Summary:

Schrödinger equation for harmonic oscillator

$\downarrow$ separating asymptotic Gauß function

Kummer differential equation

$$y\frac{d^2\phi}{dy^2}+(c-y)\frac{d\phi}{dy}-a\phi=0 \quad \xleftarrow{z=\frac{y}{b},\ \lim_{b\to\infty}} \quad z(1-z)\frac{d^2\phi}{dz^2}+[c-(a+b+1)z]\frac{d\phi}{dz}-ab\phi=0$$

hypergeometric differential eq.

$$a=\frac{1}{4}-\frac{\varepsilon}{2},\quad c=\frac{1}{2}$$

$\downarrow$ $\downarrow$

confluent hypergeometric functions

$${}_1F_1(a;c;y)=\sum_{\nu=0}^{\infty}\frac{(a)_\nu}{(c)_\nu}\frac{y^\nu}{\nu!} \quad \xleftarrow{z=\frac{y}{b},\ \lim_{b\to\infty}} \quad {}_2F_1(a,b;c;z)=\sum_{\nu=0}^{\infty}\frac{(a)_\nu(b)_\nu}{(c)_\nu}\frac{z^\nu}{\nu!}$$

hypergeometric functions

Pochhammer symbol:

$$(a)_\nu=\frac{\Gamma(a+\nu)}{\Gamma(a)}=\begin{cases}1 & ;\nu=0\\ a(a+1)\cdot\ldots\cdot(a+\nu-1) & ;\nu\in\mathbb{N}\end{cases}$$

$\downarrow$ $\downarrow$

asymptotic behaviour:

convergence radius: $R \xrightarrow[\text{test}]{\text{ratio}} \infty$

$${}_1F_1(a;c;y)\longrightarrow\frac{\Gamma(c)}{\Gamma(c-a)}(-y)^{-a}+\frac{\Gamma(c)}{\Gamma(a)}y^{a-c}\boxed{e^y} \quad y\to\infty$$

$z=\frac{y}{b},\ \lim_{b\to\infty}$

origin for quantization condition $(a)_n \overset{!}{=} -n$

$n\in\mathbb{N}_0$ as otherwise Dirichlet boundary cond. not valid

asymptotic behaviour:

convergence radius: $R=1$

$z<1$ $\quad z>1$

$${}_2F_1(a,b;c;z)=\ldots\ {}_2F_1(\ldots;\ldots;\tfrac{1}{z})$$

$$\ldots\ldots$$

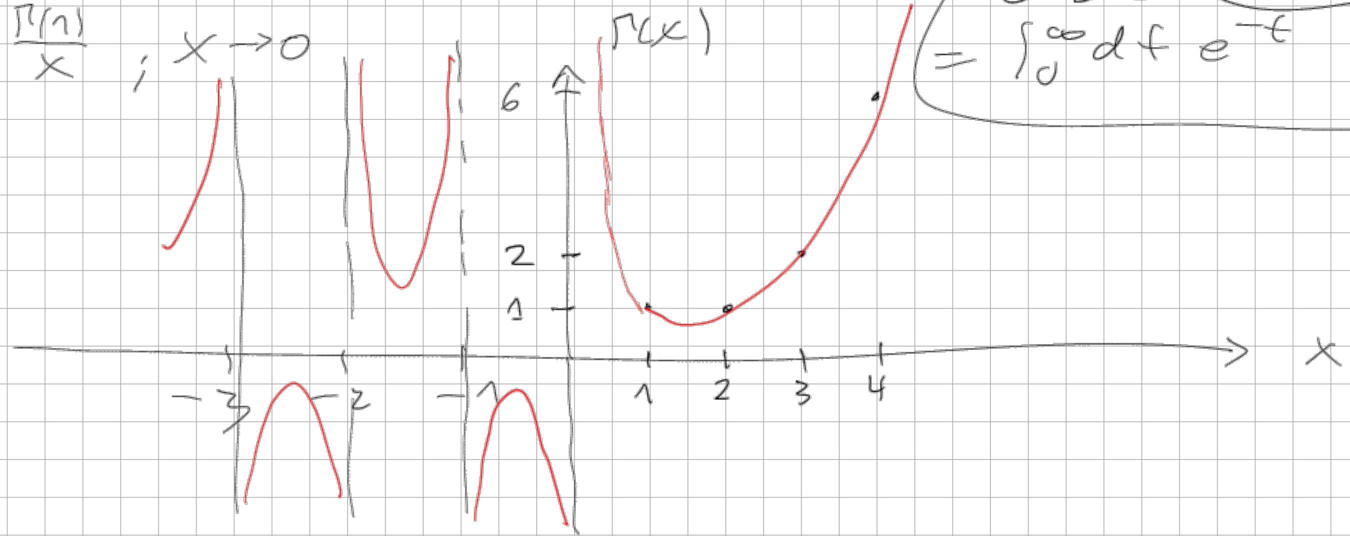
Gamma function: $\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}$; $x > 0$

$$\Gamma(x+1) = \int_0^\infty dt\, \underset{u}{t^x}\, \underset{v'}{e^{-t}} \quad \underline{\text{partial integration}} \quad \underbrace{\left[-\underset{u}{t^x}\,\underset{v}{e^{-t}}\right]_0^\infty}_{=0} - \int_0^\infty dt\, \underset{u'}{x t^{x-1}}\, \underset{v}{(-e^{-t})} = x \cdot \Gamma(x)$$

$$\Gamma(n+1) = n \cdot \Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n!\; \Gamma(1) = n!$$

$$\Gamma(1) = \int_0^\infty dt\, e^{-t} = 1$$

$$\underbrace{\boxed{\Gamma(x) = \frac{\Gamma(x+1)}{x}}}_{\text{functional identity}} \approx \frac{\Gamma(1)}{x} \; ; \; x \to 0$$



2. Time-Dependent Non-Degenerate Perturbation Theory

Problem:

$$\hat{H} = \underbrace{\hat{H}_0}_{\text{unperturbed Hamiltonian}} + \underbrace{\hat{V}}_{\text{perturbation}}$$

$\Rightarrow$ determine eigenstates and eigenvalues of $\hat{H}$

2.1 Schrödinger Equation:

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$$

assumption: $\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

orthonormality: $\langle \psi_n^{(0)} | \psi_m^{(0)} \rangle = \delta_{n,m}$

basis: completeness $\sum_n |\psi_n^{(0)}\rangle\langle\psi_n^{(0)}| = 1$

family of operators: $\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{V}$

$\lambda = 0$: $\hat{H}(\lambda=0) = \hat{H}_0$, $\lambda = 1$: $\hat{H}(\lambda=1) = \hat{H}$

λ: "artificial smallness parameter"

$$\hat{H}(\lambda)\,|\psi_n(\lambda)\rangle = E_n(\lambda)\,|\psi_n(\lambda)\rangle \qquad (1)$$

example for non-perturbative result in physics:

$$f(\lambda) = \underset{\neq 0}{e^{-\frac{1}{\lambda}}} \overset{\text{Taylor}}{=} 0 + 0\cdot\lambda + 0\cdot\lambda^2 + \dots$$

$$T_c^{BCS} \sim \frac{\hbar\omega_D}{k_B}\, e^{-\frac{1}{\underbrace{N(E_F)}_{\text{density of electrons at Fermi edge}}\, g}}$$

g ← electron-phonon coupling

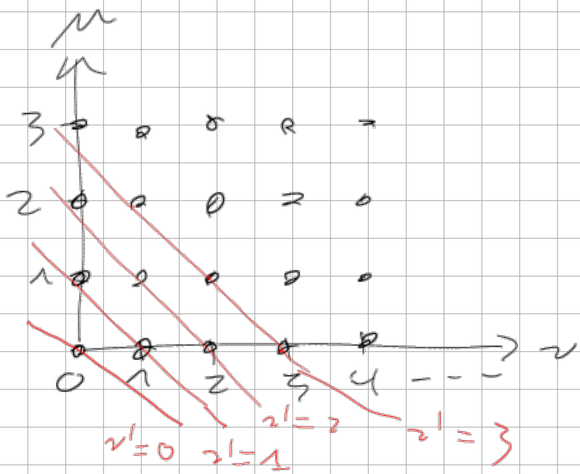
Assumption: Taylor expansion

$$E_n(\lambda) = \sum_{\nu=0}^{\infty} E_n^{(\nu)}\lambda^\nu = E_n^{(0)} + E_n^{(1)}\lambda + E_n^{(2)}\lambda^2 + \dots \qquad (2)$$

$$|\psi_n(\lambda)\rangle = \sum_{\nu=0}^{\infty} |\psi_n^{(\nu)}\rangle\lambda^\nu = |\psi_n^{(0)}\rangle + |\psi_n^{(1)}\rangle\lambda + |\psi_n^{(2)}\rangle\lambda^2 + \dots \qquad (3)$$

(2) and (3) in (1):

$$\hat{H}_0 \sum_{\nu=0}^{\infty} |\psi_n^{(\nu)}\rangle\lambda^\nu + \hat{V}\underbrace{\sum_{\nu=0}^{\infty} |\psi_n^{(\nu)}\rangle\lambda^{\nu+1}}_{\nu'=\nu+1\;\; = \sum_{\nu'=1}^{\infty} |\psi_n^{(\nu'-1)}\rangle\lambda^{\nu'}} = \underbrace{\left(\sum_{\nu=0}^{\infty} E_n^{(\nu)}\lambda^\nu\right)\cdot\left(\sum_{\mu=0}^{\infty} |\psi_n^{(\mu)}\rangle\lambda^\mu\right)} \qquad (4)$$



Cauchy product

$$= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} E_n^{(\nu)} |\psi_n^{(\mu)}\rangle \lambda^{\underbrace{\nu+\mu}_{=\nu'}} \Rightarrow \nu = \nu' - \mu$$

$$= \sum_{\nu'=0}^{\infty} \left(\sum_{\mu=0}^{\nu'} E_n^{(\nu'-\mu)} |\psi_n^{(\mu)}\rangle \right) \lambda^{\nu'}$$

(4) for $\nu = 0$: $\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

(4) for $\nu = 1, 2, 3, \dots$:

$$\hat{H}_0 |\psi_n^{(\nu)}\rangle + \hat{V} |\psi_n^{(\nu-1)}\rangle = \sum_{\mu=0}^{\nu} E_n^{(\nu-\mu)} |\psi_n^{(\mu)}\rangle$$

$\nu = 1$: $\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{V} |\psi_n^{(0)}\rangle = E_n^{(1)} |\psi_n^{(0)}\rangle + E_n^{(0)} |\psi_n^{(1)}\rangle$ (5)

$\nu = 2$: $\hat{H}_0 |\psi_n^{(2)}\rangle + \hat{V} |\psi_n^{(1)}\rangle = E_n^{(2)} |\psi_n^{(0)}\rangle + E_n^{(1)} |\psi_n^{(1)}\rangle + E_n^{(0)} |\psi_n^{(2)}\rangle$ (6)

In each order ν we can not determine $E_n^{(\nu)}$ and $|\psi_n^{(\nu)}\rangle$ completely.
We need something in addition!

2.2 Normalization:

$$\langle \psi_n | \psi_m \rangle = \delta_{n,m} \quad (7)$$

(3) in (7): $\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \langle \psi_n^{(\nu)} | \psi_m^{(\mu)} \rangle \lambda^{\underbrace{\nu+\mu}_{\nu'}} = \delta_{n,m}$

$\xrightarrow{\text{Cauchy product}}$ $\sum_{\nu'=0}^{\infty} \left(\sum_{\mu=0}^{\nu'} \langle \psi_n^{(\nu'-\mu)} | \psi_m^{(\mu)} \rangle \right) \lambda^{\nu'} = \underline{\delta_{n,m}}$

$\nu' = 0$: $\langle \psi_n^{(0)} | \psi_m^{(0)} \rangle = \delta_{n,m}$

$\nu' = 1, 2, 3, \ldots$: $\sum_{\mu=0}^{\nu'} \langle \psi_n^{(\nu'-\mu)} | \psi_m^{(\mu)} \rangle = 0$

$\nu' = 1$: $\langle \psi_n^{(0)} | \psi_m^{(1)} \rangle + \langle \psi_n^{(1)} | \psi_m^{(0)} \rangle = 0$ (8)

$\nu' = 2$: $\langle \psi_n^{(0)} | \psi_m^{(2)} \rangle + \langle \psi_n^{(1)} | \psi_m^{(1)} \rangle + \langle \psi_n^{(2)} | \psi_m^{(0)} \rangle = 0$ (9)

Solution ansatz: $|\psi_n^{(1)}\rangle = \sum_l C_{nl}^{(1)} \underbrace{|\psi_l^{(0)}\rangle}_{\hat{=}\ \text{basis}}$ (10)

$|\psi_n^{(2)}\rangle = \sum_l C_{nl}^{(2)} |\psi_l^{(0)}\rangle$ (11)

2.3 First order:

(10) in (5): $\underbrace{\hat{H}_0 \sum_l C_{nl}^{(1)} |\psi_l^{(0)}\rangle}_{\sum_l C_{nl}^{(1)} \underbrace{\hat{H}_0 |\psi_l^{(0)}\rangle}_{= E_l^{(0)} |\psi_l^{(0)}\rangle}} + \hat{V} |\psi_n^{(0)}\rangle = E_n^{(1)} |\psi_n^{(0)}\rangle + \sum_l C_{nl}^{(1)} E_n^{(0)} |\psi_l^{(0)}\rangle \Big| \langle \psi_m^{(0)} |$

$$\underbrace{\sum_l C_{nl}^{(1)} E_l^{(0)} \underbrace{\langle \psi_m^{(0)} | \psi_l^{(0)} \rangle}_{= \delta_{ml}}}_{C_{nm}^{(1)} E_m^{(0)}} + \underbrace{\langle \psi_m^{(0)} | \hat{V} | \psi_n^{(0)} \rangle}_{= V_{mn}} = E_n^{(1)} \underbrace{\langle \psi_m^{(0)} | \psi_n^{(0)} \rangle}_{= \delta_{mn}} + \underbrace{\sum_l C_{nl} E_n \underbrace{\langle \psi_m^{(0)} | \psi_l^{(0)} \rangle}_{= \delta_{ml}}}_{E_n^{(0)} C_{nm}^{(1)}}$$

$$V_{mn} = E_n^{(1)} \delta_{mn} + C_{nm}^{(1)} \left[E_n^{(0)} - E_m^{(0)} \right]$$

$m = n$: $E_n^{(1)} = V_{nn}$

$m \neq n: \quad c^{(1)}_{nm} = \frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}}$ ‖

Intermediat result:

$$|\psi_n^{(1)}\rangle = \sum_l c^{(1)}_{nl}\,|\psi_l^{(0)}\rangle = \underbrace{c^{(1)}_{nn}}\,|\psi_n^{(0)}\rangle + \sum_{l \neq n} \frac{V_{ln}}{E_n^{(0)} - E_l^{(0)}}\,|\psi_l^{(0)}\rangle$$

still undetermined from Schrödinger equation

first order of normalization: (10) in (8)

$$\sum_l \left\{ \langle \psi_n^{(0)} | \, c^{(1)}_{ml} \, |\psi_l^{(0)}\rangle + \langle c^{(1)}_{nl}\,\psi_l^{(0)} \, | \, \psi_m^{(0)}\rangle \right\} = 0$$

$$\sum_l \left(c^{(1)}_{ml}\,\delta_{nl} + c^{(1)*}_{nl}\,\delta_{lm} \right) = 0$$

$$c^{(1)}_{mn} + c^{(1)*}_{nm} = 0$$

scalar product: sesquilinear

$\langle \psi_1 | c\,\psi_2 \rangle = c\,\langle \psi_1 | \psi_2 \rangle$

$\langle c\,\psi_1 | \psi_2 \rangle = c^*\,\langle \psi_1 | \psi_2 \rangle$

$\underline{m \neq n:}$

$$\frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}} - \frac{V^*_{nm}}{E^{(0)}_{\cancel{m}\,n} - E^{(0)}_{\cancel{n}\,m}} = 0 \quad ✓$$

$$V^*_{nm} = \langle \psi_n^{(0)} | \hat{V} | \psi_m^{(0)} \rangle^* = \langle \psi_m^{(0)} | \hat{V} | \psi_n^{(0)} \rangle^* \overset{*}{=} V_{mn}$$

$m = n: \quad c^{(1)}_{nn} + c^{(1)*}_{nn} = 0 \Rightarrow c^{(1)}_{nn} = \underline{i\,\gamma_n^{(1)}}, \quad \gamma_n^{(1)} \in \mathbb{R}$

$$|\psi_n(\lambda)\rangle = \underbrace{|\psi_n^{(0)}\rangle + \lambda\, i\,\gamma_n^{(1)}\,|\psi_n^{(0)}\rangle}_{\text{up to first order in } \lambda:\ = e^{i\gamma_n^{(1)}\lambda}\,|\psi_n^{(0)}\rangle} + \lambda \sum_{l \neq n} \frac{V_{ln}}{E_n^{(0)} - E_l^{(0)}}\,|\psi_l^{(0)}\rangle + \dots$$

$$= \underbrace{e^{i\gamma_n^{(1)}\lambda}}_{\text{just a phase}} \left\{ |\psi_n^{(0)}\rangle + \lambda \sum_{l \neq n} \frac{V_{ln}}{E_n^{(0)} - E_l^{(0)}} |\psi_l^{(0)}\rangle + \cdots \right\}$$

without loss of generality: $\gamma_n^{(1)} = 0$