2. Experimental Tests of General Relativity

static and isotropic gravitational field (metric)

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

Newton

$$\underbrace{B(r) = 1 - \frac{2a}{r}}_{} = \underbrace{1 - 2\frac{a}{r} + \underline{2(\beta-\gamma)\left(\frac{a}{r}\right)^2} + \dots}_{}$$

$$A(r) = \frac{1}{1-\frac{2a}{r}} = 1 + \underline{2\gamma\frac{a}{r}} + \dots$$

deviation from Newton

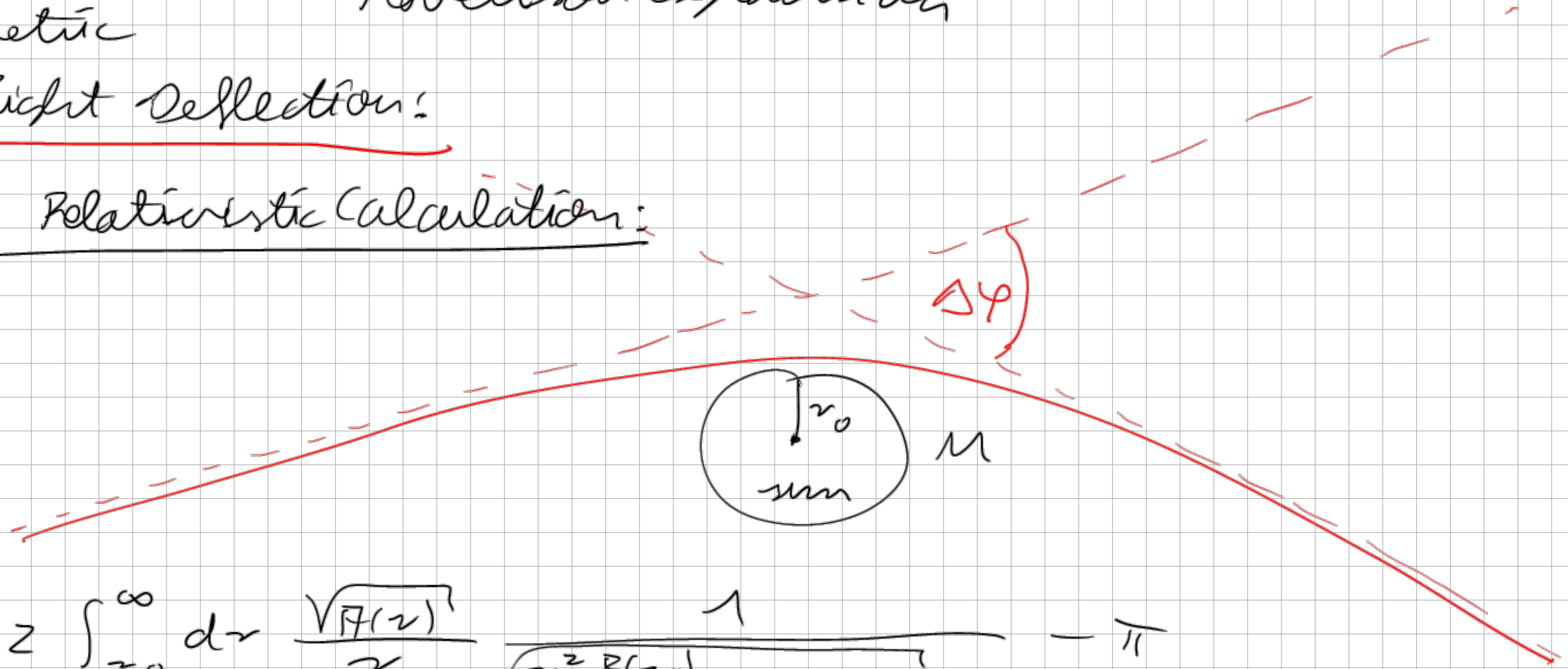
$$a = \frac{r_S}{2} = \frac{GM}{c^2}$$

Schwarzschild metric

Robertson expansion

2.1 Light Deflection:

2.1.1 Relativistic Calculation:



$$\underline{\Delta\varphi} = 2\int_{r_0}^{\infty} dr\, \frac{\sqrt{A(r)}}{r}\, \frac{1}{\sqrt{\frac{r^2 B(r_0)}{r_0^2 B(r)} - 1}} - \pi$$

$$= 2\int_{r_0}^{\infty} dr \left\{ \frac{r_0}{r}\frac{1}{\sqrt{r^2-r_0^2}} + \frac{\gamma a\, r_0}{r^2\sqrt{r^2-r_0^2}} + \frac{a}{(r+r_0)\sqrt{r^2-r_0^2}} \right\} - \pi$$

1-order expansion in a

only γ appears in light deflection

$$= 2\left[\arccos\frac{r_0}{r} + \frac{\gamma a}{r_0}\frac{\sqrt{r^2-r_0^2}}{r} + \frac{a}{r_0}\sqrt{\frac{r-r_0}{r+r_0}}\right]_{r_0}^{\infty} - \pi$$

$$= 2\left\{\cancel{\underbrace{\arccos 0}_{\pi/2}} + \frac{\gamma a}{r_0} + \frac{a}{r_0} - \underbrace{\arccos 1}_{=0}\right\} \cancel{-\pi}$$

$$= \frac{2(\gamma+1)a}{r_0}$$

$$r_0 = R_\odot = 7\cdot 10^5\,\text{km} \Rightarrow a = \frac{r_{s,\odot}}{2} = 1.5\,\text{km}$$

calculation: $2\pi \mathrel{\hat{=}} \underbrace{360\cdot 3600''}_{\text{arcsec}}$

$$\Delta\varphi = \frac{4\cdot 1.5\;\; 360\cdot 3600''}{7\cdot 10^5\;\;\cdot 2\pi}\cdot\frac{\gamma+1}{2} = 1.77''\,\frac{\gamma+1}{2}$$

$$\frac{\gamma+1}{2} \begin{cases} = 1 & (\gamma=1:\ \text{Einstein}) \\ = \frac{1}{2} & (\gamma=0:\ \text{Newton}) \end{cases}$$

2.1.2 Nonrelativistic Calculation:

- Johann Georg von Soldner
  born 1776, astronomer in Berlin 1797–1808, died 1828
- Relevant publication (1804): "Über die Ablenkung eines Lichtstrahls von seiner geradlinigen Bewegung

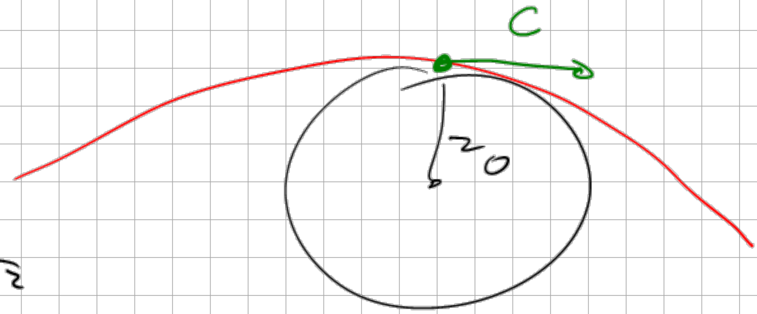
durch die Attraktion eines Weltkörpers, an welchem er vorbeigeht."

Starting point: nonrelativistic trajectory

$$\frac{d\varphi}{dr} = \frac{\frac{\ell}{m r^2}}{\sqrt{\frac{2E}{m} + \frac{2GM}{r} - \frac{\ell^2}{m^2 r^2}}}$$

(see your mechanics lecture)

$$\Delta\varphi = \int_{r_0}^{\infty} dr \, \frac{\frac{\ell}{m r^2}}{\sqrt{\frac{2E}{m} + \frac{2GM}{r} - \frac{\ell^2}{m^2 r^2}}} - \pi$$



- energy expressed by r_0

$$\left.\frac{d\varphi}{dr}\right|_{r_0} = \infty \Rightarrow \frac{2E}{m} = -\frac{2GM}{r_0} + \frac{\ell^2}{m^2 r_0^2}$$

- angular momentum evaluated at r_0

photon with mass m
moves with light velocity $\Bigg\} \; \ell = m r_0 c \longrightarrow \frac{\ell}{m} = r_0 c$

- abbreviation: $a = \frac{GM}{c^2}$

$$\Rightarrow \Delta\varphi = \int_{r_0}^{\infty} dr \, \frac{\frac{r_0}{r}}{\sqrt{r^2 - r_0^2 - 2\frac{a}{r_0}(r - r_0) r}} - \pi$$

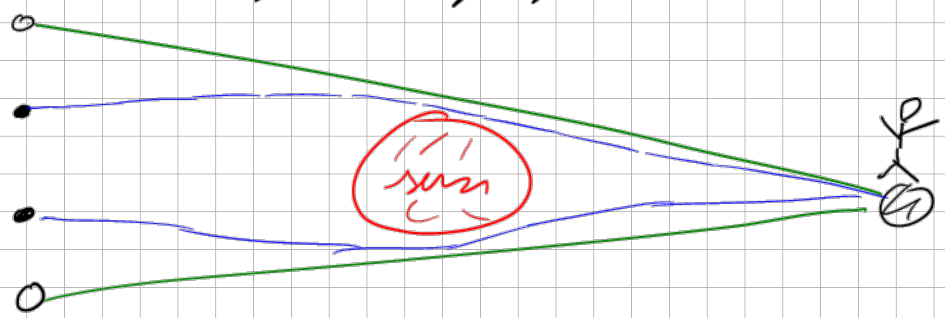
- Expansion for small a:

$$\Delta\varphi = \int_{r_0}^{\infty} dr \left\{ \frac{r_0}{r} \frac{1}{\sqrt{r^2 - r_0^2}} + \frac{a}{(r+r_0)\sqrt{r^2-r_0^2}} \right\} - \pi$$

$\hat{=}$ relativistic case with $\gamma = 0$

2.1.3 Solar Eclipse:

- 25.05.1919: two British expeditions
- idea: to compare two photographs from region at sky



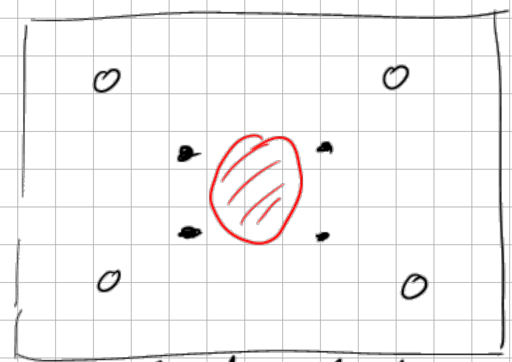


photo plate

o : observation of stars in absence of sun

• : " " " in presence of sun $\hat{=}$ light deflection

recalculate $\Delta\varphi$ from shift of star position on photo plate

Sources of error:

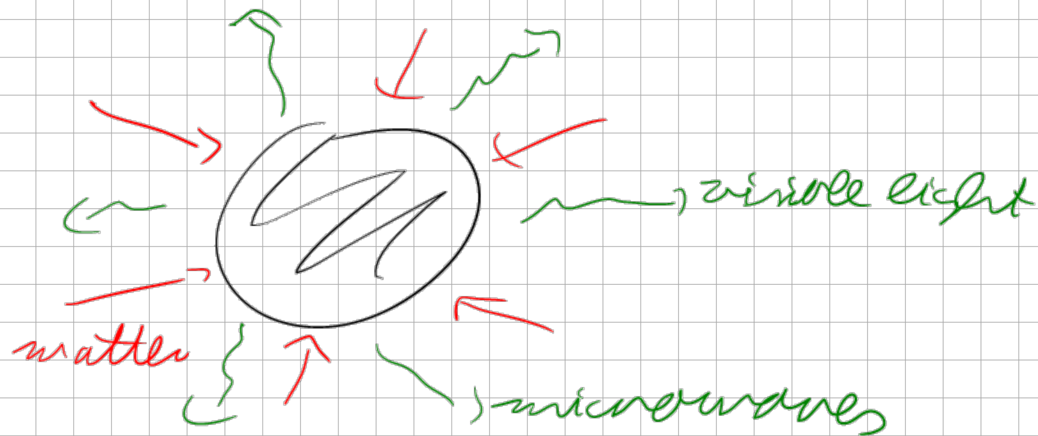
- photographic and optical errors
- short period of sun eclipse: 7.5 minutes
- not every sun eclipse can be used as only bright stars can be observed

- effect on photo plate only fraction of mm

$\Rightarrow$ imprecise method: $\gamma = 1.0 \pm 0.1$

2.1.4 Microwave Deflection:

- observe quasars: quasi-stellar radio source, far distant from us, large redshift
- large luminosity both in visible light but also in microwaves
  - $\rightarrow$ large energy conversion
- one hypothesis $\hat{=}$ plausible model
  quasar = massive black hole of mass of $10^6 - 10^{10}\ M_\odot$



Advantages of observing microwaves from quasars

- much more precise method
- Shapiro in 1967:

  $\underbrace{3C}_{\text{third Cambridge catalogue}}\ \underbrace{273}_{\text{number}}$ and $\underbrace{3C}_{\text{third Cambridge catalogue}}\ \underbrace{279}_{\text{number}}$

$\longrightarrow$ pass sun on October 8

- Resolution: wave length $\lambda = 5$ cm, interference on base length $L = 10$ m

$$\frac{5\cdot 10^{-2}}{10} \frac{360\cdot 3600''}{2\pi} \approx 10^{3}{}''$$

- Improved resolution:

VLBI: very long baseline radio interferometer
connection of several, world-wide radiotelescopes
> atomic clocks
> precise location

$\Rightarrow \lambda = 3\,cm,\ L = 10.000\,km$

$$\frac{3\cdot 10^{-2}}{10^{4}\cdot 10^{3}} \cdot \frac{360\cdot 3600''}{2\pi} \approx 5\cdot 10^{-4}{}''$$

$\Rightarrow \gamma = 0.99984 \pm 0.00028$

Error sources:

- sun corona: electrons at sun surface get excited by microwaves
  - → model about electron density at sun surface
  - → deflections from that are frequency dependent
  - → concentrate on frequency-independent effect: Einstein gravity
- earth rotation has to be taken into account

2.1.5 Gravitational Lense:

- 1979 Walsh et al.: two quasars have same microwave spectrum

→ indicates: two pictures of the same source
→ quasar-twin

See also Nature 426, 810 (2003)

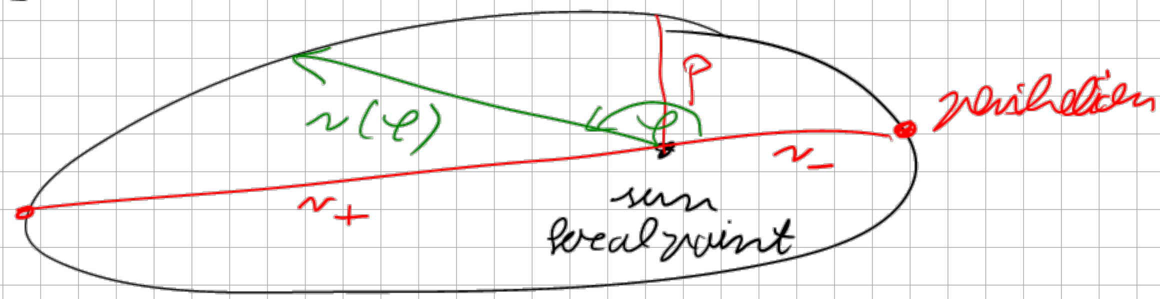
2.2 Perihelion Rotation:

- trajectory of a planet under the impact of a $1/r$ gravity leads to closed orbit: ellipse
- Einstein theory: additional $1/r^3$ potential → no closed orbit

⇒ perihelion rotation

2.2.1 Angle dependence of Perihelion:

Polar coordinates:
ellipse



$$r(\varphi) = \frac{p}{1 + \varepsilon \cos\varphi}$$

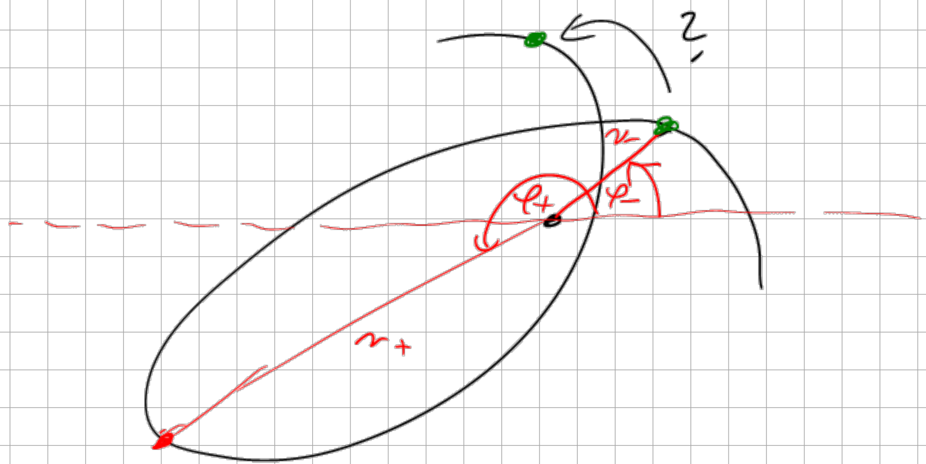
p ← parameter; ε: excentricity ($\varepsilon = 1$: circle)

$$r_- = r(\varphi = 0) = \frac{p}{1+\varepsilon}, \qquad r_+ = r(\varphi = \pi) = \frac{p}{1-\varepsilon}$$

$$\Rightarrow \quad \frac{2}{p} = \frac{1}{r_+} + \frac{1}{r_-}, \qquad \varepsilon = \frac{r_+ - r_-}{r_+ + r_-}$$

describing a rosetta motion of perihelion shift

$r_+ = r(\varphi_+)$, $r_- = r(\varphi_-)$



trajectory of massive particle

$$\varphi_+ - \varphi_- = \int_{\varphi_-}^{\varphi_+} d\varphi$$

$$= \int_{r_-}^{r_+} dr\, \frac{d\varphi}{dr} = \int_{r_-}^{r_+} dr\, \frac{\sqrt{A(r)}}{r^2} \frac{1}{\sqrt{\frac{F^2}{B(r)\tilde{\ell}^2} - \frac{C^2}{\tilde{\ell}^2} - \frac{1}{r^2}}}, \quad \tilde{\ell} = \frac{\ell}{m}$$

$\Delta\varphi = 2(\varphi_+ - \varphi_-) - 2\pi \;\hat{=}$ perihelion rotation

2.2.2 Elimination of Constants of Motion:

aphelion

$$\left.\frac{dr(\varphi)}{d\varphi}\right|_{\varphi = \varphi_\pm} = 0$$

perihelion

$$\frac{F^2}{B(r_+)\hat{\ell}^2} - \frac{C^2}{\tilde{\ell}^2} - \frac{1}{r_+^2} = 0 = \frac{F^2}{B(r_-)\hat{\ell}^2} - \frac{C^2}{\tilde{\ell}^2} - \frac{1}{r_-^2}$$

$$\frac{F^2}{\hat{\ell}^2} = \frac{\frac{1}{r_+^2} - \frac{1}{r_-^2}}{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}, \quad \frac{C^2}{\hat{\ell}^2} = \frac{\frac{1}{r_+^2 B(r_-)} - \frac{1}{r_-^2 B(r_+)}}{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}$$

reexpress radicant in terms $r_\pm$:

$$\frac{1}{B(r)}\frac{F^2}{\hat{\ell}^2} - \frac{C^2}{\hat{\ell}^2} = \frac{r_-^2\left[\frac{1}{B(r)} - \frac{1}{B(r_-)}\right] - r_+^2\left[\frac{1}{B(r)} - \frac{1}{B(r_+)}\right]}{r_+^2 r_-^2\left[\frac{1}{B(r_+)} - \frac{1}{B(r_-)}\right]} \underset{r=r_\pm}{\uparrow}= \frac{1}{r_\pm^2} \checkmark$$

2.2.3 Robertson Expansion:

$$A(r) = 1 + \gamma\frac{2a}{r} + \ldots, \quad B(r) = 1 - \frac{2a}{r} + 2(\beta-\gamma)\frac{a^2}{r^2} + \ldots$$

Aim: effect of first order in a

Note: $B(r)$ in radicant has to be used up to a^2 in order to get finally a first order result in a

Result for radicant:

$$\frac{F^2}{B(r)\hat{\ell}^2} - \frac{C^2}{\hat{\ell}^2} - \frac{1}{r^2} = \underset{\sim}{C}\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)$$

$$= 1 - (2-\beta+\gamma)\,a\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \ldots$$

2.2.4 Calculation of Integral:

$$\Delta\varphi = \frac{2}{\sqrt{C}}\int_{r_-}^{r_+} dr\,\frac{1}{r^2}\left(1+\gamma\frac{a}{r}\right)\frac{1}{\sqrt{\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)}} - 2\pi$$

substitution:

$$\frac{1}{r} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{1}{2}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\sin\psi(r)$$

$$dr = -\frac{r^2}{2}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\cos\psi \, d\psi$$

$$\sin\psi(r_+) = 1 \Rightarrow \psi(r_+) = \pi/2$$

$$\sin\psi(r_-) = -1 \Rightarrow \psi(r_-) = -\pi/2$$

$$\Delta\varphi = \frac{2}{\sqrt{C}} \int_{-\pi/2}^{+\pi/2} d\psi \left(1 + \gamma \frac{a}{r}\right) - 2\pi$$

first in a

$\Rightarrow$ elementary integral

$$\Delta\varphi = \frac{6\pi a}{P} \; \frac{2 - \beta + 2\gamma}{3}$$

Note: For Newton case one would expect $\Delta\varphi = 0 \;\hat{=}\; \beta = \gamma = 0$

This is not the case here. Relativistic effects are included due to geodesic equation.

2.2.5 Discussion:

Largest perihelion shift in solar system: mercury

ellipse parameter: $P = 5.5 \; 10^7$ km , $a_\odot = 1.5$ km

$\beta = \gamma = 1$: $\Delta\varphi = 0.104''$ per circulation around sun

accumulated effect per century: 415 surroundings of mercury around sun $\Rightarrow$ $\boxed{\Delta\varphi = 43'' \text{ per century}}$

Already known in 1882 by Newcomb

Total observed perihelion shift of mercury: $574.6''$

subtract gravitational effects from all other planets in solar system:

Venus:	$277.856''$
Earth:	$90.038''$
Mars:	$2.536''$
Jupiter:	$153.584''$
Saturn	$7.302''$
Uranus	$0.141''$
Neptun	$0.042''$
	$531.5''$

$$\begin{array}{r} 574.6'' \\ -\ 531.5'' \\ \hline 43.1'' \end{array}$$ per century

This was a not understood discrepancy.

Postulating yet unknown planet Vulcanus within mercury

$\Rightarrow$ But explained by General Relativity