Harmonic Oscillator:

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{m}{2}\omega^2x^2\right\}\psi(x)=E\,\psi(x) \quad \Big|\cdot\frac{2m}{\hbar^2}$$

$$\left\{-\frac{d^2}{dx^2}+\underbrace{\frac{m}{2}\omega^2\frac{2m}{\hbar^2}}_{=\left(\frac{m\omega}{\hbar}\right)^2=\frac{1}{\ell^4}}x^2\right\}\psi(x)=\frac{2mE}{\hbar^2}\psi(x)$$

$$\frac{2mE}{\hbar^2} \longrightarrow 2\,\underbrace{\frac{m\omega}{\hbar}}_{\frac{1}{\ell^2}}\underbrace{\frac{E}{\hbar\omega}}_{=\varepsilon}$$

$$\ell=\sqrt{\frac{\hbar}{m\omega}}$$ oscillator length

$$\left\{-\frac{d^2}{dx^2}+\frac{x^2}{\ell^4}\right\}\psi(x)=2\frac{\varepsilon}{\ell^2}\psi(x)$$ Weber differential equation

coordinate transformation to dimensionless coordinate $\underline{y=\frac{x^2}{\ell^2}}$

$$\psi(x)=\varphi\left(y=\frac{x^2}{\ell^2}\right), \quad \varphi(y)=\psi\left(x=\ell\sqrt{y}\right)$$

$$\frac{d}{dx}=\frac{dy}{dx}\frac{d}{dy}=\frac{2x}{\ell^2}\frac{d}{dy}, \quad \frac{d^2}{dy^2}=\ldots$$

$$\longrightarrow \quad y\frac{d^2\varphi}{dy^2}+\frac{1}{2}\frac{d\varphi}{dy}+\left(\frac{\varepsilon}{2}-\frac{1}{4}y\right)\varphi=0$$

Procedure: split off asymptotic solution in order to obtain standard differential equation

$$\psi(x)\sim e^{-\frac{x^2}{2\ell^2}}, \quad |x|\to\infty \quad \hat{=} \quad \varphi(y)\sim e^{-\frac{y}{2}}, \quad y\to\infty$$

$$\varphi(y)=\underbrace{e^{-\frac{y}{2}}}_{\text{asymptotic form}}\;\underbrace{\phi(y)}_{\text{rest (remainder)}}$$

$$\frac{d\varphi}{dy} = e^{-\frac{y}{2}}\left(-\frac{\phi}{2} + \frac{d\phi}{dy}\right), \quad \frac{d^2\varphi}{dy^2} = ----$$

$$\underline{y\frac{d^2\phi}{dy^2}} + \left(\frac{1}{2} - y\right)\frac{d\phi}{dy} + \left(\frac{\varepsilon}{2} - \frac{1}{4}\right)\phi = 0$$

$$y\frac{d^2\phi}{dy^2} + (c - y)\frac{d\phi}{dy} - a\phi = 0$$

Kummer differential equation

free parameters

$$a = \frac{1}{4} - \frac{\varepsilon}{2}, \quad c = \frac{1}{2}$$

Hypergeometric differential equation:

$$z(1-z)\frac{d^2\phi}{dz^2} + [c - (a+b+1)z]\frac{d\phi}{dz} - ab\phi = 0 \qquad (*)$$

three parameters

$\hat{=}$ large variety of functions

$$\phi(z) = z^{\sigma}\sum_{\nu=0}^{\infty} c_\nu z^\nu \qquad (1)$$

index expansion coefficients

$$\frac{d\phi(z)}{dz} = \sum_{\nu=0}^{\infty} c_\nu(\nu+\sigma)z^{\nu+\sigma-1} \qquad (2)$$

$$\frac{d^2\phi(z)}{dz^2} = \sum_{\nu=0}^{\infty} c_\nu(\nu+\sigma)(\nu+\sigma-1)z^{\nu+\sigma-2} \qquad (3)$$

(1) – (3) in (*):

$$\sum_{\nu=0}^{\infty} c_\nu(\nu+\sigma)(\nu+\sigma-1)\boxed{(1-z)}\,z^{\overbrace{\nu+\sigma-1}^{=\nu'}}$$

$\nu = 0, \ldots, \infty$

$\nu' = -1, 0, \ldots, \infty$

$$+\sum_{r=0}^{\infty} c_r (r+\sigma)\boxed{[c-(a+b+1)z]}\, z^{r+\sigma-1} - \sum_{r=0}^{\infty} c_r\, ab\, z^{r+\sigma} = 0$$

$$\sum_{r=0}^{\infty} z^{r+\sigma}\Big(c_r \{-[(r+\sigma)(r+\sigma-1) + a+b+1 \quad + ab]\}$$
$$+ c_{r+1}\{c(r+1+\sigma) + (r+\sigma+1)(r+\sigma)\}\Big)$$
$$+ c_0\, \sigma(c+\sigma-1)\, z^{\sigma-1} = 0$$

coefficients of different powers vanish

1) indicial equation: $\sigma(c+\sigma-1) = 0 \Rightarrow \boxed{\sigma_1 = 0,\ \sigma_2 = 1-c}$

2) recurrence relation: $c_{r+1} = \dfrac{(r+\sigma+a)(r+\sigma+b)}{(r+\sigma+c)(r+\sigma+1)}\, c_r$

$$\Rightarrow c_r = \boxed{}\, c_{r-1} = \boxed{}\, c_{r-2} = \dots = \boxed{}\, c_0$$

r times recurrence relation

Pochhammer symbol $(a)_r = a\cdot(a+1)(a+2)\cdot\ldots\cdot(a+r-1)$; $(a)_0 = 1$

special case: $(1)_r = 1\cdot 2\cdot 3\cdot\ldots\cdot(r-1)\cdot r = r!$ factorial

$$c_r = \frac{(a+\sigma)_r\,(b+\sigma)_r}{(c+\sigma)_r\,(1+\sigma)_r} \Rightarrow \boxed{\phi(z) = z^{\sigma}\sum_{r=0}^{\infty}\frac{(a+\sigma)_r(b+\sigma)_r}{(c+\sigma)_r(1+\sigma)_r}\, z^r}$$

1. Case: $\sigma_1 = 0$

$$\phi_1(z) = {}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty}\frac{\boxed{(a)_r\,(b)_r}}{\boxed{(c)_r}}\frac{z^r}{r!}$$

hypergeometric function

$b \neq -n$; $a = -n$; $n = 0, 1, 2, \ldots$ $\Rightarrow$ polynomial

$c \neq -n$; $n = 0, 1, 2, \ldots$

generalisation:

$$ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_r\cdots(\alpha_p)_r}{(\beta_1)_r(\beta_2)_r\cdots(\beta_q)_r}\frac{z^r}{r!} $$

2. Case: $\sigma_2 = 1-c$

$$ \phi_2(z) = z^{1-c}\;{}_2F_1(a+1-c,\, b+1-c;\, 2-c;\, z) $$

general solution:

$$ \phi_1(z) = A\;{}_2F_1(a,b;c;z) + B\,z^{1-c}\;{}_2F_1(a+1-c,\, b+1-c;\, 2-c;\, z) $$

Remark: All elementary functions can be expressed in terms of hypergeometric functions.

→ Gradshteyn, section 9.12

$$ z(1-z)\frac{d^2\phi}{dz^2} + [c-(a+b+1)z]\frac{d\phi}{dz} - ab\,\phi = 0 \quad \Big| : b $$

hypergeometric differential equation

↓

1. step: linear transformation: $y = bz$, $z = \frac{y}{b}$

$$ y\left(1-\cancel{\frac{y}{b}}\right)\frac{d^2\phi}{dy^2} + \left[c - y - \cancel{\frac{a+1}{b}y}\right]\frac{d\phi}{dy} - a\phi = 0 $$

2. step: limit $b \to \infty$

$$ y\frac{d^2\phi}{dy^2} + (c-y)\frac{d\phi}{dy} - a\phi = 0 $$

Kummer differential equation

$$ \phi(y) = \lim_{b\to\infty}\left\{ \underbrace{A\;{}_2F_1\left(a,b;c;\frac{y}{b}\right)} + \underbrace{B\left(\frac{y}{b}\right)^{1-c}\;{}_2F_1\left(a+1-c,\, b+1-c;\, 2-c;\, \frac{y}{b}\right)} \right\} $$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{1}{n!} \frac{y^n}{b^n} + B y^{1-c} \frac{1}{b^{1-c}} \sum_{n=0}^{\infty} \frac{(a+1-c)_n (b+1-c)_n}{(2-c)_n} \frac{y^n}{b^n} \frac{1}{n!}$$

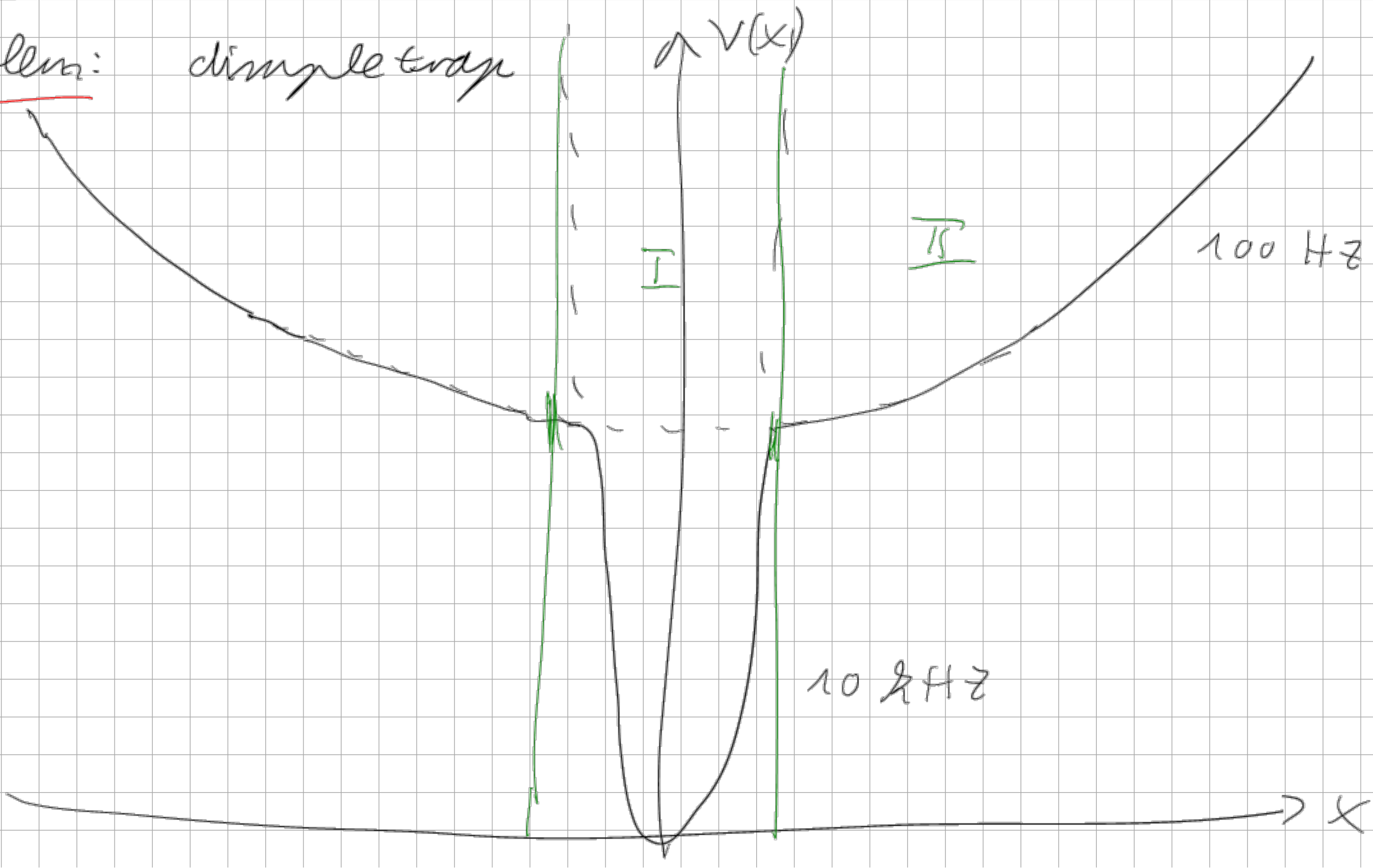
B'

$$\frac{b(b+1)\cdot \ldots \cdot (b+n-1)}{b \cdot b \cdot \ldots \cdot b} \xrightarrow{b \to \infty} 1 \qquad = \frac{(b+1-c)(b+2-c)\ldots(b+n-c)}{b \; b \ldots b} \to 1$$

$$\phi(y) = A \underbrace{\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{y^n}{n!}}_{= {}_1F_1(a;c;y)} + B' y^{1-c} \, {}_1F_1(a+1-c; 2-c; y)$$

confluent hypergeometric function

Possible problem: dimple trap



Now: back to harmonic oscillator $\quad a = \frac{1}{4} - \frac{\varepsilon}{2}, \quad c = \frac{1}{2}$

$$\phi(y) = A\,{}_1F_1\left(\frac{1}{4} - \frac{\varepsilon}{4}; \frac{1}{2}; y\right) + B\sqrt{y}\,{}_1F_1\left(\frac{3}{4} - \frac{\varepsilon}{4}; \frac{3}{2}; y\right)$$

boundary condition has to be implemented

$$\boxed{{}_1F_1(a; c; y) \xrightarrow{y \to \infty} \frac{\Gamma(c)}{\Gamma(c-a)} (-y)^{-a} + \frac{\Gamma(c)}{\Gamma(a)} e^{y} y^{a-c}}$$

asymptotic formula

to be shown next time

→ it can not be that the series goes to infinity

→ series has to truncate

Case 1:

$$\frac{1}{4} - \frac{\varepsilon}{2} \overset{!}{=} -n, \ n \in \mathbb{N}_0 \Rightarrow \varepsilon_n = 2\left(n + \frac{1}{4}\right) \Rightarrow E_n = \hbar\omega\left(2n + \frac{1}{2}\right)$$

→ energy eigenvalues with even quantum numbers

$$\phi_n(y) = N_n\,{}_1F_1\left(-n; \frac{1}{2}; y\right), \quad \varphi_n(y) = e^{-\frac{y}{2}} \phi_n(y) = e^{-\frac{y}{2}}\,{}_1F_1\left(-n; \frac{1}{2}; y\right)$$

$$\psi_n(x) = \varphi_n\left(\frac{x^2}{\ell^2}\right) = N_n\, e^{-\frac{x^2}{2\ell^2}}\,{}_1F_1\left(-n; \frac{1}{2}; \frac{x^2}{\ell^2}\right)$$

Case 2:

$$\frac{3}{4} - \frac{\varepsilon}{4} \overset{!}{=} -n, \ n \in \mathbb{N}_0 \Rightarrow \varepsilon_n\ 2\left(n + \frac{3}{4}\right) \Rightarrow E_n = \hbar\omega\left(2n + 1 + \frac{1}{2}\right)$$

→ odd quantum numbers

$$\phi_n(y) = N_n\sqrt{y}\,{}_1F_1\left(-n; \frac{3}{2}; y\right) \Rightarrow \ldots \Rightarrow \psi_n(x) = N_n \frac{x}{\ell} e^{-\frac{x^2}{2\ell^2}}\,{}_1F_1\left(-n; \frac{3}{2}; \frac{x^2}{\ell^2}\right)$$

Summary: $E_n = \hbar\omega(n+\frac{1}{2})$; $n \in \mathbb{N}_0$

(8.953.1) $H_{2n}(\xi) = (-1)^n \frac{(2n)!}{n!} \, {}_1F_1(-n; \frac{1}{2}; \xi^2)$; $n \in \mathbb{N}_0$

(8.953.2) $H_{2n+1}(\xi) = (-1)^n \frac{(2n+1)!}{n!} \, \xi \, {}_1F_1(-n; \frac{3}{2}; \xi^2)$; $n \in \mathbb{N}_0$

$\Rightarrow \psi_n(x) = N_n \, e^{-\frac{x^2}{2\ell^2}} H_n(\frac{x}{\ell})$, $H_n(-x) = (-1)^n H_n(x)$

$N_n = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n! \, \ell}}$ normalization constant

$H_0(x) = 1$, $H_1(x) = \underline{2x}$, $H_2(x) = 4x^2 - 2$, ----

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \underbrace{e^{-x^2}}_{\text{generating function}}$