2.4 Lense-Thirring Effect:

Rotating massive object: mass M and angular momentum L → Kerr metric

$$ds^2 = \eta_{\mu\nu}\, dx^\mu x^\nu + \cancel{\text{Schwarzschild metric due to mass } M} + \underbrace{2 h_{0i}\, dx^i\, c\, dt}_{\sim \text{ mass } M \text{ and angular momentum } L} + \ldots$$

geodetic precession: consequence of Schwarzschild metric on spin

now: effect on spin precession?

$$\vec{h} = (h_{0i}), \quad h_{0i} = -\frac{4GMR_E^2}{5c^3}\, \varepsilon_{imn}\, \frac{\Omega_m x_n}{r^3}$$

($\vec{\Omega}$: angular frequency vector of Earth rotation)

center-of-mass motion

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma_{\mu\nu}^{\lambda}\, \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

spin motion

$$\frac{ds^\lambda}{d\tau} = -\Gamma_{\mu\nu}^{\lambda}\, s^\mu \frac{dx^\nu}{d\tau}$$

of first order in h

$$(x^\lambda) = (c\tau, 0, 0, 0)$$

$$\cancel{\frac{d^2x^0}{d\tau^2}} = -\Gamma_{ij}^{0}\, \cancel{\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \quad ✓$$

$$\cancel{\frac{d^2x^i}{d\tau^2}} = -2\Gamma_{0j}^{i}\, \cancel{\frac{dx^j}{d\tau}\frac{dx^0}{d\tau}} \quad ✓$$

$$(s^\lambda) = (0, s^i)$$

$$\cancel{\frac{ds^0}{d\tau}} = -\Gamma_{ij}^{0}\, \cancel{s^i u^j} \quad ✓$$

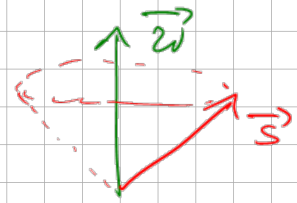
$$\frac{ds^i}{d\tau} = -\underbrace{\Gamma_{j0}^{i}}_{\text{see last time}} s^j \underbrace{u^0}_{=c} - \cancel{\Gamma_{0j}^{i} s^0 \underbrace{u^j}_{=\frac{dx^j}{d\tau}}}$$

$$\frac{ds^i}{d\tau} = -\frac{c}{2}\left(\partial_j h_{0i} - \partial_i h_{0j}\right) s^j$$

→ equation of motion for spin

→ spin precession!

$$\hookrightarrow \frac{d\vec{S}}{d\tau} = \vec{\omega} \times \vec{S}$$ spin precession



$$\vec{\omega} = -\frac{c}{2} \operatorname{rot} \vec{h}$$

$$\frac{dS^i}{d\tau} = \varepsilon_{ijk}\, \omega_j\, S^k = -\frac{c}{2} \underbrace{\varepsilon_{ijk}\, \varepsilon_{jmn}}_{} \partial_m h_{0n}\, S^k$$

$$= \varepsilon_{kij}\, \varepsilon_{mnj} = \delta_{km}\delta_{in} - \delta_{kn}\delta_{im}$$

$$= -\frac{c}{2}\left(\partial_k h_{0i} - \partial_i h_{0k}\right) S^k \quad \checkmark$$

2.4.5 Precession Frequency:

result of straight-forward calculation:

$$\vec{\omega}(\vec{r}) = \frac{2GMR_E^2}{5c^2} \frac{3(\vec{\Omega}\cdot\vec{r})\,\vec{r} - \vec{\Omega}\, r^2}{r^5}$$

angular frequency vector for spin precession

angular frequency vector of earth rotation

1) estimate for order of magnitude:

$$\omega \sim \frac{GMR_E^2}{c^2} \frac{\Omega}{R_E^3} \qquad \underline{\underline{g = GM/R_E^2}} \qquad \frac{gR_E}{c^2}\Omega = 7\cdot 10^{-10}\,\Omega$$

→ small effect

2) $\vec{\omega} = \vec{\omega}(\vec{r})$: depends on position on earth

$$\vec{\omega} = \frac{2GM\Omega}{5c^2 R_E} \cdot \begin{cases} 2\,\vec{e}_z\,; & \vec{r} = R_E\,\vec{e}_z \quad \text{(north pole)} \hat{=} \text{ parallel to earth rotation} \\ -\vec{e}_z\,; & \vec{r} = R\,\vec{e}_x \quad \text{(equator)} \hat{=} \text{ opposite to earth rotation} \end{cases}$$

$|\vec{r}| = R_E$

north pole: $\Delta\varphi = \omega \cdot 1a = \frac{4}{5}\, 7\cdot 10^{-10}\, \frac{2\pi}{1d}\, 365d\, \frac{360\cdot 3600''}{2\pi} = 0.26''$

$\uparrow$ angle change during one year

$\approx \frac{1}{34}$ geodetic precession

$\rightarrow$ predicted by Lense and Thirring in 1918

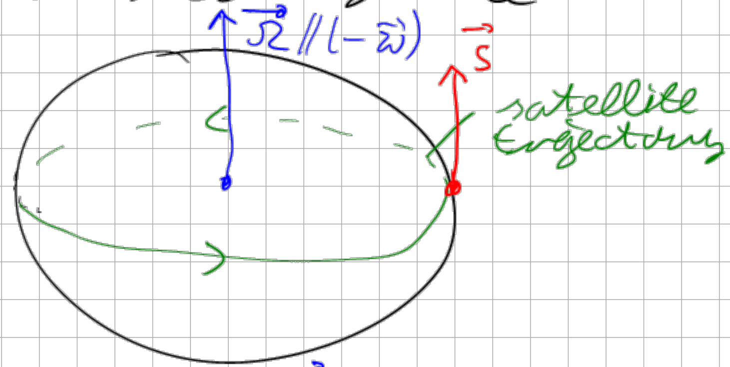
2.4.6 Gravity B Experiment:

- How to detect an effect, which is smaller than geodetic precession?
- Idea: choose a satellite trajectory, where geodetic precession vanishes.
- Note: Geodetic precession affects only spin component which is lying within the trajectory plane of satellite

  $\rightarrow$ Choose spin component perpendicular to satellite plane!

- 1. Case: equatorial plane

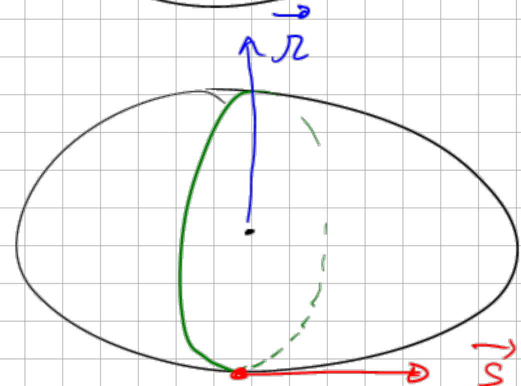
$$\frac{d\vec{S}}{d\tau} = \vec{\omega} \times \vec{S} \Rightarrow \text{no Lense-Thirring effect}$$



- 2 Case: trajectory plane with north/south pole

$$\vec{r}(t) = R_E \begin{pmatrix} \cos\omega_0 t \\ 0 \\ \sin\omega_0 t \end{pmatrix}$$

$$\vec{\omega}(t) = \frac{2}{5} \frac{GM}{c^2 R_E} \Omega \begin{pmatrix} 3\sin\omega_0 t \cos\omega_0 t \\ 0 \\ 3\sin^2\omega_0 t - 1 \end{pmatrix}$$



Due to $|\vec{\omega}(t)| \ll \omega_0$:

$$\langle \vec{\omega}(t) \rangle = \frac{\omega_0}{2\pi} \int_0^{\frac{2\pi}{\omega_0}} dt\, \vec{\omega}(t) = \frac{1}{5} \frac{GM}{c^2 R_E} \Omega \vec{e}_z$$

→ time averaging reduces effect by 1/4

→ $\frac{1}{34} \cdot \frac{1}{4} \approx \frac{1}{136}$ smaller than geodetic precession

Publication of gravity probe B results in autumn 2009: Lense-Thirring effect

3 Gravitational Waves and Their Detection:

- directly after the formulation of general relativity in 1915 Albert Einstein predicted the existence of gravitational waves. To this end ~~field~~ field equations of general relativity had to be linearised.
- By working out the implications a lot of similarities between gravitational waves and electromagnetic waves occur.

Electromagnetism (James Clark Maxwell)	Gravity (Albert Einstein)
linear theory for vector potential A^μ	nonlinear theory for metric $g_{\mu\nu}$, linearization is needed: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\lvert h_{\mu\nu} \rvert \ll 1$
inhomogeneous wave equation for A^μ	inhomogeneous equation for $h_{\mu\nu}$
homogeneous solution describes propagation in vacuum with light velocity	
charge is accelerated	mass is accelerated

particular solution via Green function of wave equation

accelarted charge generates electromagnetic wave	accelerated mass generates gravitational wave
application: sending and receive electromagnetic waves → radio (11.11.1886, Heinrich Hertz, Karlsruhe)	application: detect gravitational waves → new window for astronomy

combining detection of electromagnetic and gravitational waves from astrophysical sources: multi-messenger astronomy

Plan for chapter:

- Gravitational Plane Waves
- Particles in Gravitational Wave
- Quadrupole Radiation
- Sources of Gravitational Waves
- Indirect Detection of Gravitational Waves
  Nobel Prize 1993: Joseph Taylor, Russel Hulse (binary system of neutron stars)
- Direct Detection of Gravitational Waves
  Nobel Prize 2017: Rainer Weiss, Barry Barish, Kip Thorne (LIGO detectors)

3.1 Gravitational Plane Waves:

- Linearized theory of gravity: coupled equation for $h_{\mu\nu}$
- Hilbert gauge allows to decouple equations for $h_{\mu\nu}$
  → inhomogeneous wave equations
- Homogeneous solutions: polarization of plane wave ?
- Only two degrees of freedom for polarization remain $\hat{=}$ spin $S = 2$ of "graviton"

3.1.1 Linearization of Einstein Field Equations: