2.2.5 Discussion:

perihelion rotation predicted by GRT for mercury per century: $\Delta\phi = 43''$

observed perihelion rotation: $\Delta\phi = 574.6''$

subtract effect of other planets: $\Delta\phi = 531.5''$

discrepancy since Newcomb (1882): $43.1''$

explained by GRT

consequence: $\frac{2-\beta+2\gamma}{3} = 1.000 \pm 0.001$

$= 1 \pm \ldots$

- γ determined from light deflection: $\beta = 1.000 \pm 0.003$
- β stems from $(a/r)^2$ – term of Robertson expansion: nonlinear effect of gravity

Perihelion shifts of other planets / asteroid:

(per century)	mercury	venus	earth	icarus (0.5 km radius)
GRT	$43.03''$	$8.6''$	$3.8''$	$10.3''$
observation	$43.1'' \pm 0.5''$	$8.4'' \pm 3''$	$5.0'' \pm 2''$	$9.8 \pm 0.8''$

$\underbrace{\text{mercury, venus, earth}}$: $\varepsilon \simeq 1$ nearly circular
$\to$ experimental uncertainty larger

$\underbrace{\text{icarus}}$: $\varepsilon \simeq 0.8$

Physical origin of perihelion rotation:

additionally to Newton potential $1/r$ GRT predicts

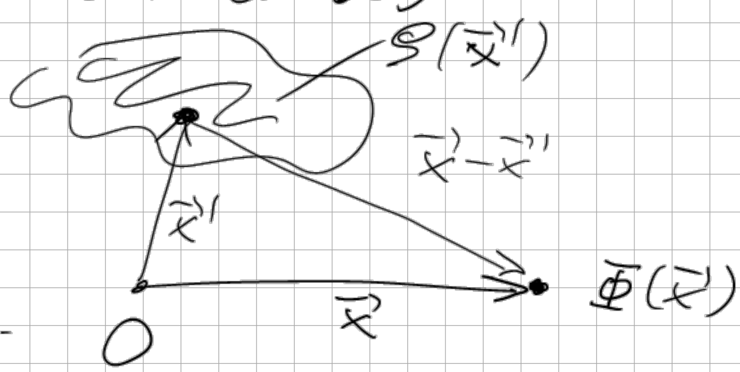
$\frac{1}{r^3}$ potential

2.2.6 Quadrupole Moment of Sun:

Newton gravity potential of a mass distribution

$$\Phi(\vec{x}) = -G \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

mass density



approximation: $|\vec{x}| \gg |\vec{x}'|$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \frac{x_j}{r^3} x'_j + \frac{1}{2} \frac{3 x_j x_k - r^2 \delta_{jk}}{r^5} x'_j x'_k + \dots$$

$$\Phi(\vec{r}) = -G \Big\{ \underbrace{\frac{M}{r}}_{\text{monopole}} + \underbrace{\frac{x_j}{r^3} \int d^3x' \, \rho(\vec{x}') \, x'_j}_{\text{dipole}} + \underbrace{\frac{1}{2} \frac{3 x_j x_k - r^2 \delta_{jk}}{r^5} T_{jk}}_{\text{quadrupole}} + \dots$$

$M = \int d^3x' \, \rho(\vec{x}')$

assume $\rho(-\vec{x}') = \rho(\vec{x}')$

$= 0$

due to symmetry

quadrupole tensor $\quad T_{jk} = \int d^3x' \, \rho(\vec{x}') \, x'_j x'_k$

consider ellipsoid: ellipsoidal coordinates

$$x' = \underbrace{a}_{\text{half axis}}\, r \sin\vartheta \cos\varphi \qquad r \in [0,1)$$
$$y' = b\, r \sin\vartheta \sin\varphi \qquad \vartheta \in [0,\pi]$$
$$z' = c\, r \cos\vartheta \qquad \varphi \in [0,2\pi]$$

assuming $\rho(\vec{x}')$ being constant in ellipsoid:

$$\Rightarrow M = \frac{4\pi}{3}\rho\, abc$$

$$(T_{ik}) = \frac{1}{5}\frac{4\pi}{3}\rho\, abc \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = \frac{M}{5}\begin{pmatrix} a^2 & & 0 \\ & b^2 & \\ 0 & & c^2 \end{pmatrix}$$

rotational ellipsoid: $a = b \neq c \Rightarrow$ effective radius $R = (a^2 c)^{1/3}$

$$\Phi(\vec{x}) = -\frac{GM}{r} - \frac{GM}{10}\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}(a^2-c^2) + \dots$$

equitorial plane: $z = 0$, $r = \sqrt{x^2+y^2}$

$$\Phi(\vec{x}) = -\underbrace{\frac{GM}{r}}_{\sim \frac{1}{r}} - \underbrace{\frac{GQ}{2r^3}}_{\sim\frac{1}{r^3}} + \dots, \quad Q = \frac{M}{5}\frac{a-c}{R}\cdot\underbrace{\frac{a+c}{R}}_{a\approx c\approx R}R^2 \quad \text{quadrupole moment}$$

$$\Rightarrow Q = q\, MR^2, \text{ dimensionless asymmetry parameter} \qquad q = \frac{2}{5}\frac{a-c}{R}$$

Identify previous GRT calculation with quadrupole moment

$$\underbrace{\frac{GM\ell^2}{m^2/c^2}}\ \sim\ \underbrace{GQ} \qquad \text{with } \frac{\ell}{m} \sim v\underbrace{p}_{\text{parameter of ellipsoid}}, \quad v^2 = \frac{GM}{p}$$

prefactor of $1/r^3$ in GRT

prefactor $1/r^3$ for quadrupol moment

$$\frac{GQ}{\frac{GMe^2}{m^2/c^2}} \sim \frac{qR^2}{pa} \Rightarrow \Delta\phi = 3\pi q \frac{R^2}{p^2}$$

helioseismology:

- sun is considered as a droplet, confined due to gravity
- rotation of sun: deformation of sphere to ellipsoid
- eigenfrequencies allow to deduce q
- 1960: Robert Leighton $\Rightarrow q = (1.7 \pm 0.4) \cdot 10^{-7}$

$$\Rightarrow \Delta\phi = 5 \cdot 10^{-5}{}''$$

This is $5 \cdot 10^{-4}$ smaller than perihelion of mercury due to GRT

Outlook: perihelion shift of stars encircling the center of milkyway since 2018

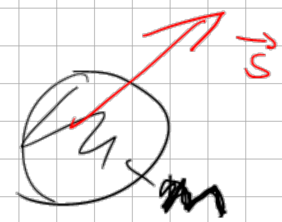
- mass of black hole $\approx 4 \cdot 10^6 M_\odot$
- one is trying to deduce spin (angular momentum) of black hole

2.3 Geodetic Precession:

What is happening to a spinning top or gyroscope (high-precision spinning top) is a static, isotropic gravitational field ?

→ spin gets a rotation due to gravity

2.3.1 Equations of Motion:



Schwarzschild metric

geodesic trajectory

spin motion

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad \frac{ds^\lambda}{d\tau} = -\Gamma^\lambda_{\mu\nu}\, s^\mu \frac{dx^\nu}{d\tau}$$

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = c^2 \quad (**) \qquad g_{\mu\nu}\, s^\mu s^\nu = -\vec{s}^{\,2}$$

⇐ normalization

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\, s^\nu = 0$$

orthogonality

$$\frac{d^2x^0}{d\tau^2} = \cancel{-\frac{B'(r)}{B(r)}\frac{dx^0}{d\tau}\frac{dr}{d\tau}}$$ ✓

$$\frac{d^2r}{d\tau^2} = -\frac{B'(r)}{2A(r)}\left(\frac{dx^0}{d\tau}\right)^2 - \cancel{\frac{A'(r)}{2A(r)}\left(\frac{dr}{d\tau}\right)^2} + \cancel{\frac{r}{A(r)}\left(\frac{d\vartheta}{d\tau}\right)^2} + \frac{r}{A(r)}\underbrace{\sin^2\vartheta}_{=1}\left(\frac{d\varphi}{d\tau}\right)^2 \quad (*)$$

$$\frac{d^2\vartheta}{d\tau^2} = \cancel{-\frac{2}{r}\frac{dr}{d\tau}\frac{d\vartheta}{d\tau}} + \cancel{\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\tau}\right)^2}$$ ✓

$$\frac{d^2\varphi}{d\tau^2} = \cancel{-\frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}} - \cancel{2\,\frac{\cos\vartheta}{\sin\vartheta}\frac{d\vartheta}{d\tau}\frac{d\varphi}{d\tau}}$$ ✓

$$\frac{ds^0}{d\tau} = -\frac{B'(r)}{2B(r)}\left(\cancel{s^0\frac{dr}{d\tau}} + s^1\frac{dx^0}{d\tau}\right)$$

$$\frac{ds^1}{d\tau} = -\frac{B'(r)}{2A(r)}s^0\frac{dx^0}{d\tau} - \cancel{\frac{A'(r)}{2A(r)}s^1\frac{dr}{d\tau}} + \cancel{\frac{r}{A(r)}s^2\frac{d\vartheta}{d\tau}} + \frac{r}{A(r)}\sin^2\vartheta\, s^3\frac{d\varphi}{d\tau}$$

$$\frac{ds^2}{d\tau} = -\frac{1}{r}\left(\cancel{s^1\frac{d\vartheta}{d\tau}} + \cancel{s^2\frac{dr}{d\tau}}\right) + \cancel{\sin\vartheta\cos\vartheta\, s^3\frac{d\varphi}{d\tau}}$$

$$\frac{ds^3}{d\tau} = -\frac{1}{r}\left(s^1\frac{d\varphi}{d\tau} + \cancel{s^3\frac{dr}{d\tau}}\right) - \cancel{\frac{\cos\vartheta}{\sin\vartheta}\left(s^2\frac{d\varphi}{d\tau} + s^3\frac{d\vartheta}{d\tau}\right)}$$

2.3.2 Satellite Trajectory in Form of a circle:

$$\left(x^0(\tau), r(\tau), \vartheta(\tau), \varphi(\tau)\right) = \left(u^0\tau, \underbrace{r}_{\text{fixed}}, \underline{\frac{\pi}{2}}, \omega_0\tau\right)$$

$$\left(\frac{dx^0}{d\tau}, \frac{dr}{d\tau}, \frac{d\vartheta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(u^0, \underline{0}, \underline{0}, \omega_0\right)$$

(*) provides a relation between three parameters u^0, r, ω_0 in our ansatz for a circular motion:

$$(*) \Rightarrow \left(\frac{u^0}{\omega_0}\right)^2 = \frac{2r}{B'(r)} \qquad (1)$$

(**) leads to another condition for 3 parameters:

$$B(r)(u^0)^2 - r^2\omega_0^2 = c^2 \qquad (2)$$

(1)+(2) in order to eliminate u^0:

$$w_0^2 = \frac{c^2}{2r\frac{B(r)}{B'(r)} - r^2} = \frac{ac^2}{r^3 - 3ar^2}, \quad a = \frac{GM}{c^2} = \frac{r_S}{2}$$

Schwarzschild: $B(r) = 1 - \frac{2a}{r}$

earth radius: $R_E = 6.400$ km

earth acceleration: $g = \frac{GM}{R_E^2}$

$\Big\}$ $a = 4.6$ mm

$$w_0^2 \approx \frac{ac^2}{R_E^3} = \frac{GM}{R_E^3}$$

known from mechanics:

$$\frac{MGm}{R_E^2} = m w_0^2 R_E$$

$$T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{R_E}{g}} \approx 1.4 \text{ h}$$

spin equation of motion for a spin in satellite:

$$\frac{dS^0}{d\tau} = -\frac{B'(r)}{2B(r)} u^0 S^1$$

$$\frac{dS^1}{d\tau} = -\frac{B'(r)}{2B(r)} u^0 S^0 + \frac{r}{B(r)} w_0 S^3$$

$$\frac{dS^2}{d\tau} = 0 \Rightarrow S^2(\tau) = \text{const.} = 0$$

four linear ordinary differential equations with constant coefficients

$$\frac{dS^3}{d\tau} = -\frac{1}{2} w_0 S^1$$

initial condition: $\left(S^0(0), S^1(0), S^2(0), S^3(0)\right) = (0, s, 0, 0)$

$$g_{\mu\nu} S^\mu_{(0)} u^\nu_{(0)} = 0$$

2.3.3 Solution for Spin Dynamics:

$\Rightarrow S^2(\tau) = 0 \Rightarrow$ spatial spin vector must lie within equatorial plane

$$\vec{r} = r \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}, \quad \vec{e}_\vartheta = \frac{\frac{\partial\vec{r}}{\partial\vartheta}}{\left|\frac{\partial\vec{r}}{\partial\vartheta}\right|} \underset{\vartheta=\pi/2}{=} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Combining remaining 3 eqs.

$$\frac{d^2 S^1}{d\tau^2} = -w^2 S^1, \quad w = w_0 \sqrt{\frac{1}{A(r)}\left[1 - \frac{r B'(r)}{2 B(r)}\right]}$$

$$\approx w_0 \left\{ 1 - \left(\gamma + \frac{1}{2}\right)\frac{a}{r} + \dots \right\}$$

w_0: angular frequency for center-of-mass motion

w: " " of spin rotation