Quantum Mechanics II — Problem Sheet 1

Problem 1: Harmonic Oscillator (16 points)

The Hamilton operator of a one-dimensional harmonic oscillator with mass M and frequency ω reads

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{M}{2}\,\omega^2 \hat{x}^2 \,, \tag{1}$$

where one demands the following commutation relations between the coordinate operator $\hat{x}$ and the momentum operator $\hat{p}$:

$$\left[\hat{x}, \hat{x}\right]_- = \left[\hat{p}, \hat{p}\right]_- = 0 \,, \qquad \left[\hat{p}, \hat{x}\right]_- = \frac{\hbar}{i} \,. \tag{2}$$

The problem is now to solve the eigenvalue problem of the Hamilton operator

$$\hat{H}|\alpha\rangle = E_\alpha |\alpha\rangle \,, \tag{3}$$

i.e. to determine how the energy eigenvalues E_α and the energy eigenfunctions $|\alpha\rangle$ depend on the quantum number α. Usually this representation-free eigenvalue problem (3) is transformed into the coordinate representation, so it amounts to solve the corresponding Schrödinger equation by taking into account the appropriate Dirichlet boundary condition. In the following, however, we proceed differently by solving the representation-free eigenvalue problem (3) directly by taking into account the commutator relations (2). To this end the two hermitian operators $\hat{x}$ and $\hat{p}$ are transformed into two new operators $\hat{a}^\dagger$ and $\hat{a}$, which are adjoint with respect to each other:

$$\hat{a}^\dagger = \sqrt{\frac{M\omega}{2\hbar}} \left(\hat{x} - \frac{i}{M\omega}\,\hat{p}\right) , \qquad \hat{a} = \sqrt{\frac{M\omega}{2\hbar}} \left(\hat{x} + \frac{i}{M\omega}\,\hat{p}\right) . \tag{4}$$

a) Deduce from (2) the commutation relations between the new operators $\hat{a}^\dagger$ and $\hat{a}$:

$$\left[\hat{a}, \hat{a}\right]_- = ? \,, \qquad \left[\hat{a}^\dagger, \hat{a}^\dagger\right]_- = ? \,, \qquad \left[\hat{a}, \hat{a}^\dagger\right]_- = ? \,. \tag{5}$$

b) Express the Hamilton operator of the harmonic oscillator (1) in terms of the new operators $\hat{a}^\dagger$ and $\hat{a}$. Show that it reduces to the form

$$\hat{H} = \hbar\omega \left(\hat{n} + \frac{1}{2}\right) \tag{6}$$

with the operator $\hat{n} = \hat{a}^\dagger \hat{a}$.

c) Determine the commutation relations for the operator $\hat{n}$:

$$\left[\hat{n}, \hat{a}^\dagger\right]_- = ? \,, \qquad \left[\hat{n}, \hat{a}\right]_- = ? \,. \tag{7}$$

d) Consider the eigenvalue problem of the operator $\hat{n}$:

$$\hat{n}|\lambda\rangle = \lambda|\lambda\rangle \,. \tag{8}$$

Why are its eigenvalues λ real?

e) Deduce from the commutation relations (7) that applying the operators $\hat{a}^\dagger$ and $\hat{a}$ to the eigenfunctions $|\lambda\rangle$ yields

$$\hat{n}\hat{a}^\dagger|\lambda\rangle = (\lambda+1)\hat{a}^\dagger|\lambda\rangle \qquad \Longrightarrow \qquad \hat{a}^\dagger|\lambda\rangle \sim |\lambda+1\rangle\,, \tag{9}$$

$$\hat{n}\hat{a}|\lambda\rangle = (\lambda-1)\hat{a}|\lambda\rangle \qquad \Longrightarrow \qquad \hat{a}|\lambda\rangle \sim |\lambda-1\rangle\,. \tag{10}$$

Discuss the result (9) and (10). Why are $\hat{a}$ and $\hat{a}^\dagger$ called ladder operators?

f) Use the inequality

$$\langle \hat{a}\lambda|\hat{a}\lambda\rangle \geq 0 \tag{11}$$

in order to show that the eigenvalues λ of the operator $\hat{n}$ are always positive. **Note:** Assume without loss of generality that the eigenfunctions $|\lambda\rangle$ are normalized: $\langle\lambda|\lambda\rangle = 1$.

g) Conclude from (10) and (11) that the eigenvalues λ are given by positive integer numbers including zero:

$$\lambda = n = 0, 1, 2, \dots\,. \tag{12}$$

Which defining relation does this imply for the ground state $|0\rangle$?

h) Show that the energy eigenvalues of the harmonic oscillator are given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,. \tag{13}$$

What is the ground-state energy?

i) Deduce for the corresponding eigenfunctions $|n\rangle$ the properties

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{14}$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{15}$$

Note: Assume without loss of generality that the eigenfunctions $|n\rangle$ are normalized: $\langle n|n\rangle = 1$.

j) Show that the eigenfunctions $|n\rangle$ can be constructed from the ground state $|0\rangle$ as follows:

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle\,. \tag{16}$$

k) Derive that two different eigenfunctions $|n\rangle$ and $|n'\rangle$ are orthogonal:

$$\langle n|n'\rangle = 0\,, \qquad n \neq n'\,. \tag{17}$$

Problem 2: Anharmonic Oscillator (8 points)

Consider the one-dimensional anharmonic oscillator with the Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{M}{2}\omega^2\hat{x}^2 + g\hat{x}^4\,, \tag{18}$$

where the anharmonicity strength g is assumed to be the smallness parameter.

Evaluate all energy eigenvalues up to second order in g by taking into account the ladder operator formalism of **Problem 1**.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until November 6 at 11.45.