; [\hat{L}_j, \hat{L}_k]_- = i\hbar\varepsilon_{jkl}\hat{L}_l$$

coordinate representation: $\hat{x}_j = x_j$, $\hat{p}_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$ $\Big\}$

2. example: spin angular momentum

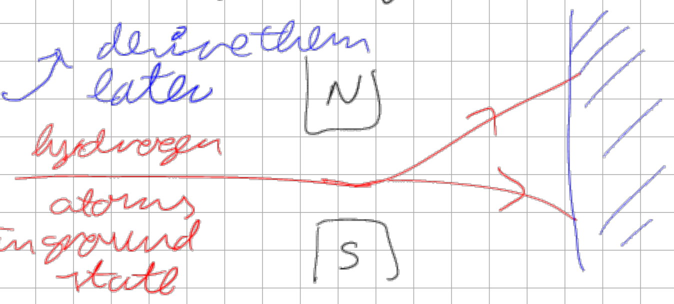
Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ↗ derive them later

Clifford algebra $[\sigma^k, \sigma^l]_+ = 2\delta_{kl}\underbrace{I}_{\text{unity matrix}}$

Lie algebra $[\sigma^k, \sigma^l]_- = 2i\,\varepsilon_{klm}\sigma^m$

spin angular momentum operators

$[\hat{S}_k, \hat{S}_l]_- = i\hbar\,\varepsilon_{klm}\hat{S}_m$



$2s+1 = 2$

$\Rightarrow s = \frac{1}{2}$

Stern-Gerlach experiment 1922

$\vec{\hat{S}} = \frac{\hbar}{2}\vec{\sigma}$

11.2 Ladder Operators:

$$\hat{\vec{J}}^2 = \sum_{k=1}^{3} \hat{J}_k^2 \quad \Rightarrow \quad [\hat{\vec{J}}^2, \hat{J}_j]_- = \dots = 0$$

$$[\hat{A}, \hat{B}\hat{C}]_- = [\hat{A},\hat{B}]_-\hat{C} + \hat{B}[\hat{A},\hat{C}]_- \qquad \boxed{[\hat{J}_i, \hat{J}_j]_- = i\hbar\, \varepsilon_{ijl}\, \hat{J}_l}$$

$\hat{J}_z$ and $\hat{\vec{J}}^2$: $\quad [\hat{\vec{J}}^2, \hat{J}_z]_- = 0$

From QM I we know then that there exist states which are eigenstates of both $\hat{J}_z$ and $\hat{\vec{J}}^2$:

$$\hat{\vec{J}}^2 |a,b\rangle = a\, |a,b\rangle \qquad (1)$$

$$\hat{J}_z |a,b\rangle = b\, |a,b\rangle \qquad (2)$$

Goal: to determine a, b based on angular momentum commutation relations

$\langle a,b|$ (1): $\quad \underbrace{\langle a,b|\hat{\vec{J}}^2|a,b\rangle} = a\, \underbrace{\langle a,b|a,b\rangle}_{=1}$

$$= \langle a,b|\hat{\vec{J}}\cdot\hat{\vec{J}}\,|a,b\rangle = \underbrace{\left(\hat{\vec{J}}\,|a,b\rangle\right)^{\dagger}}_{\langle\psi|}\cdot\underbrace{\hat{\vec{J}}\,|a,b\rangle}_{|\psi\rangle} = a \geq 0$$

In order to reveal the possible values of a, b

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y = (\hat{J}_\mp)^\dagger$$

1) $[\hat{J}_+, \hat{J}_-]_- = [\hat{J}_x + i\hat{J}_y, \hat{J}_x - i\hat{J}_y]_- = i\{i\underbrace{[\hat{J}_y, \hat{J}_x]_-}_{= -i\hbar\hat{J}_z} - i\underbrace{[\hat{J}_x, \hat{J}_y]}_{= i\hbar\hat{J}_z}\}$

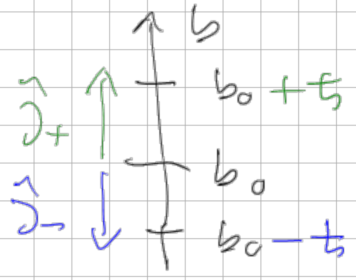
$= 2\hbar\hat{J}_z$

$\Rightarrow$ ordering of $\hat{J}_+$ and $\hat{J}_-$ matters

2) $[\hat{J}_z, \hat{J}_\pm]_- = [\hat{J}_z, \hat{J}_x]_- \pm i[\hat{J}_z, \hat{J}_y]_-$

$$= i\hbar\,\hat{J}_y \qquad\qquad = i\hbar\,\hat{J}_x$$

$$\hat{J}_z \boxed{\{\underbrace{\hat{J}_\pm |a,b\rangle}_{=2}\}} \overset{= \pm\hbar\hat{J}_\pm}{=} \hat{J}_\pm \underbrace{\hat{J}_z |a,b\rangle}_{= b|a,b\rangle} \pm \hbar \hat{J}_\pm |a,b\rangle = \underline{(b \pm \hbar)} \boxed{\{\hat{J}_\pm |a,b\rangle\}}$$



$\hat{J}_\pm |a,b\rangle$ is eigenstate of $\hat{J}_z$ with eigenvalue $b \pm \hbar$

$\Rightarrow$ $\hat{J}_\pm$ is a ladder operator

see harmonic oscillator ladder operators

$$\hat{a}^+ |n\rangle \sim |n+1\rangle, \quad \hat{a}\,|n\rangle \sim |n-1\rangle$$

3) $[\hat{\vec{J}}^2, \hat{J}_\pm]_- = [\hat{\vec{J}}^2, \hat{J}_x \pm i\hat{J}_y]_- = \ldots = 0$

$$\hat{\vec{J}}^2 \{\hat{J}_\pm |a,b\rangle\} = \hat{J}_\pm \hat{\vec{J}}^2 |a,b\rangle = a\,\hat{J}_\pm |a,b\rangle$$

$\Rightarrow$ $\hat{J}_\pm |a,b\rangle$ is eigenstate of $\hat{\vec{J}}^2$ with eigenvalue a

Conclusion from 2) and 3): $\hat{J}_\pm |a,b\rangle \underset{\uparrow}{\sim} |a, b \pm \hbar\rangle$

Note: proportionality coefficient not yet known → follows soon

Here: determine a, b

11.3 Eigenvalues:

$$\hat{J}_+^n |a,b\rangle \sim |\underbrace{a}_{\text{unchanged}}, \underbrace{b + n\hbar}_{\text{increases}}\rangle$$

What happens in the limit $n \to \infty$? It turns out that b can not increases to an arbitrarily large values as it has to fulfill the inequality

$$a \geq b^2 \; (\geq 0)$$

Let us proof this:

$$\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + i\underbrace{(\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y)}_{= -i\hbar\hat{J}_z} + \hat{J}_y^2 = \vec{\hat{J}}^2 - \hat{J}_z^2 + \hbar\hat{J}_z \qquad (**)$$

$$\hat{J}_- \hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + \hat{J}_x) = \hat{J}_x^2 + i\, i\hbar\hat{J}_z + \hat{J}_y^2 = \vec{\hat{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \qquad (*)$$

$$\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ = 2(\hat{J}_x^2 + \hat{J}_y^2) = 2(\vec{\hat{J}}^2 - \hat{J}_z^2)$$

$$\boxed{\langle a,b | \vec{\hat{J}}^2 - \hat{J}_z^2 | a,b\rangle} = \frac{1}{2}\left\{\langle a,b | \underbrace{\hat{J}_+}_{= \hat{J}_-^+} \hat{J}_- | a,b\rangle + \langle a,b | \underbrace{\hat{J}_-}_{= \hat{J}_+^+} \hat{J}_+ | a,b\rangle\right\}$$

$$= \frac{1}{2}\left\{(\hat{J}_- |a,b\rangle)^+ \hat{J}_- |a,b\rangle + (\hat{J}_+ |a,b\rangle)^+ \hat{J}_+ |a,b\rangle\right\} \geq 0$$

$$(a - b^2)\underbrace{\langle a,b | a,b\rangle}_{= 1} \geq 0 \Rightarrow a \geq b^2 \geq 0$$

$$\Rightarrow \underbrace{-\sqrt{a}}_{\text{lower boundary}} \leq b \leq \underbrace{+\sqrt{a}}_{\text{upper boundary}}$$

There must be a b_{max} such that $(\hat{J}_+)|a, b_{max}\rangle = 0$

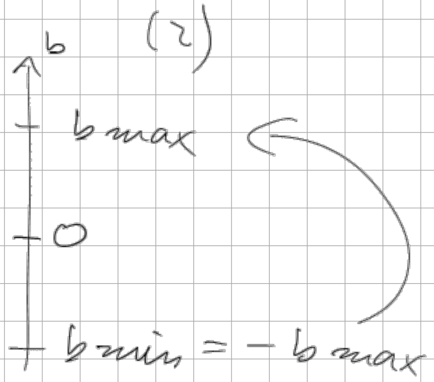
$$\hat{J}_- \hat{J}_+ |a, b_{max}\rangle \overset{(*)}{=} (\vec{\hat{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|a, b_{max}\rangle \overset{\supset}{=} 0$$

$$\Rightarrow a - b_{max}^2 - \hbar b_{max} = 0 \Rightarrow \boxed{a = b_{max}^2 + \hbar b_{max}} \qquad (1)$$

Similarly: $\hat{J}_- |a, b_{min}\rangle = 0$

$$\hat{J}_+ \hat{J}_- |a, b_{min}\rangle = 0 \overset{(**)}{=} (\hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z) |a, b_{min}\rangle$$

$$\Rightarrow \boxed{a = b_{min}^2 - \hbar b_{min}}$$



$$b_{max} = - b_{min} \quad \hat{=} \quad (1) \Leftrightarrow (2)$$

$$(\hat{J}_+)^n |a, b_{min}\rangle = (J_+)^n |a, -b_{max}\rangle \sim |a, b_{max}\rangle$$

to be determined but integer

$$\sim |a, -b_{max} + n\hbar\rangle$$

integer

$$b_{max} = -b_{max} + n\hbar \Rightarrow b_{max} = \frac{n}{2}\hbar$$

convenient notation: $j := \frac{b_{max}}{\hbar} \Rightarrow \underbrace{j = \frac{n}{2}}_{\text{either integer or half integer}}, \quad n \in \mathbb{N}$

angular momentum quantum number

Result: This is valid for **any** angular momentum operator. It is an immediate consequence of the angular momentum canonical commutation relation.

n integer implies:

- j is integer: n even
- j is half-integer: n odd

$$a \overset{(1)}{=} b_{max}^2 + \hbar b_{max} = \hbar^2 j(j+1) \text{ eigenvalue of } \hat{J}^2$$

new definition: $m = \frac{b}{\hbar} \Rightarrow b = m\hbar$

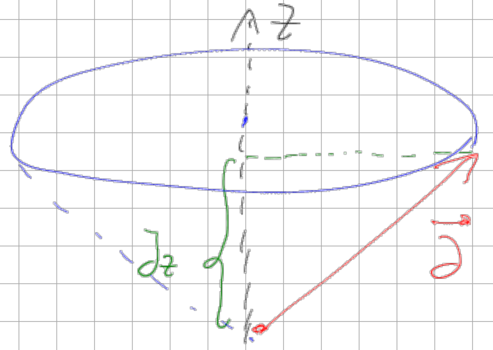
magnetic quantum number

- j integer $\hat{=}$ m integer
- j half integer $\hat{=}$ m half integer

allowed values for m: $\underbrace{-j}_{j_{min}=-j_{max}}, -j+1, \ldots, j-1, \underbrace{j}_{j_{max}}$

Condition:
$$\boxed{\begin{aligned} \hat{\vec{J}}^2 |j, m\rangle &= \hbar^2 j(j+1) |j, m\rangle \\ \hat{J}_z |j, m\rangle &= \hbar m |j, m\rangle \quad ; \quad m = -j, -j+1, \ldots, j-1, j \end{aligned}}$$

Semiclassical vector model of angular momenta:



- classically: all values are possible
- quantum mechanically: two quantization condition
  > length: $|\vec{J}| = \hbar\sqrt{j(j+1)}$
  > projection upon z-axis: $J_z = m\hbar$
- arise from $[\hat{\vec{J}}^2, \hat{J}_z]_- = 0$

$\Rightarrow$ $\vec{J}^2$ and J_z are mutually measurable, but do to $[\hat{J}_i, \hat{J}_j]_- = i\hbar\varepsilon_{ijk}\hat{J}_k$ the components J_x, J_y are not measurable

This represented pictorially by a precession of the angular momentum vector $\vec{J}$ around z-axis.