RPTU KAISERSLAUTERN-LANDAU | Winter Term 2023/2024
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Quantum Mechanics II — Problem Sheet 9

Problem 25: Selection Rules for Hydrogen Atom (12 points)

Let us consider the hydrogen atom ignoring the spins of the electron and of the proton. In order to have a non-zero transition probability from an energy state $|\psi_i\rangle$ to another one $|\psi_f\rangle$, the matrix element of the dipole moment operator $\mathbf{p}_{fi} = e\langle\psi_f|\hat{\mathbf{r}}|\psi_i\rangle$ must be non-zero. The conditions to have non-zero transition probabilities are called selection rules.

a) Which are good quantum numbers that characterize the state ψ of the hydrogen atom? To what do they physically correspond?

b) Which values can those quantum numbers have? What is the degeneracy of the state for a given energy level?

c) Recalling the definition of the angular momentum operator $\hat{\mathbf{L}}$, calculate the commutators

$$\left[\hat{L}_z, \hat{x}\right] = ?\,, \qquad \left[\hat{L}_z, \hat{y}\right] = ?\,, \qquad \left[\hat{L}_z, \hat{z}\right] = ?\,. \tag{1}$$

d) Using the results of **c)** and taking the expectation values of those between two different states $|\psi_i\rangle$ and $|\psi_f\rangle$, derive the selection rules to have $\mathbf{p}_{fi} = e\langle\psi_f|\hat{\mathbf{r}}|\psi_i\rangle \neq \mathbf{0}$.

e) These are not the only selection rules. Recall the definition of the square of the angular momentum operator $\hat{\mathbf{L}}^2$. What is the action of this operator on an eigenstate $|\psi\rangle$? Prove that

$$\left[\hat{\mathbf{L}}^2, \left[\hat{\mathbf{L}}^2, \hat{\mathbf{r}}\right]\right] = 2\hbar^2 \left(\hat{\mathbf{r}}\hat{\mathbf{L}}^2 + \hat{\mathbf{L}}^2\hat{\mathbf{r}}\right)\,. \tag{2}$$

f) Use the previous results to find other selection rules for the transition between $|\psi_i\rangle$ and $|\psi_f\rangle$.

g) What do these selection rules correspond physically to?

Problem 26: Oscillator Model of Angular Momentum (12 points)

Consider two simple harmonic oscillators, which we call the plus type and the minus type. Thus, we have for both oscillator types annihilation and creation operators, which are denoted by $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$, respectively. Furthermore, we define the number operators $\hat{n}_\pm = \hat{a}_\pm^\dagger \hat{a}_\pm$ and assume that the usual commutation relations hold for oscillators of the same type:

$$\left[\hat{a}_\pm, \hat{a}_\pm^\dagger\right] = 1\,, \qquad \left[\hat{n}_\pm, \hat{a}_\pm\right] = -\hat{a}_\pm\,, \qquad \left[\hat{n}_\pm, \hat{a}_\pm^\dagger\right] = \hat{a}_\pm^\dagger\,. \tag{3}$$

However, we treat both oscillators as uncoupled, so we demand that any pair of operators between different oscillators commute

$$\left[\hat{a}_+, \hat{a}_-^\dagger\right] = 0\,, \qquad \left[\hat{a}_-, \hat{a}_+^\dagger\right] = 0 \tag{4}$$

and so forth. Due to (4) the number operators $\hat{n}_+$ and $\hat{n}_-$ commute, so simultaneous eigenkets of $\hat{n}_+$ and $\hat{n}_-$ can be built up with eigenvalues n_+ and n_-, respectively, yielding the eigenvalue equations

$$\hat{n}_\pm|n_+, n_-\rangle = n_\pm|n_+, n_-\rangle\,. \tag{5}$$

a) How is the vacuum ket $|0,0\rangle$ defined? Determine the most general eigenket $|n_+, n_-\rangle$ by applying $\hat{a}_\pm^\dagger$ successively to this vacuum ket.

b) Prove that the operators

$$\hat{J}_\pm = \hbar \hat{a}_\pm^\dagger \hat{a}_\mp \,, \qquad \hat{J}_z = \frac{\hbar}{2}\left(\hat{n}_+ - \hat{n}_-\right) \tag{6}$$

satisfy the angular-momentum commutation relations of the usual form

$$\left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm, \qquad \left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z \,. \tag{7}$$

c) Determine the relation between the total number operator $\hat{n} = \hat{n}_+ + \hat{n}_-$ and the square of the angular momentum vector operator $\hat{\mathbf{J}}^2$.

d) Calculate how $\hat{J}_\pm$, $\hat{J}_z$, and $\hat{\mathbf{J}}^2$ act on the eigenket $|n_+, n_-\rangle$:

$$\hat{J}_\pm|n_+, n_-\rangle = ?\,, \qquad \hat{J}_z|n_+, n_-\rangle = ?\,, \qquad \hat{\mathbf{J}}^2|n_+, n_-\rangle = ?\,. \tag{8}$$

e) Compare (8) with the action of the operators $\hat{J}_\pm$, $\hat{J}_z$, and $\hat{\mathbf{J}}^2$ upon the states $|j,m\rangle$, which are characterized by the angular quantum number j and the magnetic quantum number m:

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m+1\rangle\,, \qquad \hat{J}_z|j,m\rangle = \hbar m|j,m\rangle\,, \qquad \hat{\mathbf{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle\,. \tag{9}$$

Which relation follows between the quantum numbers $n_\pm$ and j, m?

f) Determine the state $|j,m\rangle$ by applying $\hat{a}_\pm^\dagger$ successively to the vacuum ket $|j=0, m=0\rangle$.

g) Argue why all these results imply that any object of angular quantum number j and magnetic quantum number m can be visualized as being made up of $2j$ primitive spin 1/2 particles, $j+m$ of them with spin up and the remaining $j-m$ of them with spin down.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until January 22 at 11.45.