... OF APPROXIMATION

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One gives a complete characterization of one dimensional rings with the property of approximation and a positive solution in dimension 3 for a question of M. Artin [2]. In particular, one obtains many examples of three dimensional rings with the property of approximation.

Let $(A, \mathfrak{m})$ be a local noetherian ring (all the rings are supposed to be commutative with identity). A is called a ring with the property of approximation (shortly $A \in \mathbf{AP}$) if the following condition holds (cf. [4], [10]). "Let $f = (f_1, \ldots, f_m)$ be an arbitrary system of polynomials in some variables $Y = (Y_1, \ldots, Y_N)$ with coefficients in A. Then every solution $\bar{y}$ of f in $\hat{A}$ ($\hat{A}$ denotes the completion of A) can be well approximated in the $\mathfrak{m}$-adic topology by a solution of f in A, i.e. for every positive integer c there exists a solution y of f in A such that $y \equiv \bar{y} \mod \mathfrak{m}^c \hat{A}$.

Clearly, the noetherian local complete rings are trivial examples of **AP**-rings. More general, we call an extension of rings $A \hookrightarrow B$ algebraically pure if every system of polynomials with coefficients in A has a solution in B iff it has one in A (cf. [12]). It is easy to see that $A \in \mathbf{AP}$ iff the extension $A \hookrightarrow \hat{A}$ is algebraically pure.

Which rings with the property of approximation do we know?

1) Let R be a field or an excellent henselian discrete valuation ring, then the ring of algebraic power series $R\langle T\rangle$, $T = (T_1, \ldots, T_n)$, i.e. the henselization of $R[T]_{(T)}$, is an **AP**-ring as M. Artin proved (cf. [4]).

2) Let R be a valued field of characteristic 0, or a complete valued field in characteristic $p > 0$, then the ring $R\{T\}$, $T = (T_1, \ldots, T_r)$ of convergent power series with coefficients in R is an **AP**-ring, as M. Artin, M. André and others proved (see [3], [1], [14], [7], [9]).

3) A one dimensional local, noetherian, reduced ring is an **AP**-ring iff it is henselian and universally japanese (this is an easy consequence of R. Elkik's theorem [6]).

4) A two dimensional local regular ring is an **AP**-ring iff it is henselian and universally japanese (cf [9], [13]), (conversely all **AP**-rings are henselian and universally japanese [4], [8]).

Remark that in all these examples of **AP**-rings with dimension $\geqslant 3$ the Weierstrass Preparation Theorem holds.

In [2], M. Artin put the following question.

i) Let R be a complete discrete valuation ring and $X = (X_1, \ldots, X_n)$, $T = (T_1, \ldots, T_s)$ some variables.

Does $A := R[[X]]\langle T\rangle$ have the property of approximation?

In [8], a positive answer to i) is given, but the proof is wrong. We see that for A the Weierstrass Preparation Theorem does not hold, if $n, s \geq 1$. A has also not "enough" automorphisms, i.e. a formal power series $\not\equiv 0 \bmod p$ (p denotes a local parameter in R) cannot be regularized by an automorphism of A (as it happens in $\hat{A}$). These facts make i) difficult. However the case $n = 0$ of i) is already known (see 1)) and clearly it is enough to prove i) for $s = 1$ and all $n \geq 1$.

A positive answer of i) would give some interesting examples of **AP**-rings in dimension 3 but first of all it would yield some nice applications in deformation theory, based on the following consequence of i):

"Let K be a field and $f = (f_1, \ldots, f_m)$ an arbitrary system of polynomials with coefficients in $K\langle X\rangle$, $X = (X_1, \ldots, X_s)$. Suppose f has a formal solution $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ such that $\bar{y}_i \in K[[X_1, \ldots, X_{n_i}]]$ where the natural numbers n_i satisfy $1 \leq n_1 \leq \ldots \leq n_N \leq n$. Then $\bar{y}$ can be well approximated by solutions of f in $K\langle X\rangle$ having the same property".

In the present paper, which is an improved version of [11], we show that $R\,[[X]]\,\langle T\rangle$, $n, s = 1$, is an **AP**-ring for an arbitrary complete discrete valuation ring R. This gives us many interesting examples of **AP**-rings in dimension three: let A be a two dimensional **AP**-ring which is supposed to be a domain in unequal characteristic case or arbitrary in equal characteristic case, then $A\langle T\rangle$ is also an **AP**-ring.

1. RINGS WITH THE PROPERTY OF APPROXIMATION

1.1. THEOREM. *Let $(A, \mathfrak{m})$ be an one dimensional local noetherian ring. Then A is an* **AP**-*ring iff A is henselian and universally japanese.*

The proof is given in Section 2.

1.2. REMARK. The equivalence stated by (1.1) does not hold for three dimensional local rings. Indeed, if we consider A to be the henselization of the ring R constructed by C. Rotthaus (see § 1 [15]), then $A/\omega_0 A$ is an integral domain but $\hat{A}/\omega_0\hat{A}$ is not (see § 4 [15]). Thus A is not an **AP**-ring (see [8]).

Let R be a complete discrete valuation ring and X, T some variables.

1.3. THEOREM. *$R\,[[X]]\,\langle T\rangle$ is an* **AP**-*ring.*

The proof is given in section 3.

1.4. COROLLARY. *If A is a two dimensional* **AP**-*ring which is supposed to be an integral domain in unequal characteristic case, then $A\langle T\rangle$ is* **AP**-*ring too, ($A\langle T\rangle$ denotes the henselization of the local ring $A[T]_{(T)}$).*

Proof. First we consider the case in which A is a local complete ring. Then, by the Cohen Structure Theorem, A is a finite extension of a local complete regular ring B of dimension two).

By (1.3), $B\langle T\rangle$ is an **AP**-ring and thus $A\langle T\rangle$ is also **AP**-ring, since it is a finite extension of $B\langle T\rangle$ (apply (1.2) chapter II from [9]).

Now, if A is an **AP**-ring, then the morphism $A \hookrightarrow \hat{A}$ is algebraically pure. Thus the morphism $A\langle T\rangle \hookrightarrow A[[T]]$ is still algebraically pure by

Corollary 1.12 [12]. As $\hat{A}\langle T\rangle$ is an **AP**-ring (see above), the morphism $\hat{A}\langle T\rangle \hookrightarrow \hat{A}[[T]]$ is still algebraically pure. Consequently, $A\langle T\rangle \hookrightarrow \hat{A}[[T]]$ is algebraically pure and thus $A\langle T\rangle$ is an **AP**-ring.

Q.E.D.

1.5. COROLLARY. *If A is a two dimensional noetherian locally complete domain (or more general an* **AP**-*domain), then $A\langle T\rangle$ is factorial iff $A[[T]]$ is also factorial.*

Proof. By (1.4) $A\langle T\rangle$ is an **AP**-ring and it is enough to apply (5.7) chapter V [9].

1.6. REMARK. If A is a two dimensional **AP**-ring which is supposed to be domain in unequal characteristic case, then the following statements hold

1) every prime ideal $q \subset A\langle T\rangle$ extends to a prime ideal $q\hat{A}\,[[T]]$;

2) a prime ideal $q \subset A\langle T\rangle$ is regular iff $q\hat{A}[[T]]$ is a regular prime ideal;

3) every primary decomposition of an ideal $a \subset A\langle T\rangle$, $a = q_1 \cap \cap \ldots \cap q_s$, having $p_i = \sqrt{q_i}$ as associated prime ideals, extends to a primary decomposition $a\hat{A}\,[[T]] = q_1\hat{A}\,[[T]] \cap \ldots \cap q_s\hat{A}\,[[T]]$ having $p_i\hat{A}\,[[T]] = \sqrt{q_i\hat{A}\,[[T]]}$ as associated prime ideals;

4) $A\langle T\rangle$ is an universally catenary ring.

For the proof we apply (5.1), (5.2), (5.5), chapter V [9].

1.7. REMARK. With the same methods used in the proof of Theorem 1.3 one can also prove that for any one dimensional local **AP**-ring A, which is supposed to be a domain in unequal characteristic case, $A\langle T\rangle$ has also the property of approximation.

1.8. THEOREM. *Let K be a field and X, Z, T variables, then $K\langle T\rangle[[X, Z]]$ is an* **AP**-*ring. If K is a valued field of characteristic zero or a complete valued field of characteristic $p > 0$, then $K\{T\}\,[[X, Y]]$ is also an* **AP**-*ring.*

The proof is similar to the proof of (1.3) (cf. Remark (3.3)).

2. PROOF OF THEOREM (1.1)

Let $B = A[Y]/_{(f)}$ be an A-algebra of finite type.

The set of prime ideals $q \in \operatorname{Spec} B$ such that the morphism $A \to B_q$ is not smooth form a closed set defined by an ideal $\mathbf{H}_f$. By a result of R. Elkik (see [6]), there exists a function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property.

"For every $y \in A^N$ such that $f(y) \equiv 0 \mod \mathfrak{m}^{d(s,c)}$ and $\mathbf{H}_f(y) \supset \mathfrak{m}^c$, there exists a solution $\tilde{y}$ of f in A such that $\tilde{y} \equiv y \mod \mathfrak{m}^c$".

Now, let $f = (f_1, \ldots, f_m)$ be an arbitrary system of polynomials from $A[Y]$, $Y = (Y_1, \ldots, Y_N)$ and $\bar{y} \in \hat{A}^N$ a „formal" solution of f. Adding some polynomials to f we may suppose that f generates the kernel of the map $\sigma : A[Y] \to \hat{A}$ given by $P \rightsquigarrow P(\bar{y})$. We consider the ideal $\mathbf{H}_f(\bar{y})$ generated in $\hat{A}$ by elements of the form $P(\bar{y})$, $P \in H_f$. We have the following cases

Case 1. ht $\mathbf{H}_f(\bar{y}) = 1$ or $\mathbf{H}_f(\bar{y}) = \hat{A}$.

Case 2. ht $\mathbf{H}_f(\bar{y}) = 0$.

Case 1. If $\mathbf{H}_f(\bar{y})$ is an $\mathfrak{m}$-primary ideal let us put $\mathbf{H}_f(\bar{y}) \supset \mathfrak{m}^s \hat{A}$. Denote $t = \max\ \{\bar{d}\,(s, c), 2s\}$ and choose $y' \in A^N$ such that $y \equiv \bar{y}$ mod $\mathfrak{m}^t \hat{A}$. By Taylor's formula, we get $f(y) \equiv 0 \mod \mathfrak{m}^{d(s,c)}$ and $\mathfrak{m}^s \hat{A} \subset \subset \mathbf{H}_f(\bar{y}) \subset \mathbf{H}_f(y) \hat{A} + \mathfrak{m}^{2s} \hat{A}$. It results $\mathfrak{m}^s \hat{A} \subset \mathbf{H}_f(y)$. $\hat{A}$ and thus $\mathfrak{m}^s \subset \mathbf{H}_f(y)$. Consequently, there exists a solution $\tilde{y} \in A^N$ of f such that $y \equiv \tilde{y}$ mod $\mathfrak{m}^c$. If $\mathbf{H}_f(\bar{y}) = \hat{A}$ then the morphism $A \to A\,[Y]/_{(f)}$ is smooth and we may apply the Implicit Function Theorem.

Case 2. We shall use the following lemma to reduce this case to the first one (case 2 can only appear if A is not reduced).

2.1. LEMMA. *Let $B \subset \hat{A}$ be an A-algebra of finite type. Then there is a B-algebra $B' \subset \hat{A}$ of finite type such that B'_p is a smooth A_p-algebra for all minimal primes $p \in$ Spec A.*

Apply (2.1) to our situation B : Im $\sigma \simeq A[Y]/_{(f)}$, the isomorphism being induced by σ.

Then there exists a B-algebra $B' \subset \hat{A}$ of finite type; let us put $B' \simeq A[Y, X]/_{(g)}$, $X = (X_1, \ldots, X_r)$, $g = (g_1, \ldots, g_s)$, the isomorphism being given by $Y \rightsquigarrow \bar{y}$, $X \rightsquigarrow \bar{x}$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_r) \in \hat{A}^r$, such that ht $H_g(\bar{y}, \bar{x}) = 1$ or $H_g(\bar{y}, \bar{x}) = \hat{A}$.

Now as in first case we get a solution $(y, x) \in A^{N+r}$ of $g = 0$. In particular, y is a solution of $f = 0$.

Proof of Lemma 2.1. Let $\mathbf{M}$ be the set of minimal primes of A and $S = A \setminus \bigcup_{p \in \mathbf{M}} p$. The inclusions $S^{-1} A \hookrightarrow S^{-1} B \hookrightarrow S^{-1} \hat{A}$ split into a product induced by the canonical maps

$$\prod_{p \in \mathbf{M}} A_p \hookrightarrow \prod_{p \in \mathbf{M}} B \otimes_A A_p \hookrightarrow \prod_{p \in \mathbf{M}} \hat{A} \otimes_A A_p .$$

(Remark that $\prod_{p \in \mathbf{M}} \hat{A} \otimes {}_A A_p \simeq \prod_{p \in \mathbf{M}} \hat{A}_{\widehat{pA}}$).

Now we see, that it is sufficient to prove that $\prod_{p \in \mathbf{M}} \hat{A}_{\widehat{pA}}$ is a filtered inductive limit of smooth finite type $\prod_{p \in \mathbf{M}} A_p$-algebras or equivalently to prove that $\hat{A}_{\widehat{pA}}$ is a filtered inductive limit of smooth finite type A_p-algebras for all $p \in \mathbf{M}$. Indeed, then there exists a smooth $S^{-1} A$-algebra $\tilde{B} \subset S^{-1} \hat{A}$ of finite type which contains S^{-1} B and we may choose $B' \subset \hat{A}$ to be a B-algebra of finite type such that S^{-1} $B' \simeq \tilde{B}$.

Finally, it remains to prove that $\hat{A}_{\widehat{pA}}$ is a filtered inductive limit of smooth finite type A_p-algebras, $p \in \mathbf{M}$. Let $k \hookrightarrow K$ be the residue

field extension of $A_p \to \hat{A}_{\widehat{pA}}$ and $K' \subset K$ a finitely generated k-extension. K' is a separable extension of k (A universally japanese implies K/k separable). Choose $x = (x_1, \ldots, x_t) \in K''$ algebraically independent over k and $y \in K'$ algebraically separable over $k(x)$ such that $K' = k(x, y)$. Let $X = (X_1, \ldots, X_t)$, Y be variables and $F(Y) \in A_p[X, Y]$ be a polynomial, which lifts the irreducible polynomial of y over $k(x)$. Let $(\tilde{x}, \tilde{y})$ be a lifting of (x, y) to $\hat{A}_{\widehat{pA}}$. We have $F(\tilde{x}, \tilde{y}) \in p\hat{A}$ and $\frac{\partial F}{\partial Y}(\tilde{x}, \tilde{y}) \notin p\hat{A}$. $\hat{A}_{\widehat{pA}}$ being local artinian, there exists $y' \in \hat{A}_{\widehat{pA}}$ such that $F(\tilde{x}, y') = 0$ and $y' \equiv \tilde{y}$ mod $p\hat{A}$. Let q be the kernel of the map $\tau: A_p[X, Y] \to \hat{A}_{\widehat{pA}}$ given by $P \rightsquigarrow P(\tilde{x}, y)$. It results ht $(q) = 1$ because $q \subset (F) + pA_p[X, Y]$ and so $C_{K'} = A_p[X, Y]/q$ is a smooth A_p-algebra of finite type. Now we remark that $\hat{A}_{\widehat{pA}}$ is the reunion of $\{(C_{K'})_{,pC_{K'}} \mid K' \subset K$ and K'/k finite type, separable$\}$ and so $\hat{A}_{\widehat{pA}}$ is a filtered inductive limit of A_p-algebras of the type $(C_{K'})_g$, $g \notin pC_{K'}$.

3. PROOF OF THEOREM (1.3)

Let $F = (F_1, \ldots, F_m)$ be a system of m-polynomials in variables $Y = (Y_1, \ldots, Y_N)$ over $A := R\,[[X]]\,\langle T \rangle$ such that is has a solution $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ in $\hat{A} = R\,[[X, T]]$. Adding some polynomials to F we may suppose that F generates a prime ideal $\mathfrak{p}$ which is contained in the kernel of the morphism $\sigma: A\,[Y] \to \hat{A}$ given by $P \rightsquigarrow P(\bar{y})$ (for example let F be a system of generators of Ker σ). Denote $r = \text{ht}(\mathfrak{p})$. If $g = (g_1, \ldots, g_s)$ is a system of polynomials from $\mathfrak{p}$, then we consider the ideal $\Delta(g, \bar{y}) \subset \hat{A}$ generated by all $r \times r$-minors of the matrix $\left(\frac{\partial g}{\partial Y}(\bar{y})\right)$. Also we denote $\Delta(F, \bar{y})$ by $\Delta(\mathfrak{p}, \bar{y})$ or $\Delta(\mathfrak{p}, \sigma)$. We shall prove in some steps that F has a solution in A.

Step 1. Desingularization step. Reduction to the case ht $(\Delta(F, \bar{y})) \geqslant 2$

In order to get this reduction it is enough to apply the following lemma, which is in fact Proposition 3 from [13]*.

3.1. LEMMA. *Let A, A', $A \subset A'$ be noetherian factorial rings such that every prime element from A remains prime in A' and the extension $Q(A) \subset Q(A')$ is separable. Suppose A' is regular complete local ring and for every prime element q from A' the morphism $A_{qA' \cap A} \to A'_{qA'}$ is formally smooth. Then every morphism $\sigma : A[Y] \to A'$, $Y = (Y_1, \ldots, Y_N)$ can be extended to a morphism $\sigma' : A\,[Y, U] \to A'$, $U = (U_1, \ldots, U_{N'})$ such that for a prime ideal $\mathfrak{p}$, Ker $\sigma \subset \mathfrak{p} \subset$ Ker σ', ht$(\Delta(\mathfrak{p}, \sigma')) \geqslant 2$.*

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* See [13].

Indeed, we may apply (3.1) for $A' = \hat{A}$ (the hypothesis of (3.1) being fulfilled because A is an excellent henselian regular local ring) and we may suppose that F generates $\mathfrak{p}$.

Step 2. Application of Elkik's theorem

We choose an ideal $\mathfrak{a} \subset A$ of height 2 such that $\Delta(F, \bar{y}) \supset \mathfrak{a}\hat{A}$; for example we can take $\mathfrak{a} = A \cap \Delta(F, \bar{y})$, because for every prime ideal $\mathfrak{b} \subset \hat{A}$ of height two we have ht $(\mathfrak{b} \cap A) \geq 2$. Indeed, if $\mathfrak{b} \subset (p, X)\hat{A}$, then $\mathfrak{b} = (p, X)\hat{A}$ and we have $\mathfrak{b} \cap A = (p, X)$, p being a local parameter of R. If $\mathfrak{b} \not\subset (p, X)\hat{A}$, then $\mathfrak{b}$ contains a T-regular power series h. Thus the canonical map $R[[X]] \to \hat{A}/(h)$ is finite by Weierstrass Preparation Theorem and $\mathfrak{b}$ must have a nonzero intersection with $R[[X]]$ because of ht $(\mathfrak{b}/(h)) = 1$. Let $\tilde{h}$ be the monic polynomial from $R[[X]][T]$ which is a multiple of h. We have ht $(\tilde{h}, \mathfrak{b} \cap R[[X]]) = 2$ and so ht $(\mathfrak{b} \cap A) = 2$.

Now, let $v_1, \ldots, v_t$ be a system of generators of $\mathfrak{a}$ and choose some $r \times r$-minors ($r =$ ht $\mathfrak{p}$) of the matrix $\left(\frac{\partial F}{\partial Y}\right)$, $M_1 \ldots, M_s$ and some elements $\bar{u}_{ij} \in \hat{A}$ such that

$$v_i = \sum_{j=1}^{s} M_j(\bar{y})\, \bar{u}_{ij}, \quad i = 1, \ldots, t.$$

Consider the following system of equations over A

$$(+) \begin{cases} F = 0 \\ \sum_{j=1}^{s} M_j(Y)\, U_{ij} = v_i, \end{cases}$$

where $U = (U_{ij})$ are some variables. Let $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that Elkik's theorem holds (cf. [6]) in A with respect to $\mathfrak{a}$ and the system $F = 0$ (we may suppose $\nu(c, m) \geq 2$ for all c, m), i.e. if $F(\tilde{y}) \equiv 0$ mod $\mathfrak{a}^{\nu(c,m)}$ and $\Delta(F, \tilde{y}) \supset \mathfrak{a}^m$ for some $\tilde{y} \in A^N$ then there is a $y \in A^N$, $y \equiv \tilde{y}$ mod $\mathfrak{a}^c$ and $F(\tilde{y}) = 0$. The pair $(\bar{y}, \bar{u})$ is a formal solution of $(+)$ and induces for any $c \geq 1$ a formal solution of $(+)$ in the ring $\hat{A}/\mathfrak{a}^{\nu(c,1)}$ $\hat{A} \cong \widehat{(A/\mathfrak{a}^{\nu(c,1)})}$. We remark that $A/\mathfrak{a}^{\nu(c,1)}$ is an **AP**-ring (cf. (1.1)) and thus $(\bar{y}, \bar{u})$ can be approximated modulo $(p, X, T)^c$ by a solution of $(+)$ in $A/\mathfrak{a}^{\nu(c,1)}$, i.e. there exists $(\tilde{y}, \tilde{u})$ in A such that

$$\begin{cases} F(\tilde{y}) \equiv 0 \\ \sum_{j=1}^{s} M_j(\tilde{y})\, \tilde{u}_{ij} \equiv v_i \end{cases} \mod \mathfrak{a}^{\nu(c,1)}$$

and $\tilde{y} \equiv \bar{y}$, $\tilde{u} \equiv \bar{u}$ mod $(p, X, T)^c$. Consequently, there exist $d_{ik} \in \mathfrak{a}$, $k = 1, \ldots, t$ such that

$$\sum_{j=1}^{s} M_j(\tilde{y})\, \tilde{u}_{ij} = v_i + \sum_{k=1}^{t} d_{ik}\, v_k$$

and so $v_i \in \Delta(F, \tilde{y})$ for all i. Thus we have

$$F(\tilde{y}) \equiv 0 \mod \mathfrak{a}^{\nu(c,1)}$$

$$\Delta(F, \tilde{y}) \supset \mathfrak{a}.$$

Applying Elkik's theorem we get a solution $y \in A^N$ for F such that $y \equiv y$ mod $\mathfrak{a}^c$ and especially $y \equiv \bar{y}$ mod $(p, X, T)^c$.

3.3. REMARK. One can prove Theorem (1.8) with the same methods used in the proof of (1.3). Questions could only arise in first part of Step 2, but also here one can use the same idea because $K\{T\}[[X, Z]] \supset K\langle T\rangle [[X, Z]] \supset K[[X, Z]]\langle T\rangle$.

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