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ON THE PROPERTY OF APPROXIMATION OF TWO-DIMENSIONAL LOCAL RINGS

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In this paper we'll prove for a certain class of local rings the property of approximation (cf. [8]).

A local ring A is called a ring with property of approximation (and we'll write shortly $A \in AE$) if the following holds :

Let $T = (T_1, \ldots, T_n)$ be some variables, $F = (F_1, \ldots, F_m) \in A[T_1, \ldots, T_n]^m$ some polynomials and c an positive integer.

Suppose there is a $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in \hat{A}^n$, $\hat{A}$ the completion of A, such that $F(\bar{y}) = 0$, then there is a $y = (y_1, \ldots, y_n) \in A^n$ such that $F(y) = 0$ and $y = \bar{y} \bmod m_{-A}^c$.

What examples of rings with property of approximation do we know?

T h e o r e m (M. ARTIN) : *Let R be an excellent henselian discrete valuation ring, A the henselization of a ring of finite type over R in a maximal ideal, then $A \in AE$.*

(cf. [3]).

T h e o r e m (M. ARTIN) : *Let K be a valued field of characteristic 0, A an analytic algebra over K, then $A \in AE$.*

(cf. [2]).

T h e o r e m : *Let K be a complete valued field of characteristic p, A an analytic algebra over K, then $A \in AE$.*

(cf. [1], [4], [7] and [8]).

We'll prove the following theorem :

Let C be a complete discrete valuation ring with a prim element p, Y a variable.

Let A be an universal japanese ring with the following properties :

(i) $C[Y] \subseteq A \subseteq C[[Y]]$, A is local noetherian, $m_{-A} = (p, Y)$;

(ii) $\hat{A} = C[[Y]]$;

(iii) A is algebraically closed in $C[[Y]]$.

Then $A \in AE$.

P r o o f : Let $T = (T_1, \ldots, T_m)$ and let $F_1, \ldots, F_n \in A[T]$.

Suppose there is a $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_m)$, $\bar{t}_i \in \hat{A} = C[[Y]]$ such that $F_i(\bar{t}) = 0$ for all i.
We fix an integer $c > 0$.
Now we need the following lemma:
NERON's blowing up: There is a prime ideal $\gamma = (G_1, \ldots, G_N) \subseteq A[T, Z]$, $Z = (Z_1, \ldots, Z_m)$ some new variables, and a $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{m'}) \in \hat{A}_{m'}$ with the following properties:

(i) $\gamma(\bar{t}, \bar{z}) = 0$; $\gamma = \text{kernel}(A[T, Z] \to \hat{A},\ T \to \bar{t},\ Z \to \bar{z})$;

(ii) $F_i \in \gamma$ for all i;

(iii) there is a $ht(\gamma) \times ht(\gamma)$-minor δ of the JACOBIAN matrix

$$\partial(G_1, \ldots, G_N)/\partial(T, Z) \quad \text{such that p doesn't devide } \delta(\bar{t}, \bar{z}).$$

We don't want to prove this lemma here. One can get it as a direct generalization of the corresponding lemma in ARTIN's proof (cf. [3]). Also one can deduce it from a more general lemma proved in [4] or [8]. Applying this lemma we can suppose without restriction of generality that
– $F_1, \ldots, F_n$ generate the kernel of the map $A[T] \to \hat{A}$, $T \to \bar{t}$;
– there is a $r \times r$-minor δ of the JACOBIAN matrix $\partial(F_1, \ldots, F_n)/\partial(T)$ such that $\delta(\bar{t})$ is not divisible by p (r is here the height of the ideal generated by the F_i).
Now we apply WEIERSTRASS preparation theorem and get

$$\delta^2(\bar{t}) = \bar{h}(Y^s + k_{s-1}Y^{s-1} + \ldots + k_0)$$

with a unite $\bar{h} \in C[[Y]]$ and $k_i \in pC$.
Let $g = Y^c\delta^2(\bar{t})\bar{h}^{-1} \in C[Y]$.
We divide the F_i by g using again the preparation theorem and get

$$F_i = gF_{i1} + F_{i2} \text{ with } F_{i2} \in C[Y, T].$$

Similarly we get

$$\bar{t}_i = g\bar{t}_{i1} + \bar{t}_{i2},\ \bar{t}_{i2} \in C[Y].$$

Now we have

$$0 = F_i(\bar{t}) = gF_{i1}(\bar{t}) + F_{i2}(\bar{t}),$$

so $F_{i2}(\bar{t})$ must be divisible by g.
Let's denote the JACOBIAN matrix associated to $F = (F_1, \ldots, F_n)$ by $J(F)$ and similarly the corresponding matrixes of $F_1 = (F_{11}, \ldots, F_{n1})$ by $J(F_1)$ and of $F_2 = (F_{12}, \ldots, F_{n2})$ by $J(F_2)$, then we have $J(F) = J(F_1)g + J(F_2)$.
Now we choose in $J(F_2)$ the minor corresponding to δ and denote it by δ_{F_2}.
Then $\delta_{F_2} = \delta + gH$ for a suitable $H \in A[T]$.
So $\delta_{F_2}(\bar{t}) = \delta(\bar{t}) + gH(\bar{t}) = \delta(\bar{t})(1 + Y^c\delta(\bar{t})\bar{h}^{-1}H(\bar{t}))$, i.e.

$$\delta_{F_2}(\bar{t}) = \delta(\bar{t}) \cdot \text{unit}.$$

So we get finally that $Y^c\delta^2_{F_2}(\bar{t})$ divides $F_{i2}(\bar{t})$ for all i (because g was dividing $F_{i2}(\bar{t})$).

Now let's remember that $\bar{t}_i = g\bar{t}_{i1} + \bar{t}_{i2}$, g and $\bar{t}_{i2} \in C[Y]$, and F_{i2}, $\delta^2_{F_2} \in C[Y, T]$. So we can apply the classical approximation theorem from M. ARTIN (cf. [3]) and get the existence of $\tilde{t}_{i1}, \tilde{t}_i \in C\langle Y\rangle$ such that $\tilde{t}_i = g\tilde{t}_{i1} + \bar{t}_{i2}$, $\tilde{t}_{i1} \equiv \bar{t}_{i1} \bmod (p, Y)^c$ and $Y^c\delta^2_{F_2}(\tilde{t})$ divides $F_{i2}(\tilde{t})$.
So we have

$$F_i(\tilde{t}) = gF_{i1}(\tilde{t}) + F_{i2}(\tilde{t}) = gF_{i1}(\tilde{t}) + Y^c\delta^2_{F_2}(\tilde{t})L$$

for a suitable L.
On the other hand we have

(i) $\delta(\tilde{t}) - \delta(\bar{t}) = g \cdot b$, i.e. $\delta(\tilde{t}) = (\bar{t})(1 + Y^c\delta(\bar{t})\bar{h}^{-1} b)$.

This means $Y^c\delta^2(\tilde{t}) = g \cdot \text{unit}$.

(ii) $\delta_{F_2}(\tilde{t}) = \delta(\tilde{t}) + gH(\tilde{t}) = \delta(\tilde{t}) \cdot \text{unit} + gH(\tilde{t}) = \delta(\tilde{t}) \cdot \text{unit}$.

So we have finally

$$Y^c\delta^2(\tilde{t}) \text{ divides } F_i(\tilde{t}) \text{ for all } i.$$

Now we can apply NEWTON's lemma to all these F_i defining. We get the existence of a $t \in A^m$ such that $F_i(t) = 0$ for all F_i appearing in the definition of δ and $\tilde{t} \equiv t \bmod \delta(\tilde{t})(p, Y)^c$. Using the JACOBIAN criterion one can show now similarly to the proof of M. ARTIN that for all c big enough this implies $F_i(t) = 0$ for all i;
Remark: With almost the same idea one can prove that for any excellent henselian discrete valuation ring C the ring $C[[Y]]$ has the property of approximation.
Remark: Let C be a complete discrete valuation ring with residue field of characteristic 0, $\{w_i\}_{i\in I}$ a family of elements of $C[[Y]]$, then the algebraic closure of $C[Y, \{w_i\}_{i\in I}]$ in $C[[Y]]$ is a ring satisfying the assumptions of the above theorem. This is not true if C is of characteristic $p > 0$, cf. [9].

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