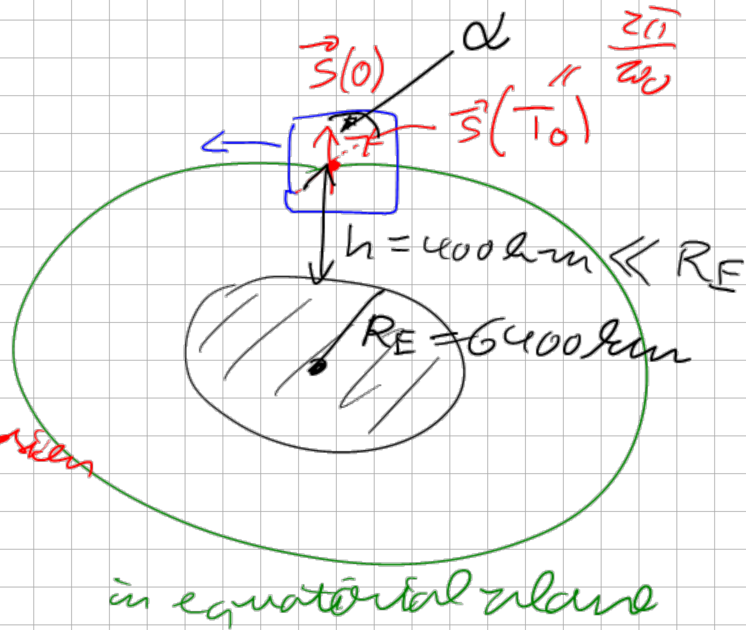
2.3 Geodetic Precession:

precession of spinning top freely falling in a gravitational field



center-of-mass affects spin precession

geodetic equations	spin precession
$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma_{\mu\nu}{}^{\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ (1)	$\frac{ds^\lambda}{d\tau} = -\Gamma_{\mu\nu}{}^{\lambda} s^\mu \frac{dx^\nu}{d\tau}$
$(x^0(\tau), r(\tau), \vartheta(\tau), \varphi(\tau)) = (u^0\tau, r, \frac{\pi}{2}, \omega_0\tau)$	$s^2(\tau) = 0 \Rightarrow$ spin vector in equatorial plane $\frac{d^2s^1}{d\tau^2} + \omega^2 s^1 = 0$, $\omega \simeq \omega_0\left[1 - \underbrace{(\gamma + \frac{1}{2})\frac{GM}{c^2r}}_{\text{leading difference is due to gravity}} + \dots\right]$
$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = c^2$ (2)	$g_{\mu\nu}s^\mu s^\nu = -s^2$

$\Rightarrow \omega < \omega_0$

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}s^\nu = 0$$

(1) $\left(\frac{u^0}{\omega_0}\right)^2 = \frac{2r}{B'(r)}$

(2) $B(r)(u^0)^2 - r^2(\omega_0)^2 = c^2$

$\Rightarrow \omega_0^2 \approx \frac{GM}{R_E^3}$ gravitational force = centrifugal force

$$(s^0(0), s^1(0), s^2(0), s^3(0)) = (0, s, 0, 0)$$

$$\downarrow$$

$$(s^0(\tau), s^1(\tau), s^2(\tau), s^3(\tau))$$

$$= \Big(-\frac{B'(r)a^0}{2B(r)\omega}s \sin\omega\tau, \underbrace{\boxed{s\cos\omega\tau, 0, -\frac{\omega_0}{2\omega}s\sin\omega\tau}}\Big)$$

$s^1 = s^r$ $s^2 = s^\vartheta$ $s^3 = s^\varphi$

$\hat{=}$ precession of spin in equatorial plane

2.3.4 Physical Interpretation:

Angle between spin vectors $(s^\lambda(0))$ and $(s^\lambda(T_0))$ in equatorial plane

$$-\cos\alpha = \frac{g_{ij}\, s^i(\tau=0)\, s^j(\tau=\frac{2\pi}{\omega_0})}{\sqrt{g_{ij}\, s^i(\tau=0)\, s^j(\tau=0)\ \ g_{kl}\, s^k(\frac{2\pi}{\omega_0})\, s^l(\frac{2\pi}{\omega_0})}} = -\cos\Big(2\pi \underbrace{\frac{\omega}{\omega_0}}\Big)$$

ratio of two angular frequencies

- $A(r) = B(r) = 1: \quad \omega = \omega_0 \quad \Rightarrow \quad \alpha = 2\pi$
- $A(r), B(r) \neq 1:$

$$\Delta\alpha = 2\pi\frac{\omega}{\omega_0} - 2\pi = -3\pi\frac{GM}{c^2 r}\frac{1+2\gamma}{3}$$

$\uparrow$ spin lacks behind

$$|\Delta\alpha| = 3\pi\frac{4.6\cdot 10^{-3}}{6.4\cdot 10^{6}}\frac{360\cdot 3600}{2\pi}'' = 1.4\cdot 10^{-3}\,''$$

one year observation:

$$\frac{\Delta t}{T_0} = \frac{365\cdot 24}{1.4} = 6257$$

$$|\Delta\alpha|_{\text{4oE}} = 6257 \cdot 1.4 \cdot 10^{-3}{}'' = 8.8''$$ (per year)

2.3.5 Gravity Probe B Experiment:

- satellite mission: test prediction of GRT
- NASA and Stanford University (Francis Everitt)
- costs: 1 billion dollar
- 4 spinning tops: gyroscopes
  - \> quartz sphere of size of a table tennis ball (3.8 cm)
  - \> in vacuum: 10.000 rotations per minute
  - \> cooled down to 1.8 K: surface made of Niob, is superconducting
  - \> changes of rotation axis measurable via SQUIDS
  - \> accuracy: they claim to measure changes of $1/40.000.000^\circ = 0.0324''$

    comparison: resolve a pinhead at a distance of 1000 km

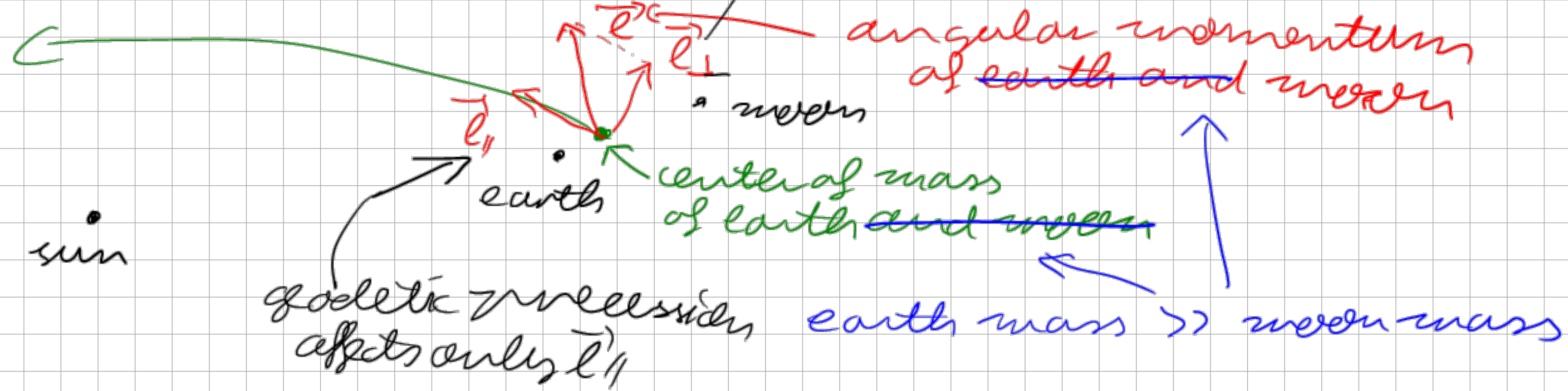
cooling with ^4He (theoretical limit of cooling is 0.7 K)

- mission beginning: 2004
- April 14, 2007: Spring Conference APS

  → geodetic effect confirmed with a precision of better than 1%

2.3.6 De Sitter Precession of Moon:

$\hat{=}$ s^2 does not change



moon motion plane is rotated only 5° with respect to earth motion plane:
$\vec{l}_\perp$ large, $\vec{l}_\parallel$ small

=> geodetic precession yields a small rotation of moon motion plane with respect to earth motion plane

$$|\Delta\alpha| = 3\pi \frac{1.5}{1.5\cdot 10^8}\cdot\frac{360\cdot 3600''}{2\pi} = 0.02''$$

per century: $2''$ (prediction from de Sitter, 1916)

All other planets in our solar system also lead to geodetic precession of earth-moon system: 10^7 larger

only possible provided that moon orbit is precisely known

1969: Apollo mission, mirrors installed, distance between earth and moon detectable

=> precession of moon-motion was checked at precision of 0.7%.

2.4 Lense-Thirring Effect:

Precession of spinning top due to the rotation of a gravitational field

$\Rightarrow$ quite small: magnitude 1/100 times smaller than geodetic precession for experimental detection

2.4.1 Kerr Metric:

static gravitational field of mass M and angular momentum L

$$ds^2 = \left(1 - \frac{2ar}{\rho^2}\right)c^2 dt^2 + \boxed{4ab\,\frac{r\sin^2\vartheta}{\rho^2}\,c\,dt\,d\varphi}$$

Note: rotation axis is z-axis

$$-\frac{\rho^2}{\Delta}\,dr^2 - \rho^2 d\vartheta^2 - \sin^2\vartheta\left(r^2 + \cancel{b^2} + \cancel{\frac{2ar}{\rho^2}\,b^2\sin^2\vartheta}\right)d\varphi^2$$

$$\rho^2 = r^2 + \cancel{b^2\cos^2\vartheta}\,,\quad \Delta = r^2 - 2ar + \cancel{b^2}$$

$$a = \frac{GM}{c^2}\,,\qquad b = \frac{L}{cM}$$

We are looking the leading correction to Schwarzschild metric

$$ds^2 = \underbrace{ds^2_{SS}}_{\text{leading term}} + \underbrace{\boxed{4ab\,\frac{\sin^2\vartheta}{r^2}(c\,dt)(r\,d\varphi)}}_{\text{subleading term: first correction due to } L} + \mathcal{O}(b^2)$$

What is the order of magnitude of that correction?

$$L \sim I\Omega \overset{I\sim MR^2}{\sim} MR^2\Omega \overset{V=R\Omega}{\sim} MRV$$

$$\boxed{} \sim \underbrace{\frac{GM}{c^2}}_{a} \cdot \frac{1}{c\cancel{M}} \cdot \cancel{MR}V \frac{1}{R^{\cancel{2}}} = \underbrace{\frac{a}{R}}_{\text{mass effect}} \underbrace{\frac{V}{c}}_{\text{angular momentum effect}}$$

2.4.2 Cartesian Coordinates:

$$\left.\begin{aligned} x &= r\sin\vartheta \underline{\cos\varphi} \\ y &= r\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\} \quad \tan\varphi = \frac{y}{x} \quad \Rightarrow d\varphi = \underline{\cos^2\varphi}\left(\frac{dy}{x} - \frac{y}{x^2}dx\right)$$

$$ds^2 = ds_{SS}^2 + 4\cdot ab \frac{\cancel{\sin^2\vartheta}}{r^2} \frac{x^2}{r^2\cancel{\sin^2\vartheta}} r\left(\frac{dy}{x} - \frac{y}{x^2}dx\right)(c\,dt)$$

$$= ds_{SS}^2 + 4\frac{G\cancel{M}}{c^2}\frac{L}{c\cancel{M}} \cdot \frac{1}{r^3}(x\,dy - y\,dx)\,c\,dt + \dots$$

$$L = I\Omega, \quad I = \frac{2}{5}MR_E^2$$

$$ds^2 = ds_{SS}^2 + 4\frac{G}{c^3}\frac{2}{5}MR_E^2 \frac{\boxed{x\,dy - y\,dx}}{r^3}\Omega c\,dt + \dots$$

$$\vec{\Omega} = \Omega\vec{e}_z, \quad \vec{\Omega}\times\vec{r} = \Omega\begin{pmatrix}-y\\x\\0\end{pmatrix}$$

$$(\vec{\Omega}\times\vec{r})\cdot d\vec{r} = -\Omega\boxed{(y\,dx - x\,dy)}$$

$$ds^2 = ds^2_{SS} \underbrace{- \frac{8GMR_E^2}{5c^3} \frac{\vec{\Omega} \times \vec{r}}{r^3} \cdot d\vec{r}\, c\,dt} + \dots$$

Here this was determined from Kerr metric by perturbation theory.
Last exercise sheet in last semester: This was obtained by perturbatively solving Einstein field equations.

2.4.3 Christoffel symbols:

geodetic precession ✓

Lense-Thirring effect

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \cancel{\text{connection in a}} + 2h_{0i}\, dx^i\, c\,dt + \dots$$

$$\vec{h} = (h_{0i}),\quad h_{0i} = -\frac{4GMR_E^2}{5c^3}\varepsilon_{imn}\frac{\Omega_m x_n}{r^3}$$

$$(g_{\mu\nu}) = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} 1 & h_{0j} \\ h_{0i} & -\delta_{ij} \end{pmatrix}$$

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu})$$

all derivatives with respect to time vanish as we deal with a stationary gravitational field

$$\Gamma_{0ij} = \Gamma_{i0j} = \frac{1}{2}(\partial_i h_{0j} - \partial_j h_{0i}) + \dots$$

$$\Gamma_{ij0} = \Gamma_{ji0} = \frac{1}{2}(\partial_i h_{0j} + \partial_j h_{0i}) + \dots$$

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & h_{0j} \\ h_{0i} & -\delta_{ij} \end{pmatrix} + \dots$$

$$\Gamma_{\lambda\mu}{}^{\kappa} = g^{\kappa\nu}\,\Gamma_{\lambda\mu\nu}$$

$$\Gamma_{i0}{}^{j} = \Gamma_{0i}{}^{j} = -\Gamma_{0ij} + \dots = \frac{1}{2}\left(\partial_j h_{0i} - \partial_i h_{0j}\right) + \dots$$

$$\Gamma_{ij}{}^{0} = \Gamma_{ji}{}^{0} = \Gamma_{ij0} + \dots = \frac{1}{2}\left(\partial_i h_{0j} + \partial_j h_{0i}\right) + \dots$$

2.4.4 Spin Precession:

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma_{\mu\nu}{}^{\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \qquad \frac{ds^\lambda}{d\tau} = -\underbrace{\Gamma_{\mu\nu}{}^{\lambda}}_{\text{first order in } h \,\sim\, \Omega \,\Rightarrow\, \text{small}} s^\mu \underbrace{\frac{dx^\nu}{d\tau}}_{\text{center-of-mass motion affects the spin-motion}}$$

center of mass motion enters only in zeroth order

$$\Rightarrow (x^\lambda) = (c\tau, 0, 0, 0), \quad (u^\lambda) = \left(\frac{dx^\lambda}{d\tau}\right) = (c, 0, 0, 0)$$

$$\left.\begin{aligned}
\frac{d^2x^0}{d\tau^2} &= -\Gamma_{\mu\nu}{}^{0}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma_{ij}{}^{0}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \quad \checkmark \\
\frac{d^2x^i}{d\tau^2} &= -\Gamma_{\mu\nu}{}^{i}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\left(\Gamma_{j0}{}^{i} + \Gamma_{0j}{}^{i}\right)\frac{dx^j}{d\tau}\frac{dx^0}{d\tau} \quad \checkmark
\end{aligned}\right\} \tau = t$$

TODO: solve spin equation $\Rightarrow$ precession of spin