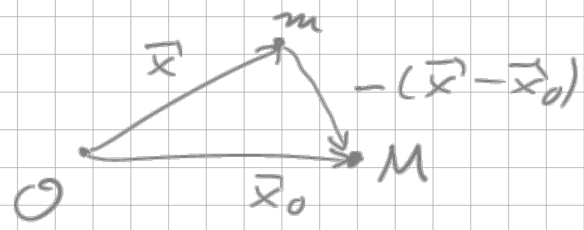
1. Summary of General Relativity I

The general relativity of Albert Einstein is up to now the established theory of gravity.

1.1 Gravity Theory Before General Relativity:

- First gravity theory stems from Isaac Newton
  - published in 1687 in book "Philosophiae Naturalis Principia Mathematica" (Mathematical Principles of Natural Philosophy)
  - Based on concept of absolute space and time
- Newton gravitational potential:

$$\Phi(\vec{x}) = -G\,\frac{M}{|\vec{x}-\vec{x}_0|} \quad \xleftarrow{} \text{origin of gravity (}M\text{)}$$



$$\vec{F}(\vec{x}) = -\underbrace{m}_{\text{test mass}}\,\mathrm{grad}\,\Phi(\vec{x}) = -\frac{GMm}{|\vec{x}-\vec{x}_0|^2}\,\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|}$$

$G = (6.673 \pm 0.010)\, 10^{-11}\ m^3/kg\, s^2$ (Newton gravitational constant)

the most unprecisely known natural constant

- Advantage: explains all phenomena of celestial mechanics in our solar system
- Disadvantage: only describes gravitostatics. It claims that changing $\vec{x}_0$ temporally has an effect instantaneously.
- Only Albert Einstein circumvented this disadvantage by formulating a

self-consistent theory of gravity. Only possible due to changing our understanding of space and time, i. e. the arena in which any physical process occurs

1.2 Relativity Theories of Albert Einstein:

- "Relativity" $\hat{=}$ observer dependence for measurements
  this sounds obvious, but has turned out to have fundamental consequences of our description of nature
- Special relativity (1905): theory (of space and time) without gravity
  - space-time is flat
  - characterized by Minkowski metric

laboratory time

$$\underbrace{c^2 d\tau^2}_{\substack{\text{proper time}\\ =\text{time of moving observer}}} = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta \quad , \quad (\xi^\alpha) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}, \quad (\eta_{\alpha\beta}) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

  - Newton mechanics is generalised to Einstein mechanics
  - Formal procedure:

$$\begin{matrix}\text{physical theory} \\ \text{in rest frame}\end{matrix} \xrightarrow[\text{Lorentz transformation}]{\text{boost with}} \begin{matrix}\text{physical theory} \\ \text{in laboratory frame}\end{matrix}$$

- General Relativity (1915): theory with gravity
  - space-time is curved (Riemann differential geometry)
  - curvature of space-time = gravity

$$c^2\, d\tau^2 = \underbrace{g_{\mu\nu}}\, dx^\mu\, dx^\nu$$

metric of space-time = Einstein's gravity potentials generalizing the Newton gravity potential Φ (10 degrees of freedom)

1.3 General Relativity:

- Equivalence principle: inertial mass = gravity mass
- Einstein elevator: gravity can be (locally) transformed away by a suitable coordinate transformation
- Formal procedure:

physical theory in a local co-moving frame $\xrightarrow[\text{transformation}]{\text{general coordinate}}$ physical theory in a local frame with gravity

- Example: motion of a point particle

$$\frac{d^2 \xi^\alpha}{d\tau^2} = 0 \longrightarrow \frac{d^2 x^\lambda}{d\tau^2} = - \underbrace{\Gamma_{\mu\nu}}{}^{\lambda} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad \text{(geodesic equation)}$$

Christoffel symbols = first derivatives of the metric = gravitational forces

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right)$$

- Gravitational potentials $g_{\mu\nu}$ are determined by Einstein field equations = second order nonlinear partial differential equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}}_{\text{curvature of space-time}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\substack{\text{symmetric energy momentum tensor}\\ \text{of all fields apart from gravity}}}$$

curvature scalar: $R = R_\mu{}^\mu$

Ricci tensor: $R_{\mu\nu} = R_{\lambda\mu\nu}{}^\lambda$

curvature tensor $R_{\lambda\mu\nu}{}^{\varkappa} = \partial_\lambda \Gamma_{\mu\nu}{}^{\varkappa} - \partial_\mu \Gamma_{\lambda\nu}{}^{\varkappa} + \Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\lambda\sigma}{}^{\varkappa} - \Gamma_{\lambda\nu}{}^{\sigma}\Gamma_{\mu\sigma}{}^{\varkappa}$

1.4 Schwarzschild metric:

- Solving Einstein field equations for statitic, isotropic space time:

$$c^2 d\tau^2 = B(r)\, c^2 dt^2 - A(r)\, dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

Newton

Robertson expansion: $B(r) = 1 - 2\frac{GM}{c^2 r} + 2(\beta-\gamma)\left(\frac{GM}{c^2 r}\right)^2 + \ldots$

$$A(r) = 1 + 2\gamma \frac{GM}{c^2 r} + \ldots$$

Newton: $\beta = \gamma = 0$, Einstein: $\beta = \gamma = 1$

- Schwarzschild solution (1916):

Schwarzschild radius

$$B(r) = 1 - \frac{2a}{r}, \quad A(r) = \frac{1}{1 - \frac{2a}{r}}, \quad a = \frac{GM}{c^2} = \frac{r_s}{2}, \quad r_{s,\odot} = \frac{2\,GM_\odot}{c^2} = 3\,\text{km}$$

Newton

$M_\odot = 2 \cdot 10^{30}\,\text{kg}$

A black hole is a massive object whose Schwarzschild radius is larger than its radius. The gravity field is then so large that neither particles nor electromagnetic waves can leave a black hole.

- Geodesic equation for Schwarzschild metric:

- 4 ordinary (nonlinear) differential equations: $t(\lambda)$, $r(\lambda)$, $\vartheta(\lambda)$, $\varphi(\lambda)$

massive particle $m > 0$: $\lambda = \tau$ proper time

massless particle $m = 0$: λ is some trajectory parameter

- integrable: 4 integrals of motion

$$g_{\mu\nu}(x(\lambda)) \frac{dx^{\mu}(\lambda)}{d\lambda} \frac{dx^{\nu}(\lambda)}{d\lambda} = \begin{cases} c^2 & ; m>0 \\ 0 & ; m=0 \end{cases}$$

energy, square of angular momentum vector, z-component of angular momentum vector

- Trajectory: eliminate trajectory parameter λ

massless particle ($m=0$)	massive particle ($m>0$)
$\frac{d\varphi}{dr} = \frac{\sqrt{A(r)}}{r^2} \frac{1}{\sqrt{\frac{F^2}{\tilde{\ell}^2}\frac{1}{B(r)} - \frac{1}{r^2}}}$ (*)	$\frac{d\varphi}{dr} = \frac{\sqrt{A(r)}}{r^2} \frac{1}{\sqrt{\frac{F^2}{\tilde{\ell}^2}\frac{1}{B(r)} - \frac{c^2}{\tilde{\ell}^2} - \frac{1}{r^2}}}$
$\tilde{\ell} = \lim_{m\to 0} \frac{\ell}{m}$, F: some integral of motion	$\tilde{\ell} = \frac{\ell}{m}$
$\Rightarrow$ light deflection	$\Rightarrow$ perihelion rotation

2. Experimental Test:

- 2 tests for geodesic equation: light deflection, perihelion rotation
- 2 tests for spinning particles: geodesic precession, Lense-Thirring precession

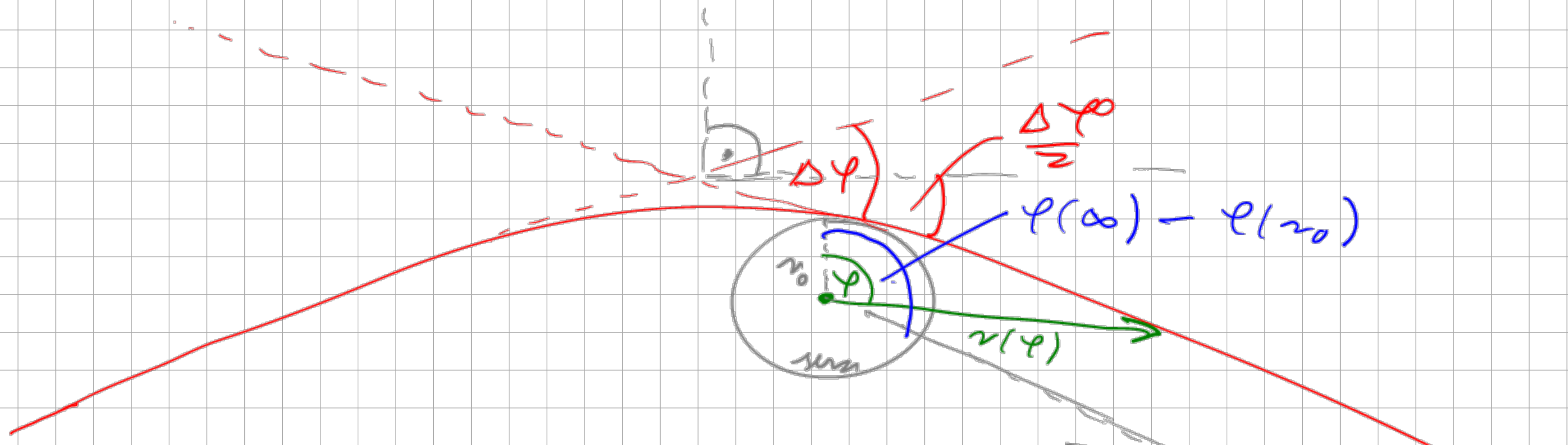
2.1 Light Deflection:

Light ray is deflected in presence of a gravitating object. Here focus on the effect of ~~the~~ sun as light deflection was first observed during sun eclipse in 1919 (Sir Eddington). More precisely observable is the deflection of microwave radiation from quasars.

2.1.1 Calculation of light deflection:

Starting point: (*)

$$\varphi(\infty) - \underbrace{\varphi(r_0)}_{\text{radius of sun}} \overset{(*)}{=} \int_{r_0}^{\infty} dr \, \frac{\sqrt{A(r)}}{r^2} \, \frac{1}{\sqrt{\frac{F^2}{\ell^2}\frac{1}{B(r)} - \frac{1}{r^2}}}$$



$$\frac{\Delta\varphi}{2} + \frac{\pi}{2} = \varphi(\infty) - \varphi(r_0) \implies \Delta\varphi = 2\left[\varphi(\infty) - \varphi(r_0)\right] - \pi$$

$$\Delta\varphi = 2\int_{r_0}^{\infty} dr \, \frac{\sqrt{A(r)}}{r^2\sqrt{\frac{F^2}{\ell^2}\frac{1}{B(r)} - \frac{1}{r^2}}} - \pi$$

1. Idea: eliminate integrals of motion by geometric quantities

$$\left.\frac{d\varphi}{dr}\right|_{r_0} \underset{\uparrow}{=} \infty \implies \frac{F^2}{\ell^2}\frac{1}{B(r_0)} = \frac{1}{r_0^2} \implies \frac{F^2}{\ell^2} = \frac{B(r_0)}{r_0^2}$$

sun radius r_0
is minimal distance

$$\Delta\varphi = 2\int_{r_0}^{\infty} dr \, \frac{\sqrt{A(r)}}{r} \, \frac{1}{\sqrt{\frac{r^2 B(r_0)}{r_0^2 B(r)} - 1}} - \pi \qquad (1)$$

2. Idea: gravity of sun is weak, so the Schwarzschild is not needed, Eddington expansion is sufficient

$$A(r) = 1 + \gamma \frac{2a}{r} + \dots, \quad B(r) = 1 - \frac{2a}{r} + \dots \quad (2)$$

"a is small" in the sense of $a/r_0 \ll 1$

(2) in (1), expanded in first order of a:

$$\Delta\varphi = 2\int_{r_0}^{\infty} dr \left\{ \frac{r_0}{r} \frac{1}{\sqrt{r^2 - r_0^2}} + \frac{\gamma a r_0}{r^2\sqrt{r^2 - r_0^2}} + \frac{a}{(r+r_0)\sqrt{r^2 - r_0^2}} \right\} - \pi$$

$$= 2\left[\arccos\frac{r_0}{r} + \frac{\gamma a}{r_0} \frac{\sqrt{r^2 - r_0^2}}{r} + \frac{a}{r_0}\sqrt{\frac{r-r_0}{r+r_0}} \right]_{r_0}^{\infty} - \pi$$