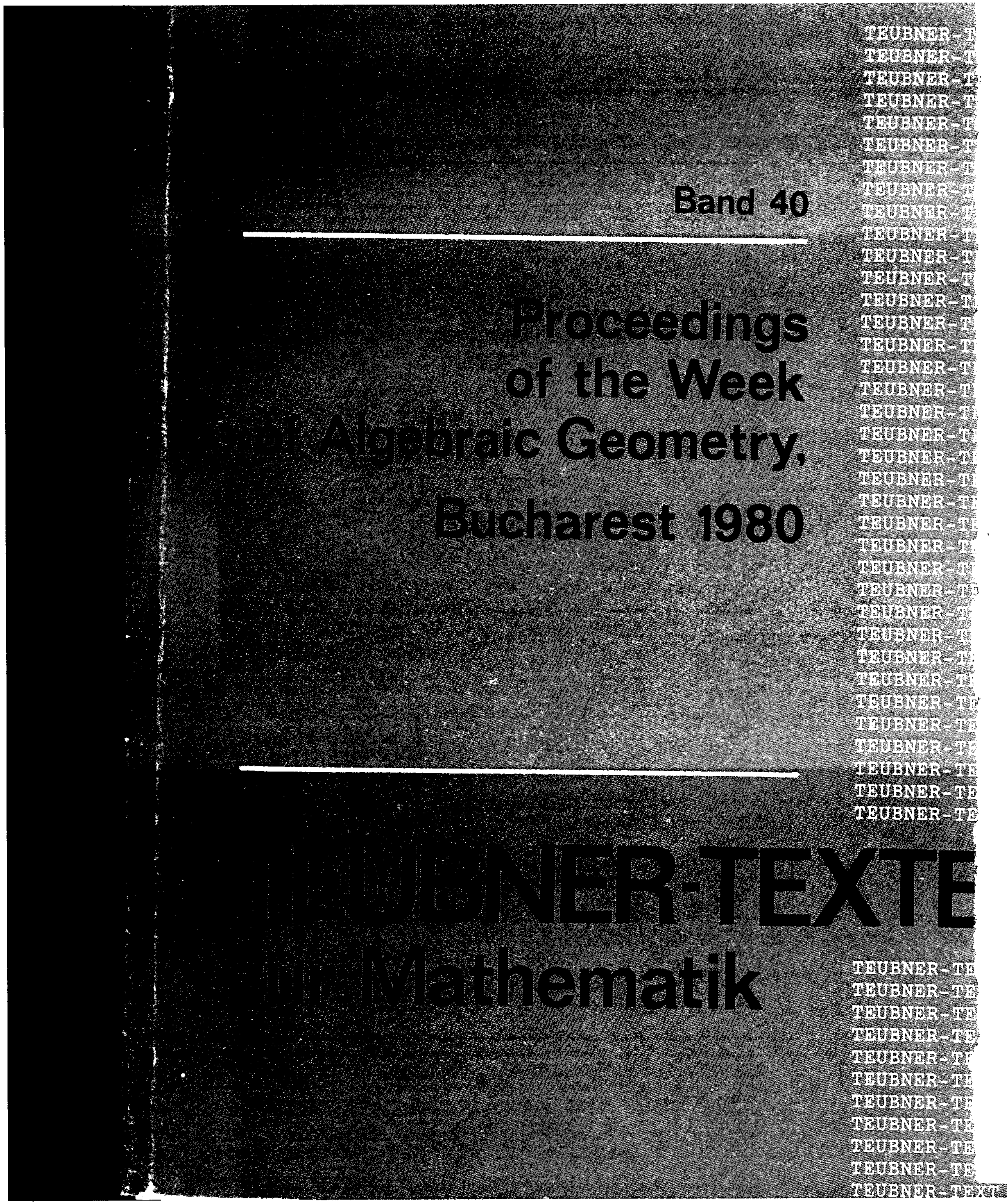

INVARIANTS OF SINGULARITIES AND NEWTON POLYHEDRON

Gerhard Pfister
Berlin, GDR

The idea of my talk is to explain with some examples how to compute discrete invariants of singularities using the Newton polyhedron. We will restrict ourselves to the hypersurface case and mainly to the results of Danilov ([2],[3]), Kouchnirenko ([8]) and Varchenko ([13]).

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial, $f(0) = 0$ and 0 an isolated singular point of the set of zeros of f.
Let $T = \{t \in \mathbb{C},\ |t| < \delta\}$, $B = \{x \in \mathbb{C}^n,\ |x| < \varepsilon\}$ for suitable δ, ε $(0 < \delta \ll \varepsilon \ll 1)$ and denote $B \cap f^{-1}(T \setminus \{0\})$ resp. $B \cap f^{-1}(t)$ by X and X_t, respectively. We know (Milnor) that $f : X \longrightarrow T \setminus \{0\}$ is a locally trivial C^∞-fibration and the fibre X_t is a bouquet of μ $(n-1)$-spheres.
We are interested in the following invariants:

- The <u>Milnor number</u> $\mu = \dim H^{n-1}(X_t, \mathbb{C})$,
- the characteristic polynomial resp. the ζ-function of the monodromy $h : H^*(X_t, C) \to H^*(X_t, C)$ (induced by the action of $\pi_1(T \setminus \{0\})$), $\zeta_f(z) = (1-z) \det(Id - zh)^{(-1)^{n-1}}$ (because, for isolated singularities, $H^q(X_t, C) = 0$ if $q \neq 0, n-1$).
- the <u>Hodge-numbers</u> of the Milnor fibre X_t and especially the intersection form on $H_c^{n-1}(X_t, R)$ (we consider $H_c^{n-1}(X_t)$ as the dual space of $H^{n-1}(X_t)$ via the perfect pairing $(\xi, \eta) = \int_{X_t} \xi \wedge \eta$; the intersection form is defined by $S(\xi, \eta) = (\xi, j(\eta))$ and $j : H_c^n(X_t) \to H^n(X_t)$ the canonical map),
- the dimension m(f) of the μ-constant stratum of the semi-universal deformation of f (<u>the modality</u>)

1. The Newton Polyhedron

For simplicity we suppose f to be convenient, i. e. $f(0,\dots,x_i,0,\dots) \neq 0$ for all i.
Let Γ_+ be the convex hull in $\mathbb{R}^n_+$ of $\bigcup_{k\in \operatorname{supp} f} (k+\mathbb{R}^n_+)$, supp f is the set of all indices of $\mathbb{Z}^n$ such that the corresponding co-efficient of f is different from 0.
Let Γ_f be the union of compact faces of Γ_+. We call Γ_f the Newton boundary of the Newton polyhedron Γ_+ of f.
In order to get nice results we add some more assumptions on f.
Let f_o be the principal part of f, i.e. if $f = \sum a_k x^k$ (k a vector of n positive integers, $x = (x_1,\dots,x_n)$ and $x^k = x_1^{k_1} \dots x_n^{k_n}$), then $f_o = \sum_{k\in\Gamma_f} a_k x^k$. f is said to be

non-degenerate if for any face $\Delta \subset \Gamma_f$ the restrictions of $x_i \frac{\partial f_o}{\partial x_i}$ to Δ have no common zero in $(\mathbb{C}\setminus\{0\})^n$ (equivalent to this condition is to require that the corresponding ideal in the conic ring $\mathbb{C}[P_\Delta]$, which is generated by all monomials x^k with k in the cone P_Δ defined by Δ, is primary to the ideal $(x)\cap\mathbb{C}[P_\Delta]$, cf. [8]).
Now let us suppose f to be non-degenerate. To the Newton polyhedron of f we will now associate the Newton number (Kouchnirenko [8]) and the ζ-function ζ_{Γ_f}(Varchenko [14]):

- $\nu_{\Gamma_f} = \sum_{k=1}^{n} (-1)^{n-k} k!\, V_k + (-1)^n$, V_n the k-dimensional volume of the compact polyhedron Γ_- behind Γ_f (the cone over Γ_f with the origin as a vertex). V_k is the sum of k-dimensional volumes of $\Gamma_- \cap L_I$ with $I \subseteq \{1,\dots,n\}$, $\# I = k$ and $L_I = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n_+,\ x_i = 0 \text{ if } i \notin I\}$.

- $\zeta_{\Gamma_f}(z) = \prod_{1} \zeta^l(z)^{(-1)^{l-1}}$, $\zeta^l(z) = \prod_{\substack{I\subset\{1,\dots,n\}\\ \# I = l}} \prod_{j=1}^{j(I)} (1-z^{m_j(I)})^{(l-1)!\,v(\Gamma_j(I))}$

  with $j(I)$, $m_j(I)$ and $v(\Gamma_j(I))$ defined as follows:
  Let $\Gamma_1(I),\dots,\Gamma_{j(I)}$ be the (l-1)-dimensional faces of

$\Gamma_f \cap L_I$ and $\sum_{i\in I} a_{ij} x_j = m_j(I)$ the unique equations defining the hyperplanes $L_j \supseteq \Gamma_j(I)$ (unique by requiring a_{ij}, $m_j(I)$ to be integers with greatest common divisor 1 and $m_j(I)$ positive). ($\Gamma_i(I)$) is the (l-1)-dimensional volume of $\Gamma_j(I)$ in L_j (requiring that the (l-1)-dimensional cube defined by a base of the lattice $L_j \cap \mathbb{Z}^n$ has volume 1).

The following theorem holds:

<u>Theorem:</u> (1) $\mu = \nu_{\Gamma_f}$ (Kouchnirenko, [8])

(2) $\zeta_f = \zeta_{\Gamma_f}$ (Varchenko, [13])

For more general statements and results (especially if f is degenerate) cf. [8], [14]. For corresponding results for complete intersections cf. [5], [6], [7], and [9].
Different proofs of the theorem of Kouchnirenko were given by Oka cf. [10] and in [9], too.
There are also some similar results for the case of non-isolated singularities given by Oka (cf. [11]).

2. Idea of the Proof of Theorem (1)

The idea of Kouchnirenko is to consider the Milnor number as the dimension of the vector space $\mathbb{C}[[x_1,\dots,x_n]]/(\partial f/\partial x_1,\dots,\partial f/\partial x_n)$, to connect this vector space with $\mathbb{C}[[x_1,\dots,x_n]]$, which is easier to handle, via an exact sequence - the Koszul complex $C^*(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})$ - and to choose a good filtration on $\mathbb{C}[[x_1,\dots,x_n]]$ which is related to the Newton boundary and allows to compute the dimensions of the graded modules in the sequence in terms of the Newton boundary.
More precisely, choose a homogeneous map $h : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ such that $h(\Gamma_f) = 1$ (if $\beta_1,\dots,\beta_r$ are vectors in the normal direction to the compact faces $\Delta_1,\dots,\Delta_r$ of maximal dimension

such that $\langle \alpha, \beta_i \rangle = 1$, if $\alpha \in \Delta_i$, then $h(\alpha) = \min_j \langle \alpha, \beta_j \rangle$.
Choose a positive integer N (minimal) such that $N \cdot h \mid \mathbb{Z}^n =: \phi$ has integral values. ϕ defines a filtration on $\mathbb{C}[\![X_1,\ldots,X_n]\!]$:

$$A_q = \{ g \in \mathbb{C}[\![X_1,\ldots,X_n]\!], \quad \phi(\operatorname{supp} g) \geq q \} .$$

One can easily check that $A_{q_1} \cdot A_{q_2} \subseteq A_{q_1+q_2}$, $\bigcap_{q \geq 0} A_q = 0$ and for all integers n there are $q(n)$ such that $(X_1,\ldots,X_n)^n \subseteq A_{q(n)}$.
Let us denote the corresponding graded ring by $\bar{A}$.
The multiplication in $\bar{A}$ can be described by using the cones P_Δ defined by the faces $\Delta \subseteq \Gamma_f$ in the following way:

$$\bar{X}^{n_1} \cdot \bar{X}^{n_2} = \begin{cases} \bar{X}^{n_1+n_2} & \text{if } n_1, n_2 \in P_\Delta \text{ for a } \Delta \subseteq \Gamma_f \\ 0 & \text{otherwise} \end{cases}$$

(because $\phi(n_1+n_2) = \phi(n_1) + \phi(n_2)$ iff $n_1, n_2 \in P_\Delta$ for a suitable $\Delta \subseteq \Gamma_f$).
So the restriction to the cone P_Δ yields a canonical map $\pi_\Delta : \bar{A} \longrightarrow A_\Delta = \mathbb{C}[P_\Delta]$.
Moreover, the restrictions to the cones, equipped with a suitable sign, give us the following exact sequence

$$0 \to \bar{A} \to \bigoplus_{\Delta \in \Psi_m} A_\Delta \to \bigoplus_{\Delta \in \Psi_{m-1}} A_\Delta \to \ldots \to \bigoplus_{\Delta \in \Psi_1} A_\Delta \to 0,$$

$$\Psi_k = \{ \Delta ,\ \Delta \text{ face of } \Gamma,\ \dim \Delta = k-1,\ \Delta \not\subseteq \text{ coordinate hyperplane} \} .$$

This sequence allows to express the dimension of the graded components of $\bar{A}$ via those of the A_Δ, i.e. in terms of the Newton boundary.
The Koszul complex now connects $\mathbb{C}[\![X_1,\ldots,X_n]\!]/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ with $\mathbb{C}[\![X_1,\ldots,X_n]\!]$.
There arises a problem, as the corresponding graded Koszul complex

is not exact (because the $\frac{\partial f}{\partial x_i}$ are not on the Newton boundary).
The idea of Kouchnirenko is to first consider $f_i := X_i \frac{\partial f}{\partial x_i}$ instead of $\frac{\partial f}{\partial x_i}$.
Knowing that $\dim \mathbb{C}[\![X_1,\ldots,X_n]\!]/(X_1 \frac{\partial f}{\partial x_1}, \ldots, X_n \frac{\partial f}{\partial x_n}) = n!\, V_n$,
one can compute the Milnor number by induction using the fact that $\dim \mathbb{C}[\![X_1,\ldots,X_n]\!]/(g_1 \cdot g_1', g_2, \ldots, g_n) = \dim \mathbb{C}[\![X_1,\ldots,X_n]\!]/(g_1,\ldots,g_n) + \dim \mathbb{C}[\![X_1,\ldots,X_n]\!]/(g_1', g_2, \ldots, g_n)$:

$$\dim \mathbb{C}[\![X_1,\ldots,X_n]\!]/(X_1 \frac{\partial f}{\partial x_1}, \ldots, X_n \frac{\partial f}{\partial x_n}) = \sum_{I \subseteq \{1,\ldots,n\}} \mu_I,$$

$$\mu = \mu(f), \quad \mu_{\{1,\ldots,n\}} = 1, \quad \mu_I = \mu(f_I) \quad \text{and } f_I = \sum_{\substack{n \in \operatorname{supp} f \\ n_\alpha = 0,\ \alpha \in I}} a_n X^n .$$

Now let us consider the Koszul complexes

$C^*(f_1,\ldots,f_n, \mathbb{C}[\![X_1,\ldots,X_n]\!])$, $C^*(\bar{f}_1,\ldots,\bar{f}_n, \bar{A})$,

$C^*(f_{1_\Delta},\ldots,f_{n_\Delta}, A_\Delta)$, $\bar{f}_i$ resp. f_{i_Δ} the inertial form of f_i in $\bar{A}$ and A_Δ, respectively.
If the Koszul complex $C^*(\bar{f}_1,\ldots,\bar{f}_n, \bar{A})$ is acyclic in positive dimension, then the exact sequence

$$C^*(f_1,\ldots,f_n, \mathbb{C}[\![X_1,\ldots,X_n]\!]) \longrightarrow \mathbb{C}[\![X_1,\ldots,X_n]\!]/(f_1,\ldots,f_n) \longrightarrow 0$$

gives us, by passing to the graduations, the exact sequence

$$C^*(\bar{f}_1,\ldots,\bar{f}_n, \bar{A}) \longrightarrow \bar{A}/(\bar{f}_1,\ldots,\bar{f}_n) = gr(\mathbb{C}[\![X_1,\ldots,X_n]\!]/(f_1,\ldots,f_n)) \longrightarrow 0.$$

To prove that $C^*(\bar{f}_1,\ldots,\bar{f}_n, \bar{A})$ is exact is one of the main steps in Kouchnirenko's proof. The idea is to compare this Koszul complex with $C^*(f_{1_\Delta},\ldots,f_{n_\Delta}, A_\Delta)$. A_Δ is Cohen Macaulay and $f_{1_\Delta},\ldots,f_{n_\Delta}$ generate an m_Δ-primary ideal (f is non-degenerate) so that the last Koszul complex is acyclic in dimension $n - \dim A_\Delta$. A diagram chase shows that

$C^*(\bar f_1,\dots,\bar f_n,\bar A)$ is acyclic.

To compute the dimension $\dim \mathbb{C}[[X_1,\dots,X_n]]/(f_1,\dots,f_n) =$
$= \dim gr\,(\mathbb{C}[[X_1,\dots,X_n]]/(f_1,\dots,f_n)) = \sum_k \dim gr_k\, \bar A/(\bar f_1,\dots,\bar f_n)$,
it is useful to consider the Poincaré series ($\bar M = \bigoplus M_q \rightsquigarrow$
$P_{\bar M}(T) = \sum_q (\dim_{\mathbb{C}} M_q)\, T^q$; $P_{\bar M[\ell]} = T^l P_{\bar M}$;
$0 \to \bar M' \to \bar M \to \bar M'' \to 0$ exact with maps of degree zero implies
$P_{\bar M'} - P_{\bar M} + P_{\bar M''} = 0$).

Using the exact sequence given by the graded Koszul complex we get

$$P_{\bar A/(\bar f_1,\dots,\bar f_n)}(T) = P_{\bar A}(T)(1-T^M)^n .$$

Using the exact sequence given by the conic rings A_Δ we get

$$P_{\bar A}(T) = \sum_{\substack{\Delta \subseteq \Gamma_f \\ \Delta \not\subset \text{coordinate} \\ \text{hyperplane}}} (-1)^{n-\dim A_\Delta} P_{A_\Delta}(T).$$

For a face $\Delta \subseteq \Gamma_f$ which is a simplex one can show that

$$P_{A_\Delta}(T) = \frac{1}{(1-T^M)^{\dim A_\Delta}} \sum_{j \in P(\Delta) \setminus \bigcup_{q=1}^{\dim A_\Delta} (s_q + P(\Delta))} T^{\alpha_j}$$

(s_q the vertexes of Δ).

This means that for any face Δ

$$P_{A_\Delta}(T)\,(1-T^M)^{\dim A_\Delta}\Big|_{T=1} = (\dim A_\Delta)!\, V_\Delta ,$$

the volume of the cone over Δ with vertex 0.

Finally we get

$$\dim_{\mathbb{C}} \mathbb{C}[[X_1,\dots,X_n]]/(f_1,\dots,f_n) = P_{\bar A/(\bar f_1,\dots,\bar f_n)}(T)\Big|_{T=1}$$
$$= \sum_{\Delta \in \Upsilon_n} P_{A_\Delta}(T)(1-T^M)^n\Big|_{T=1}$$
$$= n!\, V_n ,$$

which proves Kouchnirenko's theorem.

Another possibility to prove Kouchnirenko's theorem is to consider the Milnor number as the dimension of $\Omega_f^{n-1}/d\,\Omega_f^{n-2} + f\Omega_f^{n-1}$, $\Omega_f^p = \Omega^p/df \wedge \Omega^{p-1}$ the module of relative differentials (cf. [9]).

This gives us the possibility to generalize Kouchnirenko's result because Greuel proved that in the case of complete intersections, too, one gets the Milnor number as the dimension of this vector space: Let $\underline{f} = (f_1,\dots,f\ )$ be a complete intersection, f_i algebraic power series in $x_1,\dots,x_n$ having an isolated singularity at 0, then the Milnor number $\mu(f_1,\dots,f_k) = \dim_{\mathbb{C}} \Omega_{\underline f}^{n-k} / d\Omega_{\underline f}^{n-k-1} + \sum_{j=1}^{k} f_j\, \Omega_{\underline f}^{n-k}$, $\Omega_{\underline f}^p = \Omega^p / df_1 \wedge \Omega^{p-1} + \dots + df_k \Omega^{p-1}$.

Greuel (cf. [5]) also gave the exact sequences which connect the above module with the free modules Ω^p :

$$0 \to \Omega_{\underline f'}^0 \xrightarrow{\wedge df_k} \Omega_{\underline f'}^1 \to \dots \to \Omega_{\underline f'}^p \to \Omega_{\underline f}^p \to 0$$

$p \le n-k$ and $\underline f' = (f_1,\dots,f_{k-1})$

$$0 \to \mathbb{C}[\underline f] \to \Omega_{\underline f}^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\underline f}^{n-k} \to \Omega_{\underline f}^{n-k}/d\,\Omega_{\underline f}^{n-k} \to 0$$

and

$f_1,\dots,f_k$ is a regular sequence in $\Omega_{\underline f}^{n-k} / d\,\Omega_{\underline f}^{n-k-1}$

The idea is now to extend the filtration of the Newton boundary to the $\Omega_{\underline f}^i$ in order to get exact graded sequences. Similar to Kouchnirenko we get the same troubles because the differential is not compatible with the filtration. So it is useful to consider the logarithmic complex $\hat{\Omega}^*$ given by $\hat{\Omega}^1 = \sum_i \mathbb{C}\langle X_1,\dots,X_n\rangle \frac{dX_i}{X_i}$ and $\hat{d}f = \sum x_i \frac{\partial f}{\partial x_i} \frac{dx_i}{x_i}$ equipped with the filtration of $\mathbb{C}\langle X_1,\dots,X_n\rangle$ given by the Newton boundary (as in the hypersurface case we associate to the complete intersection $\underline{f}$ the Newton polyhedron Γ_+ as the convex hull in $\mathbb{R}^n_+$ of $\bigcup_{k \in \operatorname{supp} \underline{f}} (k + \mathbb{R}^n_+)$, $\operatorname{supp} \underline{f} = \bigcup_{i=1}^{k} \operatorname{supp} f_i$, and the Newton boundary $\Gamma_{\underline{f}}$ as the union of compact faces of Γ_+).

We also consider $\Omega^p \subset \hat{\Omega}^p$ as a canonical submodule with the induced filtration.

Let us denote the graded modules corresponding to $\hat{\Omega}^*$ resp. Ω^* by $\tilde{\Omega}^*$ resp. $\bar{\Omega}^*$.

Our aim is to prove that

$$0 \longrightarrow \bar{\Omega}^0_f \xrightarrow{\wedge d\bar{f}_k} \dots \longrightarrow \bar{\Omega}^p_f \longrightarrow \bar{\Omega}^p_f \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C}[\bar{f}] \longrightarrow \bar{\Omega}^0_{\underline{f}} \xrightarrow{\bar{d}} \dots \longrightarrow \bar{\Omega}^{n-k}_{\underline{f}} \longrightarrow \bar{\Omega}^{n-k}_{\underline{f}} / d\bar{\Omega}^{n-k-1}_{\underline{f}} \longrightarrow$$

are exact and

$$\bar{f}_1,\dots,\bar{f}_k \quad \text{is a regular sequence in} \quad \tilde{\Omega}^{n-k}_{\underline{f}} / d\tilde{\Omega}^{n-k-1}_{\underline{f}}$$

($\bar{\ }$ denotes as usually the passing to the inertial forms in $\bar{A}$).

Since $\bar{\Omega}^p$ is not a free $\bar{A}$-module in general, we firstly prove that the corresponding sequences for $\tilde{\Omega}^*$ are exact and that $\bar{\Omega}^*_{\underline{f}} \subset \tilde{\Omega}^*_{\underline{f}}$ is a graded submodule. Then we can deduce the above exact sequences and we are able to compute the Poincaré series.

In the case of $k = 1$ one gets e.g.

- $P_{\bar{\Omega}^{n-1}_f/d\Omega^{n-2}_f + f\Omega^{n-1}_f}(T) = (1-T^M)\, P_{\bar{\Omega}^{n-1}_f/d\Omega^{n-2}_f}$,

- $P_{\bar{\Omega}^p} = \sum_{l=0}^{p} (-1)^l \binom{n-l}{p-l} \sum_{\Delta \in \sigma_l} (-1)^{l-\dim\Delta-1} \frac{V_\Delta}{(1-T^M)^{\dim\Delta+1}}$

with σ_l = set of simplexes of a subdivision of the Newton boundary Γ_f into simplexes such that all the edges of these simplexes are exactly the points of $\Gamma_f \cap \operatorname{supp} f$, which are contained in an l-dimensional coordinate hyperplane, but not in a lower dimensional one

and

$$V_\Delta(T) = \sum_{j \in P(\Delta) \setminus \bigcup_{q=1}^{\dim\Delta+1} (s_q + P(\Delta))} T^{d_j}$$

(s_q the vertexes of Δ).

Altogether we get

$$P_{\bar{\Omega}^{n-1}_f/d\Omega^{n-2}_f + f\Omega^{n-1}_f}(T) =$$

$$= (-1)^n + \sum_{\substack{0 \le l \le n-1 \\ \Delta \in \sigma_l}} (-1)^{n-\dim\Delta-1} (1-T)^{n-1-\dim\Delta-1} V_\Delta(T)$$

$$\mu = P_{\bar{\Omega}^{n-1}_f/d\Omega^{n-2}_f + f\Omega^{n-1}_f}(T)\Big|_{T=1} =$$

$$= \sum_{\Delta \in \sigma_{n-\dim\Delta-1}} (-1)^{n-\dim\Delta-1}\, V_\Delta(1)$$

$$= \sum_{k=n}^{n} (-1)^{n-k}\, k!\, V_k$$

gives us Kouchnirenko's formula on the right hand side.

In general we get the following result:

Let $\underline{f}$ be as above a complete intersection with isolated singularity, $\underline{f}$ convenient (definition as in the hypersurface case). Let $f_i^{(\Gamma_f)}$ be the inertial form of f_i in a graded ring induced by the Newton boundary of $\underline{f}$ and of degree m_i, $\underline{M} = (\frac{m_1}{M},\dots,\frac{m_k}{M})$,

M defined as in the hypersurface case, and let us suppose that $\underline{M} \in \mathbb{Z}^k$.

$\underline{f}$ is called <u>non-degenerate</u> if the ideal generated by the s-minors of $(x_i \frac{\partial f_j^{(\Delta)}}{\partial x_i})_{j=1,\dots,s}$ and $f_1^{(\Delta)},\dots,f_s^{(\Delta)}$ is m_Δ-primary in $\mathbb{C}[P(\Delta)]$ for any s and any face $\Delta \subseteq \Gamma_f$ ($f_i^{(\Delta)}$ is the restriction of $f_i^{(\Gamma)}$ to $\mathbb{C}[P(\Delta)]$).

If we now put the <u>Newton number</u> of $\underline{f}$ to be

$$\nu(\underline{f}) = \sum_{d=k}^{n} (-1)^{n-d}\, d!\, \sum_{|\alpha|=d-k} M_1^{\alpha_1+1}\cdot\ldots\cdot M_k^{\alpha_k+1}\, V_d + (-1)^{n-k+1}$$

(V_d defined as in the hypersurface case),
then we get the following theorem:

<u>Let $\underline{f}$ be as above and non-degenerate, then</u> $\mu(\underline{f}) = \nu(\underline{f})$.

<u>Remark</u>:

(1) If $\underline{f}$ is degenerate, one can show as Kouchnirenko that $\mu \geq \nu$.

(2) Kouchnirenko gave and example (cf. [8]) of a degenerate f with $\mu = \nu$.

(3) Briançon proved that f being degenerate on a maximal face implies $\mu > \nu$.

(4) There are "enough" non-degenerate $\underline{f}$, i.e. the set of all principle parts of $\underline{f}$ (parametrized by its coefficients) contains a Zarisk-open dense set (Kouchnirenko,(cf. [8]or[9]): and non-degenerence on maximal faces is even an open condition (Oka [10]).

3. The Hodge Numbers

For simplicity let us restrict ourselves to the case of a quasi-homogeneous hypersurface f of weight $w_1,\dots,w_n$, i.e. $f(\lambda^{w_1}X_1,\dots,\lambda^{w_n}X_n) = \lambda f(X_1,\dots,X_n)$, for the general case cf. Danilov [3].

Let V be defined by $f(x) = 1$, then the cohomology of the Milnor fibre is given by $H^*(V,\mathbb{C})$.

To get the mixed Hodge structure on $H^{n-1}(V)$, we choose a compactification $\overline{V} \supset V$ (which is almost smooth and has a pure Hodge structure; and similarly $V_\infty = \overline{V} \setminus V$), for details cf. Steenbrink [12]. Then we consider the logarithmic complex $\Omega^*_{\overline{V}}(\log V_\infty)$. This complex is locally generated by $\frac{dx_1}{x_1}$, $dx_2,\dots,dx_n$ if $V_\infty : x_1 = 0$. The two filtrations

$$W^p(\Omega^m_{\overline{V}}(\log V_\infty)) = \Omega^p_{\overline{V}}(\log V_\infty) \wedge \Omega^{m-p}_{\overline{V}},$$

$$F^p(\Omega^m_{\overline{V}}(\log V_\infty)) = \begin{cases} 0 & , m < p, \\ \Omega^m_{\overline{V}}(\log V_\infty) & , m \geq p, \end{cases}$$

give us via the spectral sequence

$$_{W[n-1]}E_1^{-p,n-1+p} = \mathbb{H}^{n-1}(\overline{V}, Gr_p^{W[n-1]}\Omega^*_{\overline{V}}(\log V_\infty)) \Longrightarrow H^{n-1}(V,\mathbb{C})$$

(which degenerates at E_2) the mixed Hodge structure $(H^{n-1}(V,\mathbb{C}),W,F)$.

Our case is easy because $W[n-1]^k\Omega^p_{\overline{V}}(\log V_\infty) = \begin{cases} \Omega^p_{\overline{V}}(\log V_\infty) & , k \geq n, \\ \Omega^p_{\overline{V}} & , k = n-1, \\ 0 & , k < n-1. \end{cases}$

This means $_{W[n-1]}E_1^{-p,n-1+p} = 0$, $p > n$,

$$_{W[n-1]}E^{-n,2n-1} = \mathbb{H}^{n-1}(\overline{V}, \Omega^*_{\overline{V}}(\log V_\infty)/\Omega^*_{\overline{V}})$$
$$= H^{n-2}(V_\infty,\mathbb{C})(-1)$$

(via the residue map R, $0 \longrightarrow \Omega^*_{\overline{V}} \longrightarrow \Omega^*_{\overline{V}}(\log V_\infty) \xrightarrow{R} i_*\Omega^{*-1}_{V_\infty} \longrightarrow 0$, $i : V_\infty \hookrightarrow \overline{V}$, if V_∞ is defined locally by $x_1=0$, then $R(f\frac{dx_1}{x_1} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = f|V_\infty\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, $i_2,\dots,i_k \in \{2,\dots,n\}$, if H has a pure Hodge structure of weight n, then H(-1) means the twisting with Tate's Hodge structure to get a morphism of

Hodge structures, i.e. if e.g. $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$, then

$H_{\mathbb{C}}(-1) = \bigoplus_{p+q=n+2} H_{\mathbb{C}}(-1)^{p,q}$ and $H_{\mathbb{C}}(-1)^{p,q} = H^{p-1,q-1}$).

$$W[n-1]\, E_1^{-n+1,2n-2} = H^{n-1}(\overline{V}, \mathbb{C}).$$

We can especially see that the non-zero d_1 in the spectral sequence is the residue map.
This means that the mixed Hodge structure is given by the long exact sequence

$$\ldots \longrightarrow H^{n-1}(\overline{V},\mathbb{C}) \longrightarrow H^{n-1}(V,\mathbb{C}) \xrightarrow{R} H^{n-2}(V_\infty,\mathbb{C})(-1) \longrightarrow \ldots,$$

i.e. $Gr_k^W\, H^{n-1}(V,\mathbb{C}) = 0$ for $k \neq n-1, n$,

$$Gr_{n-1}^W\, H^{n-1}(V,\mathbb{C}) = \mathrm{Im}(H^{n-1}(\overline{V},\mathbb{C}) \longrightarrow H^{n-1}(V,\mathbb{C})),$$

$$Gr_n^W\, H^{n-1}(V,\mathbb{C}) = \mathrm{Im}(H^{n-1}(V,\mathbb{C}) \longrightarrow H^{n-2}(V_\infty,\mathbb{C})(-1))$$

(by the way, these images are just primitive cohomology groups resp. they come from primitive cohomology groups, i.e.
$Gr_{n-1}^W\, H^{n-1}(V,\mathbb{C}) \cong \mathrm{Prim}^{n-1}(\overline{V},\mathbb{C})$, $Gr_n^W\, H^{n-1}(V,\mathbb{C}) \cong \mathrm{Prim}^{n-2}(V_\infty,\mathbb{C})(-1$

The compactification we choose in the following way:
Let $w_i = \frac{u_i}{d}$ (w_i the weights of f), u_i, d integers, d minimal.
Let P_Δ resp. $P_{\tilde\Delta}$ be the weighted projective spaces corresponding to the weights $(w_1,\ldots,w_n)$ and $(w_1,\ldots,w_n,\frac{1}{d})$, respectively: The weights (or the polyhedra Δ resp. $\tilde\Delta$ defined by them, or, what is the same, the Newton boundary Δ of f resp. $\tilde\Delta$ of $F = f-x_{n+1}^d$) define filtrations on $\mathbb{C}[X_1,\ldots,X_n]$ resp. $\mathbb{C}[X_1,\ldots,X_{n+1}]$ (defined as above via the Newton boundary) and give graduations on $\mathbb{C}[X_1,\ldots,X_n]$ resp. $\mathbb{C}[X_1,\ldots,X_{n+1}]$ by putting $|X_i| = w_i$, $i=1,\ldots,n$, and $|X_{n+1}| = 1$.
Then, by definition (cf. [2],[12])
$P_\Delta = \mathrm{Proj}\ \mathbb{C}[X_1,\ldots,X_n]$, $P_{\tilde\Delta} = \mathrm{Proj}\ \mathbb{C}[X_1,\ldots,X_{n+1}]$.

Now one can prove that $\overline{V}$ defined by $F = f-x_{n+1}^d = 0$ in $P_{\tilde\Delta}$ is a compactification of V and V_∞ is defined by $f = 0$ in P_Δ. $\overline{V}$ and V_∞ are quasi-smooth, i.e. they are locally described by conic rings of simplexes (or locally in $\mathbb{C}^n$ a quotient B/G, $B \subseteq \mathbb{C}^n$ an open ball, $G \subset GL_n(\mathbb{C})$ a finite subgroup), thus $H^k(\overline{V},\mathbb{C})$ resp. $H^k(V_\infty,\mathbb{C})$ have a pure Hodge structure of weight k (cf.[2]). For this situation Steenbrink could compute the Hodge decomposition explicitly and got the following nice result:

<u>**Theorem:**</u> <u>Let</u> $\{\underline{x}^\alpha,\ \alpha \in I \subset \mathbb{Z}^n\}$ <u>be a mononial base of</u> $\mathbb{C}[[X_1,\ldots,X_n]]/(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})$,

$$\omega_\alpha = \underline{x}^\alpha\,(f(x)-1)^{[-l(\alpha)]}\,dx_1\wedge\ldots\wedge dx_n,\quad l(\alpha) = \sum_{i=1}^n(\alpha_i+1)w_i,$$

<u>rational differential forms</u>, $\alpha \in I$, <u>and</u> η_α <u>the images of the</u> ω_α <u>in</u> $H^{n-1}(V,\mathbb{C})$ <u>via the residue map. Then the mixed Hodge structure</u> $(H^{n-1}(V), W, F)$ <u>can be described as follows:</u>

(1) $Gr_k\, H^{n-1}(V,\mathbb{C}) = 0$, $k \neq n-1, n$,

(2) $\{\eta_\alpha\}$, $p < l(\alpha) < p+1$, <u>is a base of</u> $Gr_F^p\, Gr_{n-1}^W\, H^{n-1}(V,\mathbb{C})$,

(3) $\{\eta_\alpha\}$, $l(\alpha) = p$, <u>is a base of</u> $Gr_F^p\, Gr_n^W\, H^{n-1}(V,\mathbb{C})$.

As a corollary one gets for even n the rank and signature of the intersection form S:

$$\mathrm{rank}\,(S) = \#\{\beta \in I,\ l(\beta) \notin \mathbb{Z}\},$$

$$\mathrm{sign}\,(S) = \#\{\beta \in I,\ l(\beta) \notin \mathbb{Z},\ [l(\beta)]\ \text{even}\} - \#\{\beta \in I,\ l(\beta) \notin \mathbb{Z},\ [l(\beta)]\ \text{odd}\},$$

knowing that the residue map maps the rational n forms with pols of order n-1-p to the primitive part of $F^p\, H^{n-1}(V)$, knowing the behaviour of the intersection form on primitive cohomology (i.e. if $Gr^W_{n-1} H^{n-1}(V,\mathbb{C}) = \bigoplus H^{p,q}$, then $(-1)^{n(n-1)/2}\ i^{p-q}\ S(x,\bar{x}) > 0$ if $x \in H^{p,q}$ and $S(x,y) = 0$ if $x \in H^{p,q}$, $y \in H^{r,s}$ with $(p,q) \neq (r,s)$) and knowing that the canonical mixed Hodge structure on $H^{n-1}_c(V,\mathbb{C})$ is just the dual of the mixed Hodge structure on $H^{n-1}(V,\mathbb{C})$.

In his Oslo talk in 1976, 'Mixed Hodge Structures on the Vanishing Cohomology', Steenbrink showed how to get the mixed Hodge structure of the Milnor fibre in general for a hypersurface by using limits of Hodge structures. Using toric varieties (i.e. a generalization of the weighted projective space coming from a simplex to any polyhedron), Danilov could express Steenbrink's construction in terms of the Newton polyhedron (cf. [3]). Applied to our special case (quasi-homogeneous hypersurface) we get for instance:

(1) $h^{pq}(V_\infty) = 0$ if $p \neq q$ or $p+q \neq \dim\Delta - 1$
$(h^{pq}$ = Hodge numbers = $\dim H^q(\ ,\Omega^p))$.

(2)
$$h^{pp}(V_\infty) = \begin{cases} \sum\limits_{\tau \leq \Delta} \binom{\dim\tau}{p} (-1)^{\dim\tau + p}, & 2p < \dim\Delta - 1, \\ (-1)^p\ \chi(V_\infty, \Omega^p_{V_\infty}), & 2p = \dim\Delta - 1, \\ \sum\limits_{\tau \leq \Delta} \binom{\dim\tau}{p+1} (-1)^{\dim\tau + p + 1}, & 2p > \dim\Delta - 1, \end{cases}$$

$$h^{p,\dim\Delta-1-p} = (-1)^{\dim\Delta-1-p}\left\{\chi(V_\infty,\Omega^p) + \sum_{\tau \leq \Delta} \binom{\dim\tau}{p+1} (-1)^{\dim\tau}\right\}$$

and

$$\chi(V_\infty,\Omega^p) = -\sum_{\tau \leq \Delta} \left((-1)^{\dim\tau}\binom{\dim\tau}{p+1} + \sum (-1)^k \binom{\dim\tau+1}{p+k+1}\omega(k\tau)\right\}$$

with
$\omega(k\tau)$ = # integral points of $k\tau$ lying in the interior of $k\tau$.

4. The μ-constant Stratum of the Semi-universal Deformation

In his talk at the Congress in Vancouver 1974 Arnold asked whether the μ-constant stratum of an isolated hypersurface singularity is smooth.
We can give an answer for quasihomogeneous hypersurfaces (cf. B. Martin, G. Pfister, The smoothness of the μ-constant stratum of a quasihomogeneous hypersurface, to appear).

Theorem: Let f be a quasihomogeneous polynomial (or more general semi-quasihomogeneous) defining an isolated singularity at $0 \in \mathbb{C}^n$, then the μ-constant stratum of the semi-universal deformation of f is smooth.
As a corollary we obtain that the modality of f (=dimension of the μ-constant stratum) is equal to the inner modality.
To prove this we use the results of Kouchnirenko (cf. [8]) and Greuel and Hamm (cf. [5]) connecting the Milnor number of f with the Newton number an study the behaviour of the Newton number under deformations:
Let $\Delta_f \subseteq \mathbb{R}^n_+$ be the face defined by f (i.e. the intersection of the hyperplane defined by the coefficients of f with $\mathbb{R}^n_+$).
We choose a monomial base
$$M = \{ X^\alpha, \alpha \in B \subseteq N^n \} \text{ of } \mathbb{C}[X] / \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$
and let $B_0 = \{\alpha \in B, \alpha \text{ strictly under } \Delta_f\}$.
Then the μ-constant stratum of the semi-universal deformation
$$F = f + \sum_{\alpha \in B} T_\alpha X^\alpha \text{ of f is given by}$$
the equations $T_\alpha = 0$, $\alpha \in B_0$.

References

1. V.I. Arnold: The index of a singular point of a vector field, the Petrovskii-Oleinik inequalities and mixed Hodge structures, Funkt. Anal. Priloz. 12 (1), 1-14 (1978).

2. V.I. Danilov: The geometry of toric varieties, Uspekhi Mat. Nauk, 33 (2), 85-134 (1978).

3. V.I. Danilov: Newton polyhedra and vanishing cohomology, Funkt. Anal. Priloz. 13 (2), 32-47 (1979).

4. M. Demazure; H. Pinkham; B. Tessier: Seminaire sur les Singularités des Surfaces, Lecture Notes 777.

5. G.M. Greuel; H.A. Hamm: Invarianten quasihomogener vollständiger Durchschnitte, Inv. math. 49, 67-86 (1978).

6. A.G. Khovanski: Newton polyhedra and the Euler-Jacobi formula, Uspekhi Mat. Nauk 33 (6), 245-246 (1978).

7. A.G. Khovanski: Newton polyhedra and toric varieties, Funkt. Anal. Priloz. 11 (4), 56-67 (1977) and 12 (1), 51-61 (1978).

8. A.G. Kouchnirenko: Polyedres de Newton et nombres de Milnor, Inv. Math. 32 (1), 1-32 (1976).

9. B. Martin; G. Pfister: Milnor number of complete intersections and Newton polyhedron, Preprint INCREST (Bucuresti) 71 (1979).

10. M. Oka: On the bifurcation of the multiplicity and topology of the Newton boundary, J. Math. Soc. Japan 31 (3), 435-450 (1979).

11. M. Oka: On the topology of the Newton boundary II, J. Math. Soc. Japan 32 (1), 65-92 (1980).

12. J.H.M. Steenbrink: Intersection form for quasi-homogeneous singularities, Composito Math. 34, 211-223 (1977

13. A.N. Varchenko: Zeta-function of monodromy and Newton's diagram, Inv. math. 37, 253-262 (1976).

14. A.N. Varchenko: A formula for the ranks of the homology groups of the boundary of an isolated singularity of a surface in $\mathbb{C}^3$ and the Newton diagram, Funkt. Anal. Priloz. 13 (1), 65-66 (1979).

Gerhard Pfister
Humboldt-Universität
Sektion Mathematik
108 Berlin, Unter den Linden 6
DDR