11.4. Matrix elements:

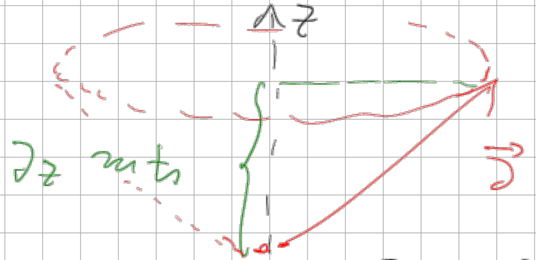
angular momentum operators: $[\hat{J}_j, \hat{J}_k]_- = i\,\varepsilon_{jkl}\,\hat{J}_l$

$\hat{\vec{J}}^2 = \sum_j \hat{J}_j^2$

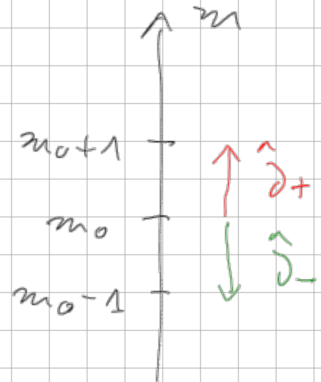
$\Rightarrow \quad \hat{\vec{J}}^2 \,|j, m\rangle = \hbar^2 j(j+1)\,|j, m\rangle$; j either half-integer or integer

$\quad\;\; \hat{J}_z \,|j, m\rangle = \hbar\, m \,|j, m\rangle$; $m = -j, \dots, j$

semiclassical vector model



$|\vec{J}| = \hbar\sqrt{j(j+1)}$



$\langle j', m' |\, \hat{\vec{J}}^2 \,| j, m\rangle = \hbar^2 j(j+1)\, \delta_{jj'}\, \delta_{m,m'}$

$\langle j', m' |\, \hat{J}_z \,| j, m\rangle = \hbar\, m\, \delta_{jj'}\, \delta_{m m'}$

Now: matrix elements for $\hat{J}_x$, $\hat{J}_y$ wanted!

$\hat{J}_\pm = \hat{J}_x \pm i\,\hat{J}_y$ ladder operators

last time: $\hat{J}_\pm \,|j, m\rangle \sim |j, m\pm 1\rangle$

last time: this was left open

$\langle j, m |\, \underbrace{\hat{J}_-}_{=\hat{J}_+^+} \hat{J}_+ \,| j, m\rangle \underset{\text{last time}}{=} \langle j, m |\, \hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \,| j, m\rangle$

$\left(\hat{J}_+ |j, m\rangle\right)^+ \;\hat{J}_+ |j, m\rangle$

$= \hbar^2 \left(j(j+1) - m^2 - m \right)$

$$\hat{J}_+ |j,m\rangle = C^+_{jm} |j, m+1\rangle \longrightarrow |C^+_{jm}|^2 = \hbar^2 (j+1) - m^2 - m$$

$$C^+_{jm} = \hbar \underbrace{\sqrt{j(j+1) - m^2 - m}}_{=\sqrt{(j-m)(j+m+1)}} \; \underbrace{e^{i\alpha}}_{=:0}$$

and correspondingly: $\hat{J}_- |j,m\rangle = C^-_{jm} |j, m-1\rangle$

$$\rightarrow C^-_{jm} = \hbar \sqrt{(j+m)(j-m+1)}$$

$$\langle j', m' | \hat{J}_\pm | j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)}\, \delta_{jj'} \delta_{m', m\pm 1}$$

11.5 Example: spin 1/2

$j = \frac{1}{2}, \quad m = \pm \frac{1}{2} \implies$ basis states $|\underbrace{\tfrac{1}{2}}_{=j}, \underbrace{\pm \tfrac{1}{2}}_{=m}\rangle$

preparatory consideration non-vanishing matrix elements:

$$\langle \tfrac{1}{2}, \tfrac{1}{2} | \hat{J}_+ | \tfrac{1}{2}, \underbrace{-\tfrac{1}{2}}_{m=}\rangle = \hbar \sqrt{\left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right)\left(\tfrac{1}{2} - \tfrac{1}{2} + 1\right)} = \hbar$$

$$\langle \tfrac{1}{2}, -\tfrac{1}{2} | \hat{J}_- | \tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar$$

$$\langle \tfrac{1}{2}, \pm\tfrac{1}{2} | \hat{J}_z | \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm \frac{\hbar}{2}$$

$$\begin{aligned} \hat{J}_+ &= \hat{J}_x + i \hat{J}_y \\ \hat{J}_- &= \hat{J}_x - i \hat{J}_y \end{aligned} \implies \begin{aligned} \hat{J}_x &= \tfrac{1}{2}(\hat{J}_+ + \hat{J}_-) \\ \hat{J}_y &= \tfrac{1}{2i}(\hat{J}_+ - \hat{J}_-) \end{aligned}$$

$$\left(\langle \tfrac{1}{2}, \tfrac{1}{2}|, \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right) \begin{pmatrix} \frac{1}{2}(\hat{J}_+ + \hat{J}_-) \\ \frac{1}{2i}(\hat{J}_+ - \hat{J}_-) \\ \hat{J}_z \end{pmatrix} \begin{pmatrix} |\tfrac{1}{2}, \tfrac{1}{2}\rangle \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} \langle \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}(\cancel{\hat{S}_+} + \hat{S}_-) | \tfrac{1}{2}, \tfrac{1}{2} \rangle & \langle \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}(\hat{S}_+ + \cancel{\hat{S}_-}) | \tfrac{1}{2}, -\tfrac{1}{2} \rangle \\ \langle \tfrac{1}{2}, -\tfrac{1}{2} | \tfrac{1}{2}(\cancel{\hat{S}_+} + \hat{S}_-) | \tfrac{1}{2}, \tfrac{1}{2} \rangle & \langle \tfrac{1}{2}, -\tfrac{1}{2} | \tfrac{1}{2}(\cancel{\hat{S}_+} + \cancel{\hat{S}_-}) | \tfrac{1}{2}, -\tfrac{1}{2} \rangle \end{pmatrix} \\ \begin{pmatrix} 0 & \langle \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2i} \hat{S}_+ | \tfrac{1}{2}, -\tfrac{1}{2} \rangle \\ \langle \tfrac{1}{2}, -\tfrac{1}{2} | \tfrac{1}{2i} \hat{S}_- | \tfrac{1}{2}, \tfrac{1}{2} \rangle & 0 \end{pmatrix} \\ \begin{pmatrix} \langle \tfrac{1}{2}, \tfrac{1}{2} | \hat{S}_z | \tfrac{1}{2}, \tfrac{1}{2} \rangle & 0 \\ 0 & \langle \tfrac{1}{2}, -\tfrac{1}{2} | \hat{S}_z | \tfrac{1}{2}, -\tfrac{1}{2} \rangle \end{pmatrix} \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \frac{\hbar}{2} \vec{\sigma} = \vec{S}$$ spin vector operator

Analogous procedure for spin 1:

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

11.6. Formal Theory of Angular Momentum Addition:

consider two angular momentum operators: $\hat{\vec{J}}_1, \hat{\vec{J}}_2$
acting on two different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$

$$[\hat{J}_{1j}, \hat{J}_{1k}]_- = i\hbar \varepsilon_{jkl} \hat{J}_{1l} \quad , \quad [\hat{J}_{2j}, \hat{J}_{2k}]_- = i\hbar \varepsilon_{jkl} \hat{J}_{2l} \qquad (1)$$

independence: $[\hat{J}_{1j}, \hat{J}_{2k}]_- = 0 \qquad (2)$

total angular momentum operator: $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$

(1) + (2): $\quad [\hat{J}_j, \hat{J}_k]_- = i\hbar \varepsilon_{jkl} \hat{J}_l$

Eigenvalue discussion with ladder operators applies to

1) $\hat{\vec{J}}_1^2, \quad \hat{J}_{1z}$
2) $\hat{\vec{J}}_2^2, \quad \hat{J}_{2z}$ $\qquad$ option A
3) $\hat{\vec{J}}^2, \quad \hat{J}_z$ $\qquad$ option B

Option A: $\hat{\vec{J}}_1^2, \hat{\vec{J}}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$ commute among themselves
$\rightarrow$ simultaneous eigenstates of all 4 operators denoted by

$$|j_1, j_2; m_1, m_2\rangle = \underbrace{|j_1, m_1\rangle}_{\in \mathcal{H}_1} \underbrace{|j_2, m_2\rangle}_{\in \mathcal{H}_2}$$

fixed j_1, j_2: $\dim(\mathcal{H}_1) = 2j_1 + 1 \quad , \quad \dim(\mathcal{H}_2) = 2j_2 + 1$

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$: $\quad \dim(\mathcal{H}) = (2j_1 + 1)(2j_2 + 1)$

$$\hat{\vec{J}}_k^2 \, |j_1, j_2; m_1, m_2\rangle = \hbar^2 j_k(j_k+1) \, |j_1, j_2; m_1, m_2\rangle \; ; k = 1, 2$$

$$\hat{J}_{kz} \, |j_1, j_2; m_1, m_2\rangle = \hbar m_k \, |j_1, j_2; m_1, m_2\rangle \; ; k = 1, 2$$

Option B: $\hat{\vec{J}}_1^2, \hat{\vec{J}}_2^2, \hat{\vec{J}}^2, \hat{J}_z$ commute, where $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$ and

$$\hat{\vec{J}}^2 = (\hat{\vec{J}}_1 + \hat{\vec{J}}_2)(\hat{\vec{J}}_1 + \hat{\vec{J}}_2) = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2 \hat{\vec{J}}_1 + \hat{\vec{J}}_1 \hat{\vec{J}}_2 + \hat{\vec{J}}_2^2$$

$$\hat{\vec{J}}_k^2 \, |j_1, j_2; j, m\rangle = \hbar^2 j_k(j_k+1) \, |j_1, j_2; m_1, m_2\rangle \; k = 1, 2$$

$$\hat{\vec{J}}^2 \, |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) \, |j_1, j_2; j, m\rangle$$

$$\hat{J}_z \, |j_1, j_2; j, m\rangle = \hbar m \, |j_1, j_2; j, m\rangle$$

Note: $\hat{\vec{J}}^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$

$$[\hat{\vec{J}}^2, \hat{J}_{1z}]_- = \ldots = \hbar \{ -\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+} \} \longleftarrow [\hat{J}_{kz}, \hat{J}_{k\pm}]_- = \pm\hbar \hat{J}_{k\pm}$$

last time

$$[\hat{\vec{J}}^2, \hat{J}_{2z}]_- = \ldots = \hbar \{ +\hat{J}_{1+}\hat{J}_{2-} - \hat{J}_{1-}\hat{J}_{2+} \}$$

$$[\hat{\vec{J}}^2, \hat{J}_z]_- = 0$$

Result 1: Option A can not be extended by $\hat{\vec{J}}^2$

Result 2: Option B " " " " " $\hat{J}_{1z}, \hat{J}_{2z}$

Result: Two bases sets of eigenstates to two maximal sets of mutually commuting operators

$$\underbrace{|j_1, j_2; j, m\rangle}_{\text{option B}} = \sum_{m_1, m_2} \underbrace{\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}} \underbrace{|j_1, j_2; m_1, m_2\rangle}_{\text{option A}}$$

$$\sum_{m_1, m_2} |j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2| = 1$$

11.7 Properties of Clebsch-Gordan Coefficients

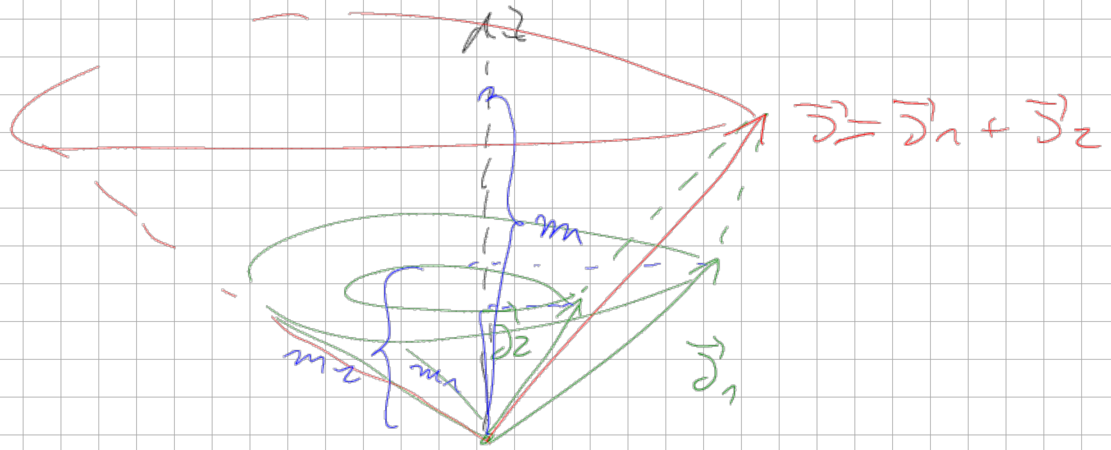
1) $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} \Rightarrow (\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}) |j_1, j_2; j, m\rangle = 0 \;\Big|\; \langle j_1, j_2; m_1, m_2|$

$$\underbrace{(m - m_1 - m_2)}_{\neq 0} \cdot \underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle}_{= 0} = 0$$

$\neq 0 \Rightarrow = 0$

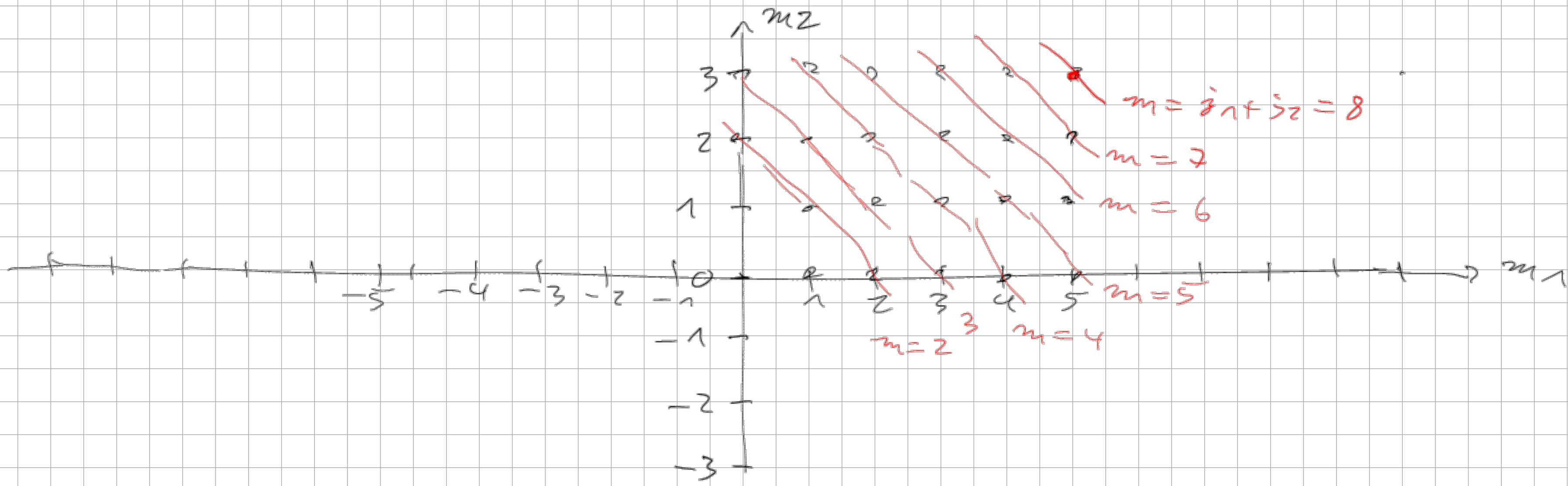
All Clebsch-Gordan coefficients vanish unless $\boxed{m = m_1 + m_2}$

Consequence for semiclassical vector model



2) degeneracy of m is determined by all pairs (m_1, m_2) with $m = m_1 + m_2$

here $j_1 = 5$, $j_2 = 3$



$j_1 = 5 \qquad j_2 = 3$

j	$2j+1$
$8 = j_1 + j_2$	17
7	15
6	13
5	11
4	9
3	7
$2 = j_1 - j_2$	5

$$77 = \underbrace{11}_{(2\cdot 5+1)} \cdot \underbrace{7}_{(2\cdot 3+1)} = 77$$

triangle rule

$j = |j_1 - j_2|, \; - - - - , \; j_1 + j_2$

$$\underbrace{\sum_{j=|j_1-j_2|}^{j_1+j_2}(2j+1)}_{\text{option B}} = \left(\sum_{j=1}^{j_1+j_2} - \sum_{j=1}^{j_1-j_2-1}\right)(2j+1)$$

$j_1 \geq j_2$

$$= \underbrace{(2j_1+1)(2j_2+1)}_{\text{option A}}$$