EXCELLENT HENSELIAN RINGS NOT CONTAINING THEIR RESIDUE FIELD

BY

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In [2] an example of a henselian discrete valuation ring not containing its residue field is given. From this construction it is not clear how to choose such a ring to be also excellent. Ch. Rotthaus developed in [4] the construction of excellent discrete valuation rings. We can also use these ideas for our example (see Section 1). In Section 2 we construct some excellent rings which do not have sufficiently many derivations.

1. We start with the prime field $\mathbf{F}_p$ of characteristic $p > 0$.

Let $\{X, Y, X_{ij}; i, j \in \mathbf{N}\}$ be a family of indeterminates ($\mathbf{N}$ the natural numbers). Let $L = \mathbf{F}_p(X_{ij}; i, j \in \mathbf{N})$ and $K = L(X^{p^{-n}}; n \in \mathbf{N})$. We consider the elements $r_i = \sum_{j \geq 1} X_{ij} Y^{j!} \in K[[Y]]$, where $i \in \mathbf{N}$. The set of the r_i is algebraically independent over $K(Y)$ (cf. [4], p. 123).

1.1. *Let A be the algebraic closure of $A_0 = L[Y, X^{p^{-n}} + r_n; n \in \mathbf{N}]$ in $K[[Y]]$. Then A is a henselian, discrete valuation ring with completion $K[[Y]]$* (cf. [1] or [2]), *and has the following properties*

i) *the residue field K of A cannot be embedded into A;*

ii) *A is excellent.*

Proof. (i) We first remark that $X \notin A$. ($X \in A$ would mean that X is algebraic over A_0, and this would give an algebraic relation between some of the r_i over $K(Y)$.) If $h: K \to A$ would be an embedding of K in A, then $X^{p^{-n}} - h(X^{p^{-n}}) = Y a_n$, with $a_n \in A$, $n \in \mathbf{N}$. Hence $a_n = Y^{p-1} a_{n+1}$, i.e. $a_0 \in \bigcap_{n \geq 1} Y^n A = (0)$. It would follow that $X \in A$.

(ii) For any $t \in \mathbf{N}$, let $L_t = L(X^{p^{-t}})$ and A_t be the algebraic closure of $A_{0t} = L[Y, X^{p^{-n}} + r_n; n = 0, \ldots, t]$ in $L_t[[Y]]$. Then A_t is a discrete valuation ring with completion $L_t[[Y]]$ (cf. [1] or [2]), and $A = \varinjlim A_t$. In 1.2 we prove that A_t is excellent, i.e. $Q(A_t) \to L_t((Y))$ is separable. The ring $B = \varinjlim L_t[[Y]]$ is a discrete valuation ring, with residue field K and completion $K[[Y]]$. In 1.3 we prove that B is excellent, i.e. $Q(B) \to K((Y))$ is separable. The inclusion $A \to B$ induces the equality $\hat{A} = \hat{B}$. The field extension $Q(A) \to Q(B)$, which is $\varinjlim (Q(A_t) \to L_t((Y)))$, is still separable. Hence $Q(A) \to Q(\hat{A})$ is separable, i.e. A is excellent.

1.2 *The ring A_t (defined above) is excellent.*

Indeed, let $s_i = X^{p^{-i}} + r_i$ with $i = 0, \ldots, t$ and $\tilde{r}_{i+1} = r_i - r_t^{p^{t-i}} = s_i - s_t^{p^{t-i}}$ with $i = 0, \ldots, t-1$. Then $A_{0t} = L[Y, \tilde{r}_1, \ldots, \tilde{r}_t, s_t]$. The morphism of $L[Y, \tilde{r}_1, \ldots, \tilde{r}_t]$-algebras $f: A_{0t} \to L[Y, \tilde{r}_1, \ldots,$

$\ldots, \tilde{r}_t, X^{p^{-t}}]$, $f(s_t) = X^{p^{-t}}$ is an isomorphism. It can be extended to an isomorphism of $L[[Y]]$-algebras $g: L_t[[Y]] \to L_t[[Y]]$, $g(X^{p^{-t}}) = X^{p^{-t}} - r_t$. Hence A_t is isomorphic with the algebraic closure C_t of $L[Y, \tilde{r}_1, \ldots, \tilde{r}_t, X^{p^{-t}}]$, hence of $L_t[Y, \tilde{r}_1, \ldots, \tilde{r}_t]$, in $L_t[[Y]]$. The ring C_t is excellent if the field extension $L_t(Y, \tilde{r}_1, \ldots, \tilde{r}_t) \xrightarrow{a} L_t((Y))$ is separable. The extension $L_t(Y) \to L_t((Y))$ is separable, and $\tilde{r}_{t+1} \notin (L_t((Y)))^p L_t(Y, \tilde{r}_1, \ldots, \tilde{r}_i) = M_i$ with $i = 0, \ldots, t-1$. Indeed, $\tilde{r}_{i+1} = r_i - r_t^{p^{t-i}} \notin M_i$ iff $r_i \notin M_i$. Since $M_i = (L_t((Y)))^p L_t(Y, r_0, \ldots, r_{i-1})$, as in [4], p. 124 it follows that $r_i \notin M_i$. Hence the extension a is separable.

1.3 PROPOSITION. *Let L be a field, X and Y indeterminates, $L_n = L(X^{p^{-n}})$ with $n \in \mathbb{N}$ and $B = \varinjlim L_n[[Y]]$. Then B is an excellent ring.*

Proof. Let $K = \varinjlim L_t = L(X^{p^{-n}};\ n \in \mathbb{N})$. Then B is a discrete valuation ring, with residue field K and completion $C = K[[Y]]$. Hence B is excellent iff $M = K((Y))$ is separable over the field of fractions $N = \bigcup L_t((Y))$ of B. We prove that $N^{1/p}$ and M are linearly disjoint over N. Let $b_1, \ldots, b_r \in N^{1/p}$ be linearly independent over N. If $0 \neq (\bar{t}_1, \ldots, \bar{t}_r) \in M^r$ and $\sum_{i=1}^{r} b_i t_i = 0$, let m be s.t. $a_i = b_i^p \in L_m((Y))$. We can suppose that $a_i \in L_0[[Y]]$ and $t_i \in K[[Y]]$. Hence it suffices to prove that there exists $0 \neq (t_1, \ldots, t_r) \in L_1[[Y]]$ with $\sum_{i=1}^{r} a_i t_i^p = 0$. To obtain it, let $a_i = \sum_{j<p} a_{ij} Y^j$ with $a_{ij} \in L_0[[Y^p]]$ and $G_j = \sum_{i=1}^{r} a_{ij} T_i^p$. Then $G_j(\bar{t}) = 0$. We prove that $G = \{G_0, \ldots, G_{p-1}\}$ has a nontrivial solution in $L_1[[Y]]$. Let $c = 1 + \max\{\mathrm{ord}_Y \bar{t}_i;\ \bar{t}_i \neq 0\}$. Let θ be a SAE-function (see [1]) of G over $L_1[[Y]]$; then $d := \theta(c) \geq c$. It suffices to show that G has a solution t' mod Y^d in $L_1[[Y]]$, which is nontrivial mod Y^c. Let $\bar{t}_i = \bar{t}_i' + Y^d \bar{t}_i''$ with $\bar{t}_i' \in L_n[Y]$ for some $n \geq 1$. Then $\bar{t}' := (\bar{t}_1', \ldots, \bar{t}_r') \neq 0$ and $G(\bar{t}') \equiv 0$ mod Y^{pd}. Let $a_{ij} = a_{ij}' + Y^{pd} a_{ij}''$ with $a_{ij}' \in L_0[Y^p]$. Then $\sum_i a_{ij}' \bar{t}_i'^p = Y^{pd} c_j'$ with $c_j' \in L_{n-1}[Y^p]$. If $\{e_h;\ h \in H\}$ is a basis of L over L^p, let $a_{ij}' = \sum_{h=1}^{s} a_{ijh}' e_h$, $a_{ijh}' \in L^p(X)[Y^p]$ and $c_j' = \sum_{h=1}^{s} c_{jh}' e_h$, $c_{jh}' \in L^p(X^{p^{-n+1}})[Y^p]$. Then $\sum_i a_{ijh}' \bar{t}_i'^p = Y^{pd} c_{jh}'$. Let $a_{ijh} \in L_1[Y]$ with $a_{ijh}^p = a_{ijh}'$ and $c_{jh} \in L_n[Y]$ with $c_{jh}^p = c_{jh}'$. Hence $\sum_i a_{ijh} \bar{t}_i' = Y^d c_{jh}$.

Let $v_1, \ldots, v_m$ be a basis of L_n over L_1 and let $t_i' = \sum_{k=1}^{m} t_{ik} v_k$ and $c_{jh} = \sum_{k=1}^{m} c_{jhk} v_k$ with $t_{ik}, c_{jhk} \in L_1[[Y]]$. Then $\sum_i a_{ijh} t_{ik} = Y^d c_{jhk}$. Since $\mathrm{ord}_Y \bar{t}_i' \leq c-1$ if $\bar{t}_i' \neq 0$, there exist $k \in \{1, \ldots, m\}$ s.t. $t' = (t_{1k}, \ldots, t_{rk}) \neq 0$ and $\mathrm{ord}_Y t_{ik} \leq c - 1$ for some $i \in \{1, \ldots, r\}$. It follows that $G(t') \equiv 0$ mod Y^{pd} and $t' \neq 0$ mod Y^c.

REMARK. If $[L : L^p] = \infty$, it is known that $K^p[[Y]][K] = \varinjlim k[[Y]]$, where k runs over the finite extensions of K^p in K, is a discrete valuation ring, which is *not excellent* (cf. Nagata, Local Rings, E3).

2. In [3] a list of henselian discrete valuation rings with residue class field $\mathbb{C}$ (particularly, they have the approximation property), having various small groups of automorphisms, independent from the modules of derivations is given. Here we shall determine the module of derivations for positive characteristics and for some two-dimensional henselian regular local rings.

If B is an A-algebra, we denote by $\mathrm{Der}_A(B)$ the module of A-derivations of B in B. Let $L \subseteq K$ be an algebraic, separable extension of fields of characteristic p.

2.0. If V is a K-algebra and discrete valuation ring of parameter T and residue class field K (i.e. $\hat{V} = K[[T]]$), then $\mathrm{Der}_L(V) = Ve\dfrac{\mathrm{d}}{\mathrm{d}T}$, where if $\dfrac{\mathrm{d}}{\mathrm{d}T}(V) \subseteq V$, then $e = 1$; else $e = 0$.

If $p = 0$, see [3] for example.

If $p > 0$, let K be the field from [4], II, $u = \sum_{i \geq 1} A_i T^{i!}$, $v = \sum_{i \geq 1} B_i T^{i!}$ and w be either u or $Tu + v$. Then the algebraic (a fortiori separable) closure V of $K[T, w]$ in $K[[T]]$ is an excellent henselian discrete valuation ring, by [3], 1.1 and [4], p. 124. If $w = u$, then $\dfrac{\mathrm{d}w}{\mathrm{d}T} \in K[T]$, hence $\mathrm{Der}_L(V) = V\dfrac{\mathrm{d}}{\mathrm{d}T}$. If $w = Tu + v$ then $\dfrac{\mathrm{d}w}{\mathrm{d}T} - u \in K[T]$; hence $\dfrac{\mathrm{d}w}{\mathrm{d}T} \notin V$, since u and v are algebraically independent over $K(T)$.

2.1. Let X and Y be two indeterminates and $E = K[[X, Y]]$. Let $g = \{g_i;\ i \in I\} \subseteq K[[X]] = E'$ and $h = \{h_j;\ j \in J\} \subseteq K[[Y]] = E''$. Let R be the (separable) algebraic closure of $R' = K[X, g]$ in E' and A the (separable) algebraic closure of $A' = R[Y, h]$ in E. Let S (resp. B) be the (separable) algebraic closure of $S' = K[Y, h]$ (resp. $S[X, g]$) in E'' (resp. E). Then $A = B$. If $p > 0$, we take the separable algebraic closure. Propositions 1.1 and 1.2 from [3] hold also in this case.

The rings R, S are discrete valuation rings, by [3], 1.1. Hence $\mathrm{Der}_L(R) = Re'\dfrac{\mathrm{d}}{\mathrm{d}X}$ and $\mathrm{Der}_L(S) = Se''\dfrac{\mathrm{d}}{\mathrm{d}Y}$, with $e', e'' = 0, 1$.

Moreover

i) *the ring A is local, noetherian, regular, henselian and $\hat{A} = K[[X, Y]]$.*

ii) $\mathrm{Der}_L(A) = Ae'\dfrac{\partial}{\partial X} \oplus Ae''\dfrac{\partial}{\partial Y}$.

Indeed, the first assertion follows directly from [3], 1.2.

In order to prove ii), observe that $\dfrac{\mathrm{d}}{\mathrm{d}X}(R) \subseteq R$ iff $\dfrac{\partial}{\partial X}$ [illegible] same assertion holds for Y and S. Indeed, let [illegible]

$\frac{\partial c}{\partial X} \in A'$. If $t \in A$, then there exists $P(Z) \in A'[Z]$ with $P(t) = 0$ and $\frac{dP}{dZ}(t) = 0$; hence $\frac{\partial t}{\partial X} \in A$. Conversely, let $\frac{\partial}{\partial X}(A) \subseteq A$. If $t \in R$, then $f = \frac{\partial t}{\partial X} \in A \cap E'$. Since $f \in A$, there exists $P(Z) = c_n Z^n + \ldots + c_0 \in A'[Z]$ with $c_i \in A'$ and $P(f) = 0$. Since A is regular, we can suppose that Y is not a common divisor of $c_0, \ldots, c_n$ in A (changing c_i in A). Since $A/YA = R$, we write $c_i = d_i + Ye_i$, with $d_i \in R$, $e_i \in A$ and $i = 0, \ldots, \ldots, n$. Then there is a j s.t. $d_j \neq 0$. From $0 = P(f)$ and $f \in E'$, it follows that $d_n f^n + \ldots + d_0 = 0$. If $d_n = \ldots = d_1 = 0$, then $d_0 = 0$. Hence f is algebraic (and separable) over R.

Since $\hat{A} = E$, then $\mathrm{Der}_L(A) \subseteq \mathrm{Der}_L(E) = E\frac{\partial}{\partial X} \oplus E\frac{\partial}{\partial Y}$. Let $D \in \mathrm{Der}_L(A)$, hence $D = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$, with $a = D(X)$, $b = D(Y) \in A$. Let $D \neq 0$. Suppose $\frac{\partial}{\partial Y} \in \mathrm{Der}_L(A)$. If $a \neq 0$, then $\frac{\partial}{\partial X} \in \mathrm{Der}_L(A)$, since $D(f) \in A$ for any $f \in A$. If $D(X) = 0$ for any $D \in \mathrm{Der}_L(A)$, then $\mathrm{Der}_L(A) = A\frac{\partial}{\partial Y}$. Suppose $\frac{\partial}{\partial Y} \notin \mathrm{Der}_L(A)$; then there is a $f \in S$ $(\subseteq E'')$ s.t. $\frac{df}{dY} \notin S$, i.e. $\frac{\partial f}{\partial Y} \notin A$. Since $b\frac{\partial f}{\partial Y} = D(f) \in A$, it follows that $b = 0$. Since $a \neq 0$, it results that $\frac{\partial}{\partial X} \in A$; hence $\mathrm{Der}_L(A) = \frac{\partial}{\partial X}$.

REMARK. Let g, R be as above and $h = \{h_j(X, Y); j \in J\} \subseteq E$. If A is the (separable) algebraic closure of $R[Y, h]$ in E, we can find an R' in E', defined as R, and $h' \subseteq E''$ s.t. A is also the (separable) algebraic closure of $R'[X, h']$ in E.

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