Quantum Mechanics II — Problem Sheet 10

Problem 27: Two Spin 1/2-Particles (5 points)

$\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ are the spin operators of two spin 1/2-particles, for instance the two electrons in the helium atom.

a) Find the mutual eigenstates $|s_1, s_2; s, m_s\rangle$ of the total spin operator $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$, its z-component $\hat{S}_z$ as well as $\hat{\mathbf{S}}_1^2$ and $\hat{\mathbf{S}}_2^2$.

b) Show that those states are also eigenstates of the operators $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ and determine the corresponding eigenvalues.

c) Show that the operator

$$\hat{P} = \frac{3}{4} + \frac{\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2}{\hbar^2} \tag{1}$$

represents a projection operator in the space of spin states. Onto which subspace does the operator $\hat{P}$ project?

Problem 28: Hamiltonian of Two Spin 1/2-Particles (3 points)

The Hamilton operator of two spin 1/2-particles is given by

$$\hat{H} = -J\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \mu \left(\hat{S}_{1z} + \hat{S}_{2z} \right) . \tag{2}$$

Calculate the eigenvalues and determine the eigenstates in the basis $\{|s_1, s_2; s, m_s\rangle\}$.

Problem 29: LS Coupling (6 points)

Calculate for the total angular momentum of the electron $\mathbf{J} = \mathbf{L} + \mathbf{S}$ with $s = 1/2$ and $l \geq 1$ the mutual eigenstates $|l, 1/2; j, m_j\rangle = |j, m_j\rangle$ of the operators $\hat{\mathbf{J}}^2, \hat{J}_z, \hat{\mathbf{L}}^2, \hat{\mathbf{S}}^2$ as linear combinations of the eigenstates $|l, 1/2; m_l, m_s\rangle = |l, m_l\rangle|1/2, m_s\rangle$ of the operators $\hat{\mathbf{L}}^2, \hat{L}_z, \hat{\mathbf{S}}^2, \hat{S}_z$. To this end proceed as follows:

a) Show that the quantum number j can only have the two values $l + 1/2$ and $l - 1/2$.

b) Verify for the eigenstates the following expressions:

$$\left| l \pm \frac{1}{2}, m_j \right\rangle = \sqrt{\frac{l \pm m_j + 1/2}{2l+1}}\, |l, m_j - 1/2\rangle |1/2, 1/2\rangle \pm \sqrt{\frac{l \mp m_j + 1/2}{2l+1}}\, |l, m_j + 1/2\rangle \, |1/2, -1/2\rangle \, . \tag{3}$$

Problem 30: Two Particles With Angular Quantum Number One (5 points)

Two angular momentum operators $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ couple to the total angular momentum operator $\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$. Calculate for the angular quantum numbers $j_1 = j_2 = 1$ all Clebsch-Gordan coefficients.

Problem 31: Two Particles With Spin 1/2 and 3/2 (5 points)

Consider two angular momentum operators $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ with the angular momentum quantum numbers $j_1 = 1/2$ and $j_2 = 3/2$.

a) Which quantum numbers j and m_j are possible for the square and the z-component of the total angular momentum operator $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$?

b) Determine for the maximal value of j and for all non-negative m_j all Clebsch-Gordan coefficients.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until January 29 at 11.45.