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WEIERSTRASS categories and the property of approximation

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0. Introduction

In his paper "On the solution of analytic equations" (cf. [2]) M. ARTIN proved the following famous approximation theorem: Let k be a valued field of characteristic 0 and $f(X, Y)=(f_1(X, Y), \ldots, f_m(X, Y))$ convergent power series in $X=(X_1, \ldots, X_n)$, $Y=(Y_1, \ldots, Y_N)$ with coefficients in k. If the equation $f(X, Y)=0$ has a formal solution, $\bar{y}=(\bar{y}_1, \ldots, \bar{y}_N)$, $\bar{y}_i \in k[[X]]$ formal power series in X, then there exists for any integer $c \cap 0$ a convergent solution $y=(y_1, \ldots, y_N)$, $y_i \in k\{x\}$ convergent power series in X, of the equation $f(X, Y)=0$ such that $y_i \equiv \bar{y}_i \bmod X^c$.

With similar methods M. ARTIN proved the analogous theorem for algebraic power series (the case $n=1$ was already considered by M. J. GREENBERG, cf. [7]) in [3]. The main idea of ARTIN's proofs is the application of the WEIERSTRASS preparation theorem and the implicit function theorem (resp. NEWTON's lemma) in order to be able to apply induction on n (the number of the indeterminates). This gave us the idea to generalize ARTIN's proof to classes of rings with the preparation theorem and some other "good" properties (cf. [12], [13]). The reason for such a generalization was to get a common proof for ARTIN's approximation theorems in the algebraic and analytic case. Furthermore we wanted to prove the approximation theorem for convergent power series over a valued field of characteristic $p>0$ (For the case of the field being completely valued such an approximation theorem was also proved by M. ANDRÉ [1], U. JÄHNER [8], M. VAN DER PUT [10] with different methods). In [10] we developed the idea to consider classes of rings with the preparation theorem to the so-called WEIERSTRASS categories and proved the approximation theorem for rings of these WEIERSTRASS categories.

I. DENEF and L. LIPSHITZ pointed out that some details of this proof were incomplete resp. incorrect and invented W-systems being similar to [13] but more general (i.e. families of regular local rings with the preparation theorem and some more good properties) and proved the approximation theorem for rings of a W-system (cf. [5]).[1])

[1]) The authors would like to thank Mr. DENEF and Mr. LIPSHITZ for their interest in our WEIERSTRASS categories and for their hints with respect to some problems in our proof of the approximation theorem.

Among other things this paper aims at generalizing slightly the notion of WEIERSTRASS categories of [10] in order to get also a connection to [5] and to give a complete proof of the approximation theorem. Furthermore we want to prove ELKIK's approximation theorem (cf. [6]) for WEIERSTRASS categories. In the first three chapters we will give a general definition of a WEIERSTRASS category and its properties. Especially we will characterize smooth morphisms, prove the implicit function theorem and NEWTON's lemma. In chapter 4 we will prove ELKIK's approximation theorem for WEIERSTRASS categories.

To get an idea of a WEIERSTRASS category we will give the definition for the local case here. Let R be a field or a henselian discrete valuation ring and by $\mathbf{C}_R$ let us denote the category of all NOETHERian HENSELian local rings over R with the same residue field. A full subcategory $\mathbf{H}_R$ of $\mathbf{C}_R$ is a WEIERSTRASS category if it has the following properties:

1) Each morphism in $\mathbf{H}_R$ is a WEIERSTRASS-morphism, i.e. if for a morphism in $A \to B$ in $\mathbf{H}_R$ the morphism $A \to B/\mathfrak{m}_A B$ is finite, then $A \to B$ is A-finite.

2) $\mathbf{H}_R$ contains free objects, i.e. if $A \in \mathbf{H}_R$ and if $(T_1, \ldots, T_n)$ is a finite sequence of indeterminates, there exists the free A-algebra $A\ \{\{T_1, \ldots, T_n\}\}$ in $\mathbf{H}_R$ (this means that for any A-algebra $B \in \mathbf{H}_R$ and $t_1, \ldots, t_n \in \mathfrak{m}_B$ there exists exactly one A-morphism $A\{\{T_1, \ldots, T_n\}\} \to B$ in $\mathbf{H}_R$ mapping the T_i onto the t_i). Moreover, the kernels of the canonical morphisms $A\{\{T_1, \ldots, T_n\}\} \to A[T_1, \ldots, T_n]/(T_1, \ldots, T_n)^\nu$ are the ideals $(T_1, \ldots, T_n)^\nu A\{\{T_1, \ldots, T_n\}\}$.

3) Any $A \in \mathbf{H}_R$ is a quotient of some $R\{\{T_1, \ldots, T_n\}\}$. In [10] we called a WEIERSTRASS category $\mathbf{H}_R$ excellent iff for all $A \in \mathbf{H}_R$ the morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ is formally smooth.[2] In this paper we will consider a more general situation. We call $\mathbf{H}_R$ semi-excellent iff for all $A \in \mathbf{H}_R$ and $T = (T_1, \ldots, T_n)$ the morphism $\operatorname{Spec} A'[[T]] \to \operatorname{Spec} A[T]$ is formally smooth in all $\mathfrak{p} \in \operatorname{Spec} A[T]$ being kernels of a suitable morphism $A[T] \to B'$, $B \in \mathbf{H}_R$.

The notation of a semi-excellent WEIERSTRASS category is in principle a generalization of the excellent WEIERSTRASS categories of [10] and the W-systems of DENEF and LIPSHITZ [5]. We do not know if a semi-excellent WEIERSTRASS category is already excellent. But it is useful to have this apparently more general notion because of the following example:

(1) Let k be a quasicompletely valued field (i.e. the completion $\bar{k}$ of k with respect to the valuation is a separable extension of k) and $\mathbf{H}_k$ the category of analytic k-algebras, then $\mathbf{H}_k$ is semi-excellent (cf. [17]). But we do not know if an analytic k-algebra is excellent in case of char $(k) = p > 0$. This only seems to be known up to now if k is already complete. Further examples of excellent WEIERSTRASS categories are (cf. [10]):

(2) The category $\mathbf{H}_R$ of HENSELian rings of finite type over R, R a field or an excellent discrete valuation ring.

(3) The category $\mathbf{H}_R$ of all NOETHERian HENSELian local R-algebras that are

[2]) We denote the completion of A by $A' = \varprojlim A/I^\nu$

complete with respect to the $\mathfrak{m}_R$-adic topology such that, for all $A \in \mathbf{H}_R$, $A/\mathfrak{m}_R^c A$ is HENSELian of finite type over $R/\mathfrak{m}_R^c \subset$ for all $c \geqq 1$, where R is a complete discrete valuation ring of characteristic 0.

(4) Let $\{R_\alpha\}_{\alpha \in I}$ be a filtered system of fields or a complete discrete valuation ring such that the corresponding residue field extensions are separably, and $R = \varinjlim_{\alpha \in I} R_\alpha$ be a field or an excellent discrete valuation ring, $\mathbf{H}_R$ the category of all R-algebras $\varinjlim_{\alpha \in I} R_\alpha [[X_1, \ldots, X_\alpha]]$ and their quotients.

The main result of this paper will be the following theorem (cf. Theorem 6.):

Semi-excellent WEIERSTRASS categories have the property of approximation, i.e. ARTIN's approximation theorem holds: Let $(A, \mathfrak{m})$ be a local ring from a semi-excellent WEIERSTRASS category and $f = (f_1, \ldots, f_m)$ an arbitrary system of polynomials in some variables $Y = (Y_1, \ldots, Y_N)$ with coefficients in A (or, more general by $f_i \in A\{\{Y\}\}$). Then every solution $\bar{y}$ of f in A' (the completion of A) can be well approximated in the $\mathfrak{m}$-adic topology by a solution of f in A (i.e. for every positive integer c there exists a solution y of f in A such that $y \equiv \bar{y} \bmod \mathfrak{m}^c A$). We would like to express our gratitude to Miss BEHRENDT and Miss SUBIRGE for typing the manuscript of this paper.

1. Basic definitions

We denote by C the category of all HENSELian pairs (A, I_A) such that

$$\bigcap_{\nu=0}^{\infty} I_A^\nu = 0 .$$

We call a morphism $(A, I) \to (B, J)$ in C a WEIERSTRASS *morphism* if it has the following property: (W): For any closed ideal $K \subseteqq B$ (with respect to the I-adic topology) such that the morphisme $A \to B/K + IB$ is finite, the morphisme $A \to B/K$ is A-finite.

Remark. In most cases the property (W) implies the stronger property (W'): For any separated B-module E (with respect to the I-adic topology) of finite type such that E/IE is of finite type over A, the module E is of finite type over A.

Proposition 1. *Property (W) implies (W') in the following cases*

(1) *A' and B are* NOETHER*ian rings*

(2) *A is complete with respect to the I-adic topology*

More precisely, assume that A' is NOETHER*ian, let $(A, I) \to (B, J)$ be a* WEIERSTRASS *morphism in C and let E be a B-module of finite type with annihilator ideal $N \subseteqq B$ such that all ideals*

$$BI^\nu + N/N \subseteqq B/N \subseteqq \mathrm{End}_B(E) \quad (\nu = 1, 2, \ldots)$$

are closed in the I-adic topology of $\mathrm{End}_B(E)$, the following properties are equivalent

(i) *E is of finite type over A*

(ii) *E/IE is of finite type over A*

Proof of (2). If $p_1, \ldots, p_N \in E$ and

$$E = Ap_1 + \ldots + Ap_N + IE ,$$

then $E = Ap_1 + \ldots + Ap_N$ since A is complete. For the latter assertion we infere from (ii) the property

$$E' = A'p_1 + \ldots + A'p_N ,$$

hence $\mathrm{End}_{A'}(E')$ is of finite type over A' and by hypothesis $B'/NB' \subseteqq \mathrm{End}_{B'}(E') \subseteqq \mathrm{End}_{A'}(E')$. Therefore B'/NB' is finite over A' and $B'/NB' + IB' = B'/N + IB'$ is finite over A, hence B/N is finite over A by (W) and since E is of finite type over B/N, it is also of finite type over A.

Statement (1) is a special case of the latter assertion. Now we consider a subcategory $\mathbf{H}$ of $\mathbf{C}$, and we define the notion of a free pair with respect to $\mathbf{H}$ in the obvious way as follows: If (A, I), (B, J) are pairs in $\mathbf{H}$ and $T = (T_1, \ldots, T_n)$ is a finite sequence of elements of J, we call (B, J) free over (A, I) with generators $T_1, \ldots, T_n$ if for any morphism $(A, I) \to (C, K)$ in $\mathbf{H}$ and any sequence $(t_1, \ldots, t_n)$, $t_i \in K$, there exists exactly one A-morphism

$$f: (B, J) \to (C, K) \quad \text{such that} \quad f(T_i) = t_i .$$

Obviously, (B, J) is uniquely determined by this property and we will denote it by $(A, I)\ \{\{T_1, \ldots, T_n\}\} = (A\{\{T_1, \ldots, T_n\}\}, \ I\{\{T_1, \ldots, T_n\}\})$ or simply by $A\{\{T_1, \ldots, T_n\}\}$ if there is no confusion about the ideal J.

Now the essential notion of our paper is the following notion of a Weierstrass category.

Definition. *A full subcategory* $\mathbf{H}$ *of* $\mathbf{C}$ *is called* a Weierstrass *category, W-category for short, if it satisfies the following axioms*

(W 0) *For each morphism* $(A, I) \to (B, J)$ *in* $\mathbf{H}$ *the rings* $B/I^{\nu}B$ *are* Noetherian *and* $A/I \to B/J$ *is surjective*

(W 1) *Each morphisme in* $\mathbf{H}$ *is a* Weierstrass *morphism*

(W 2) $\mathbf{H}$ *is closed with respect to finite morphisms in* $\mathbf{C}$, *i.e. if* $(A, I) \in \mathbf{H}$ *and if* $(A, I) \to (B, J)$ *is a finite morphism in* $\mathbf{C}$ *and if* $A/I \to B/J$ *is surjective, then* (B, J) *belongs to* $\mathbf{H}$

(W 3) $\mathbf{H}$ *contains free objects. If* $(A, I) \in \mathbf{H}$ *and if* $(T_1, \ldots, T_n)$ *is a finite sequence of indeterminates, there exists the free pair* $(A, I)\ \{\{T_1, \ldots, T_n\}\}$ *in* $\mathbf{H}$, which *moreover satisfies the property: The kernels of the canonical morphisms* $A\{\{T_1, \ldots, T_n\}\} \to A[T_1, \ldots, T_n]/(T_1, \ldots, T_n)^{\nu}$ (*which contain* $(T_1, \ldots, T_n)^{\nu}$) *are the ideals* $(T_1, \ldots, T_n)^{\nu} A\{\{T_1, \ldots, T_n\}\}$.

Furthermore we define a semi-excellent W-category $\mathbf{H}$ *as a W-category, where all of its objects are noetherian local rings* (A, I) *such that for any set of indeterminates* $T = (T_1, \ldots, T_n)$ *the morphism* $\operatorname{Spec} A'[[T]] \to \operatorname{Spec} A[T]$ *is formally smooth in all* $\mathfrak{p} \in \operatorname{Spec} A[T]$ *being kernels of a suitable morphism* $A[T] \to B'$, $B \in \mathbf{H}$.

2. Smooth morphisms in Weierstrass categories

In the following $\mathbf{H}$ denotes a fixed W-category. We first determine the structure of formally smooth morphisms in $\mathbf{H}$. Recall that a morphism $(A, I)\to(B, J)$ is called formally smooth if for any local Artinian ring $(\bar{R}, \bar{\mathfrak{m}})$ and any (small) extension $(R, \mathfrak{m})\to(\bar{R}, \bar{\mathfrak{m}})$ of Artin-rings it holds that: If

$$\begin{array}{ccc} (A,I) & \longrightarrow & (R,\mathfrak{m}) \\ \downarrow & \overset{\mu}{\nearrow} & \downarrow \\ (B,J) & \xrightarrow[\bar{\mu}]{} & (\bar{R},\bar{\mathfrak{m}}) \end{array}$$

is a commutative diagram of morphisms (without the dotted arrow), the morphism $\bar{\mu}$ can be lifted to an (A, I)-morphism $\mu : (B, J)\to(R, \mathfrak{m})$.

We want to describe the structure of formally smooth morphisms in $\mathbf{H}$. To do this we first somewhat generalize the construction of free objects. Let (A,I) be a pair in $\mathbf{H}$ and let E be a projective A-module of finite presentation

$$E=AT_1+\ldots+AT_n/\lambda_1(T)\,A+\ldots+\lambda_r(T)\,A$$

($\lambda_1, \ldots, \lambda_r$ linear forms in T). By $(A,I)\ \{\{E\}\}=(A\{\{E\}\},\ I\{\{E\}\})$ we denote the following pair:

$$A\{\{E\}\}=A\{\{T\}\}/\lambda_1A\{\{T\}\}+\ldots+\lambda_rA\{\{T\}\}$$
$$I\{\{E\}\}\ =IA\{\{E\}\}+T_1A\{\{E\}\}+\ldots+T_nA\{\{E\}\}\,.$$

We want to show that $(A, I)\ \{\{E\}\}\in\mathbf{H}$. Clearly $I\{\{T\}\}=IA\{\{T\}\}+T_1A\{\{T\}\}+$ $+\ldots+T_nA\{\{T\}\}$ by axiom (W 3) (observe $I\subseteqq I\{\{T\}\}$, $T\in I\{\{T\}\}$ and $A\{\{T\}\}/$ $/I\{\{T\}\}\to A/I$ is surjective by axiom (W 0)). Furthermore $(A\{\{E\}\},\ I\{\{E\}\})$ is Henselian and therefore it remains to show that $B=A\{\{E\}\}$ is separated in the $J=I\{\{E\}\}$-adic topology. To prove this we embed $(A\{\{E\}\}, I\{\{E\}\})$ into $(A\{\{T\}\},$ $I\{\{T\}\})$. As E is projective, there exists a projection operator

$$\pi : AT_1\oplus\ldots\oplus AT_n\to AT_1\oplus\ldots\oplus AT_n$$

with the kernel $A\lambda_1(T)+\ldots+A\lambda_r(T)$. The operator π can be lifted to an (A, I)-morphism

$$\tilde{\pi} : A\{\{T\}\}\to A\{\{T\}\},\quad \tilde{\pi}(T_\nu)=\pi(T_\nu)\,.$$

If $B'=\tilde{\pi}(A\{\{T\}\})$, $J'=\tilde{\pi}(I\{\{T\}\})$, then the pair $(B', J')\subseteqq(A\{\{T\}\},$ then $I\{\{T\}\})$ is in $\mathbf{H}$ (by (W 2)) *If* $U=(U_1, \ldots, U_n)$ are indeterminates, we define a (B', J')-morphism by

$$p : B'\{\{U\}\}\to A\{\{T\}\},\quad p(U_\nu)=T_\nu\,.$$

If $f(T)\in A\{\{T\}\}$ and $\tilde{\pi}(f(T))=0$, the corresponding element $f(U)\in B'\{\{U\}\}$ is contained in the kernel of the morphism $B'\{\{U\}\}\to B'$, $U_\nu\mapsto\pi(T_\nu)$, hence we can write (by (W 3))

$$f(U)=\sum_{\nu=1}^{n}(U_\nu-\pi(T_\nu)\,g_\nu'(U)$$

($g'_\nu(U) \in B'\{\{U\}\}$) and we obtain (applying the morphism p)

$$f(T) = \sum_{\nu=1}^{n} (T_\nu - \pi(T_\nu))\, g'_\nu(T)\ .$$

Therefore

$$\begin{aligned}\operatorname{Ker}(\tilde{\pi}) &= \sum_{\nu=1}^{n} (T_\nu - \pi(T_\nu)) A\{\{T\}\} \\ &= \sum_{\varrho=1}^{r} \lambda_\varrho(T)\, A\{\{T\}\}\end{aligned}$$

and

$$(B, J) = (B', J') \in \mathbf{H}\ .$$

Now we can describe the structure of formally smooth morphisms.

Theorem 1. *For any pair $(A, I) \in \mathbf{H}$ we have*

(1) *If E is a projective A-module of finite type, the morphism $(A, I) \to (A, I)\{\{E\}\}$ is formally smooth.*

(2) *If $(A, I) \to (B, J)$ is a formally smooth morphism in $\mathbf{H}$ and if A is* Henselian *with respect to the ideal $I_1 = J \cap A$, the pair (A, I_1) is contained in $\mathbf{H}$ and there exists a projective E-module of finite type such that $(B, J) = (A, I_1)\{\{E\}\}$.*

To prove the first part of the theorem we can replace everything by the I-adic and $I\{\{E\}\}$-adic completion respectively. But the algebra, respectively

$$A\{\{E\}\}' = A'\|T\|/\lambda_1 A'\ \|T\| + \ldots + \lambda\, A'\nu T\|$$

is obviously formally smooth over A'.

Proof of assertion (2). We replace (A, I) by (A, I_1) (contained in $\mathbf{H}$ by axiom (W 2)), hence we can assume: $A/I \cong B/J$.

Step I. *The module $\bar{E} = J/J^2 + IB$ is projective and of finite type over $\bar{A} = A/I$.* Obviously $\bar{E}$ is an $\bar{A}$-module of finite type since $\bar{B} = B/IB$ is Noetherian. We have to show that for any epimorphism $M \to N$ of $\bar{A}$-modules of finite type any homomorphisme $\bar{p} : \bar{E} \to N$ can be lifted to a homorphism $p : \bar{E} \to M$. Let $\mathfrak{m}$ be any maximal ideal of $\bar{A}$ and assume that $\mathfrak{m}^\nu M = \mathfrak{m}^\nu N = 0$ for suitable ν. Consider the epimorphism of local Artinian algebras $R = \bar{A}/\mathfrak{m}^\nu \oplus M \to \bar{R} = \bar{A}/m \oplus N$ ($M^2 = N^2 = 0$) and the homomorphisms of rings $s : \bar{B} \to \bar{A}$, $s(b) = b \bmod J\bar{B}$ (observe $\bar{B}/J\bar{B} = \bar{A}$) and $\bar{t} : \bar{B} \to \bar{R}$, $\bar{t}(b) = (s(b) \bmod m^\nu, \bar{p}\,((b - s(b)) \bmod J^2\bar{B}))$. As $\bar{B}$ is formally smooth over $\bar{A}$, we can lift $\bar{t}$ to an homomorphism $t : \bar{B} \to R$ and t induces a lifting $p : \bar{E} \to M$ of the homomorphism $\bar{p}$. This implies that for any maximal ideal $\mathfrak{m}$ of $\bar{A}$ the $\mathfrak{m}$-adic completion of $\bar{E}$ is projective over the $\mathfrak{m}$-adic completion of $\bar{A}$, by faithfully flat descent we infere therefore that $\bar{E}$ is projective.

Step II. *Construction of a surjection* $(A, I)\ \{\{E\}\} \to (B, J)$. Since (A, I) is Henselian, we can lift idempotent elements in any finite A-algebra $\mathscr{E}$ from $\mathscr{E}/I\mathscr{E}$ to $\mathscr{E}$. Now $\bar{E}$ is a direct summand of a free $\bar{A}$ module $\bar{A}^N$ and we can lift the corresponding projection operator in $\operatorname{End}(A^N)/I\ \operatorname{End}(A^N)$ to $\operatorname{End}\ (A^N)$. Hence $\bar{E}$ can be lifted (uniquely up to an isomorphism) to a projective A-module E of finite type. Since $\bar{E}=J/J^2+IB$, we can lift the isomorphism $E/IE\cong\bar{E}$ to an A-linear homomorphism $s:E\to J$ and s induces a morphism $(A,I)\{\{E\}\}\to(B,J)$, also denoted by s, which is obviously surjective (by axiom $(W\,1)$).

Step III. *The morphism s is injective.* It is sufficient to show: for any ideal Q in A such that A/Q is local ARTINian the induced homomorphism

$$\bar{s}:A\{\{E\}\}/QA\{\{E\}\}\to B/QB$$

is injective (since the intersection of all these ideals Q is 0 by axiom $(W\,0)$). Hence we can assume that A is local ARTINian. Since B is formally smooth over A, we can construct step by step a homomorphism $t:B\to A\{\{E\}\}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{t} & A[[E]] \\ {}_s\nwarrow & & \nearrow \\ & A\{\{E\}\} & \end{array}$$

(with the canonical embedding $A\{\{E\}\}\subset A[[E]]$) is commutative, therefore s is injective, and the proof of the theorem is finished.

We want to mention the following consequence.

Corollary 1. *If $(A,I)\to(B,J)$ is a formally smooth morphism in* $\mathbf{H}$ *and $A/I\cong B/J$, the module* $\mathrm{Hom}_A(E,I)$ *acts transitively and free on the set of sections* $\mathrm{Hom}_{(A,I)}((B,J),(A,I))$, *and this set is not empty.*

In the next section we derive the characterization of formal smoothness by the JACOBIan criterion. For this reason we have to introduce the following facts:

(1) *For any morphism $(A,I)\to(B,J)$ in* $\mathbf{H}$ *the pair (B,J) can be written up to isomorphy in the form*

$$B=A\{\{T\}\}/K,\quad J=IB+\sum_{\nu=1}^{n}T_\nu B\,.$$

We have to choose representatives $t_1,\ldots,t_n$ of generators of the A-module T/J^2+IB. Then we put $T=(T_1,\ldots,T_n)$ and define $A\{\{T\}\}\to B$ by $T_\nu\to t_\nu$, $K=\ker(A\{\{T\}\}\to B)$. By axiom $(W\,1)$ we infere $B=A\{\{T\}\}/K$ and J is generated by the image of the ideal I and the elements t_ν.

(2) *If E is any $A\{\{T\}\}$-module separated with respect to the I-*adic *topology, the module of derivations is*

$$\mathrm{Der}_A(A\{\{T\}\},E)\cong\mathrm{Hom}_{A\{\{T\}\}}\left(\bigoplus_{\nu=1}^{n}A\{\{T\}\}\,dT_\nu,E\right),$$

induced by a universal derivation

$$d:A\{\{T\}\}\to\bigoplus_{\nu=1}^{n}A\{\{T\}\}\,dt_\nu$$

$$df=\sum_{\nu=1}^{n}\frac{\partial f}{\partial T_\nu}\,dT_\nu\,.$$

Proof. The pair $B=A\{\{T\}\}+A\{\{T\}\}\,\varepsilon$, $J=I\{\{T\}\}+\varepsilon A\{\{T\}\}$ defined by $\varepsilon^2=0$ belongs to $\mathbf{H}$ by axiom $(W\,2)$. The $A\{\{T\}\}$-module

$$\Theta=\{\varphi\mid\varphi\in\mathrm{Hom}_{(A,I)}((A\{\{T\}\},I\{\{T\}\},(B,J))\,\varphi\bmod\varepsilon=id_{A\{\{T\}\}}\}$$

is isomorphic to the module $\mathrm{Der}_A(A\{\{T\}\}, A\{\{T\}\})$ by associating the morphism $\varphi: f \mapsto f + \varepsilon\vartheta(f)$ to a derivation ϑ. Especially to any T_ν we can associate the morphism $T_i \mapsto T_i$ $(i \neq \nu)$, $T_\nu \mapsto T_\nu + \varepsilon$, the corresponding derivation is denoted by $\frac{\partial}{\partial T_\nu}$. If $D: A\{\{T\}\} \to E$ is any derivation, we consider $D_0(f) = D(f) - \sum_{\nu=1}^{n} \frac{\partial f}{\partial T_\nu} D(T_\nu)$. From the axiom ($W$ 3) we infer that $A\{\{T\}\}/(T^\nu)A\{\{T\}\}$ is isomorphic to $A[T]/(T)^\nu A[T]$, hence $D_0(f) \in (T)^\nu E$ for any integer ν, and as E is separated, we get $D_0 = 0$,

$$D = \sum_{\nu=1}^{n} D(T_\nu) \frac{\partial}{\partial T_\nu} \qquad \text{q.e.d.}$$

3. Quasiprojective schemes over Weierstrass categories

We want to study systems of equations of the type $F(T, U) = 0$, where $T = (T_1, \ldots, T_n)$; $U = (U_1, \ldots, U_m)$ and $F(T, U) \in A\{\{T\}\}[U]^r$. In other words, in a slightly more general formulation we study schemas of the type

$$X \to \mathrm{Spec}\,(B) \to \mathrm{Spec}(A),$$

where $(A, I) \to (B, J)$ is a morphism in $\mathbf{H}$ and Spec $(B) \to$ Spec (A) is induced by this morphism, and where $X \to$ Spec (B) is a quasiprojective schema of finite presentation over B, i.e. X is a subschema of some projective scheme $\mathbf{P}(E)$, where E is a B-module of finite presentation and where X is locally closed in $\mathbf{P}(E)$ and locally defined in $\mathbf{P}(E)$ over B by a finite number of equations.

If E is generated by say p elements, we can find a closed embedding $\mathbf{P}(E) \subset \mathbf{P}^p \times \mathrm{Spec}(B)$ of finite presentation, hence we can assume that $X \subset \mathbf{P}^p \times \mathrm{Spec}\,(B)$. Moreover, if $B = A\{\{T\}\}/K$, we can assume $X \subset \mathbf{P}^p \times \mathrm{Spec}\, A\{\{T\}\}$ to be locally closed and locally defined by K and by finite many polynomial equations with coefficients in $A\{\{T\}\}$.

We prove the Jacobian criterion for this mixed situation, then we prove the existence of sections of X over A (theorem on implicit functions) and its generalization (the analogue of Newtons Lemma). By p we denote the projection

$$p: X \to \mathrm{Spec}\,(A).$$

If $x \in X$ is a point such that $p(x) \in V(I)$, we call p *formally smooth* at x (or X formally smooth over A in the point x) if the morphism of pairs $(A, I) \to (\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ (which is not in $\mathbf{H}$) is formally smooth.

Theorem 2. *Assume that* $(A, I) \in \mathbf{H}$, $T = (T_1, \ldots, T_n)$, *and* $X \subseteq P = \mathbf{A}^m \times$ Spec $A\{\{T\}\}$ *is a locally closed subschema,* $p: X \to$ Spec (A) *the corresponding morphism. Let* $U_1, \ldots, U_m$ *be affine coordinates in* $\mathbf{A}^m$. *If* $x \in X$, $p(x) \in V(I)$, *then the schema* X *is formally smooth over* A *in* x *if and only if the following condition is satisfied:*

(J) *There exist functions* $f_1, \ldots, f_k \in A\{\{T\}\}\,[U]$ *which generate the kernel of the homomorphism* $\mathcal{O}'_{P,x} \to \mathcal{O}'_{X,x}$ *and such that the* JACOBI*an matrix* $\left(\frac{\partial f_i}{\partial T_j}(x), \frac{\partial f_i}{\partial U_h}(x)\right)$ *has the rank k.* (If $f \in \mathcal{O}'_{P,x}$, we denote by $f(x)$ the residual class of f in $\mathcal{O}_{P,x}/m_{P,x}$).

Proof. 1.) Assume the condition (J) to be satisfied and assume that R, $\bar{R} = R/tR$ are local ARTINian A-algebras such that $\mathfrak{m}_R t = 0$, and $\mu : \mathcal{O}_{X,x} \to \bar{R}$ is a morphism of local rings. We can lift the composition $\mathcal{O}_{P,x} \to \mathcal{O}_{X,x} \xrightarrow{\mu} \bar{R}$ to a morphism $\nu : \mathcal{O}_{P,x} \to R$ (by lifting the images of T_i, U_j). If the function f on P is in the kernel of $\mathcal{O}_{P,x} \to \mathcal{O}_{X,x}$, the image $\nu(f)$ has the form $\nu(f) = ta(f)$, where $a(f) \in k = R/\mathfrak{m}$ (since t is annihilated by $\mathfrak{m}_R$). For any choice of elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ the map

$$\nu'(f) = \nu(f) + t\sum_{j=1}^{n} a_j \frac{\partial f}{\partial T_j}(x) + t\sum_{h=1}^{m} b_h \frac{\partial f}{\partial U_h}(x)$$

is a morphism of A-algebras $\mathcal{O}_{P,x} \to R$. If condition (J) is satisfied, we can choose $a_1, \ldots, a_n, b_1, \ldots, b_m$ in such a way that

$$\nu'(f_1) = \ldots = \nu'(f_k) = 0\ .$$

Since ker (ν') contains some power of $\mathfrak{m}_{P,x}$ it thus contains also the kernel of $\mathcal{O}_{P,x} \to \mathcal{O}_{X,x}$ and induces a lifting $\mu : \mathcal{O}_{X,x} \to R$ of $\bar{\mu}$.

2) Assume that X is formally smooth over A in x. We will show that condition (J) is satisfied. Define

$$C = \mathcal{O}'_{P,x}, \quad B = \mathcal{O}'_{X,x}, \quad K = \mathrm{Ker}\,(\mathcal{O}_{P,x} \to \mathcal{O}_{X,x})\ ,$$

then $C/KC \cong B$ and because of formal smoothness the canonical A-morphism

$$B \to C/KC + \mathfrak{m}_C^2$$

can be lifted to an A-morphism

$$\eta : B \to C\ .$$

If $\pi : C \to B$ is the canonical morphism, the composition $\pi \circ \eta : B \to B$ coincides with id_B mod $\mathfrak{m}_B^2$, hence $\pi \circ \eta$ is an isomorphism of B and $\varepsilon = \eta \circ (\pi \circ \eta)^{-1}$ is a section of π.

Any derivation $A\{\{T\}\}[U] \to E$ over A with values in an I-adic separated C-module E extends in a unique way to a derivation $C \to E$, thus we have derivations

$$\frac{\partial}{\partial T_j} : C \to C, \quad \frac{\partial}{\partial U_h} : C \to C\ .$$

If $\vartheta : A\{\{T\}\}\,[U] \to KC/K^2C = : \bar{K}$ is defined by $\vartheta(f) = f - \varepsilon \circ \pi(f)$ mod K^2C, it is a derivation and we can extend it to C and get

$$\vartheta(f) = \sum_{j=1}^{n} \frac{\partial f}{\partial T_j}(T_j - t_j) + \sum_{h=1}^{m} \frac{\partial f}{\partial U_h}(U_h - u_h)$$

where $t_j=\varepsilon\circ\pi(T_j)$, $u_i=\varepsilon\circ\pi(U_i)$. Then ϑ induces a B-linear map

$$v:\Omega\ =:\ \bigoplus_{j=1}^{n} BdT_j\oplus\bigoplus_{h=1}^{m} BdU_h\to\bar{K}$$

$$v(dT_j)=(T_j-t_j)\bmod K^2C,\quad v(dU_h)=(U_h-u_h)\bmod K^2C\,.$$

The derivation $A\{\{T\}\}\,[U]\to\Omega$,

$$\varphi\mapsto\sum_{j=1}^{n}\pi\left(\frac{\partial f}{\partial T_j}\right)T_j+\sum_{h=1}^{m}\pi\left(\frac{\partial f}{\partial U_h}\right)dU_h$$

induces a B-linear map

$$w:\bar{K}\to\Omega$$

$\left(\text{because of } \pi\left(\frac{\partial}{\partial T_j}(K^2)\right)=0,\ \pi\left(\frac{\partial}{\partial U_h}(K^2)\right)=0\right)$. For $f\in KC$ if holds that

$$v\circ w\,(f\bmod K^2C)\equiv\left[\sum_{j=1}^{n}\pi\left(\frac{\partial f}{\partial T_j}\right)(T_j-t_j)+\sum_{h=1}^{m}\pi\left(\frac{\partial f}{\partial U_h}\right)(U_h-u_h)\right]$$
$$=\vartheta(f)=f-\varepsilon\circ\pi(f)\bmod K^2C$$
$$=f\bmod K^2C\,.$$

Therefore $v\circ w=id_{\bar{K}}$, $\bar{K}$ is a direct summand of Ω, hence a free B-module. If the functions $f_1,\ldots,f_k\in A\{\{T\}\}\,[U]$ represent a free base of $\bar{K}$, the matrix corresponding to w is $\left(\frac{\partial f_i}{\partial T_j},\frac{\partial f_i}{\partial U_h}\right)$ (evaluated at x), hence condition (J) is satisfied.

We consider the following question: Given a morphism $(A, I)\to(B, J)$ in $\mathbf{H}$ and a quasiprojective B-schema $X\xrightarrow{p}\operatorname{Spec}(B)$. Assume that a commutative diagram of morphisms

$$(1)\qquad\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec}(B)\\ {\scriptstyle\varepsilon_0}\uparrow & & \downarrow\\ \operatorname{Spec}(A/I) & \subset & \operatorname{Spec}(A)\end{array}$$

is given.

We want to extend ε_0 to a section of X over $\operatorname{Spec}(A)$. To reduce this question to a slightly simpler situation we first prove the following:

Lemma 1. *Assume that it set-theoretical holds in* (1) *that* $p\circ\varepsilon_0(V(I))\subseteq V(J)$. *If we denote the kernel of the homomorphism* $(p\circ\varepsilon_0)^*: B\to A/I$ *by* J_1, *the pair* (B, J_1) *belongs to* $\mathbf{H}$ *(and* $IB\subseteq J_1$*). Moreover there exists in embedding* $X\subseteq\mathbf{P}^m\times\operatorname{Spec}(B)$ *and a section* $\eta:\operatorname{Spec}(B)\to\mathbf{P}^m\times\operatorname{Spec}(B)$ *which coincides on* $\operatorname{Spec}(B/J_1)$ *with the morphism* ε_0.

Proof. By the assumption $V(J)\supseteq V(J_1)=p\circ\varepsilon_0(V(I))\cong V(I)$ and by axiom $(W\,0)$ the morphism $(A, I)\to(B, J)$ induces a closed embedding $V(J)\subseteq V(I)$. Therefore $V(J_1)=V(J)$, hence (B, J_1) is Henselian and the inclusion $J_1^r\subseteq J\cap J_1$ holds for some $r\gg 0$, moreover, since by the axiom $(W\,0)$ the ring B/IB is

NOETHERian, the ideals J, J_1 and $J \cap J_1$ are finitely generated modulo IB. If the elements $t_1, \ldots, t_r \in J \cap J_1$ represent a base of $J \cap J_1/IB$, we can define an A-morphism $A\{\{T_1, \ldots, T_r\}\} \to B$, $T_i \mapsto t_i$. By axiom (W 1) this morphism is finite, hence we infer that (B, J_1) belongs to $\mathbf{H}$ (by axiom (W 2)). Now we construct the schema Y as follows: The morphism ε_0 also induces a morphism η : Spec $(B/J_1) \to X$ and we can assume that $X \subseteqq \mathbf{P}^m \times$ Spec (B). The morphism is given as a point of $\mathbf{P}^m \times$ Spec (B) by an epimorphism $(B/J_1)^{m+1} \to \bar{L}$ onto an invertible (B/J_1)-module $\bar{L}$, and since (B, J_1) is HENSELian, we can lift $\bar{L}$ to an invertible B-module L and the epimorphism to an epimorphism of B-modules, $B^{m+1} \to L$. Therefore η_0 can be lifted to a B-morphism η : Spec $(B) \to \mathbf{P}^m \times$ Spec (B) q.e.d.

Lemma 2. *Let S be an affine schema and $X \subseteqq \mathbf{P}^m \times S$ a projective S-schema defined by forms $F_\alpha(T_0, \ldots, T_m) = 0$ with coefficients from $\Gamma(S)$. If $\eta : S \to X$ is a section corresponding to an invertible sheaf L on S and $m+1$, sections $\eta_0, \ldots, \eta_m$ generating L, let $U \subseteqq S \times \mathbf{A}^N$ $(N = (m+1)^2)$ be the schema defined by the equations*

$$F_\alpha\left(\eta_0 + \sum_{j=0}^{m} Y_{0j}\eta_j, \ldots, \eta_m + \sum_{j=0}^{n} Y_{mj}\eta_j\right) \otimes L^{-d_\alpha} = 0 \quad (d_\alpha = \deg F_\alpha)$$

and by the inequality $\det(\delta_{ij} + Y_{ij}) \neq 0$. *Then $\bar{f}(x) = (x, 0, \ldots, 0) \in U$ defines a section of U over S and*

$$\varepsilon(x, y_{ij}) = \left(x, \eta_0 + \sum_{j=0}^{m} y_{0j}\eta_j : \ldots : \eta_m + \sum_{j=0}^{m} y_{mj}\eta_j\right)$$

defines an S-morphism $U \to X$ such that $\varepsilon \circ \S = \eta$. Moreover ε is a locally trivial fibration with the fibre

$$Gl(m) \times \mathbf{A}^m \times \mathbf{G}_m\,.$$

Proof. Let S_0 be the open set where L is generated by η_0, by a lineare change of coordinates on $\mathbf{P}^m \times S_0$ and $\mathbf{A}^N \times S_0$ we can assume that $(\eta_0, \ldots, \eta_m) = (\eta_0, 0, \ldots, 0)$ on S_0. Then U is defined by

$$F_\alpha((1 + Y_{00})\eta_0, \quad Y_{10}\eta_0, \ldots, Y_{m0}\eta_0) = 0$$

and ε by

$$\varepsilon(x, y_{ij}) = (x, 1 + y_{00} : y_{10} : \ldots : y_{m0})\,.$$

Hence ε is a locally trivial fibration with the fibre $GL(m) \times \mathbf{A}^m \times \mathbf{G}_m$ (stabilizer of a point of $\mathbf{P}^m$ under the action of $GL(m+1)$).

For later use we note the following

Lemma 3. If *$\Phi : E \to A^n$ is a homomorphism of A-modules, where E is projective of rank k and if $x \in A$ such that*

$$x \det(E^*) \subseteqq \textit{image of } \Lambda^k\Phi^*$$

($\Phi^* : A^n \to E^*$ the dual map to Φ), there exists a homomorphism $\gamma : A^n \to E$ such that $\gamma \circ \Phi = x\, \mathrm{id}_E$.

Note that the condition about x can also be written as $x \in$ image of $(\det(E) \otimes \Lambda^k A^{n*} \to A)$ induced by Φ^*. In the case of $k > n$, the element x must be 0, hence we can put $\gamma=0$. Assumed $k\leqq n$, the condition about x implies

$$xE^*\subseteqq \text{image of} \quad \Phi^* .$$

If $p: A^m\to E^*$ is an epimorphism, we define a homomorphism $\beta: A^m\to(A^n)^*$ by $\beta(e_i)=v_i\in(A^n)^*$ such that v_i are elements with the property $\Phi^*(v_i)=xp(e_i)$. Since E^* is projective, p has a section $r: E^*\to A_m$ and we define $\gamma^*=\beta\circ r$. Then $\Phi^*\circ\gamma^*=$ $=x\,\mathrm{id}_{E^*}$ holds and for the dual map we have

$$\gamma\circ\Phi=x\,\mathrm{id}_{E^*} \qquad \text{q.e.d.}$$

Theorem 2'. (*theorem on implicit functions*). *Assume that in the diagram* (1) *it holds that*

(a) $p\circ\varepsilon_0(V(I))\subseteqq V(J)$

(b) *X is formally smooth over A in all points $\varepsilon_0(x)$ ($x\in$ Spec $(A(\mathrm{I}))$.*

Then ε_0 can be extended to a section of $X\to$ Spec (A).

By lemma 1 we can replace (B, J) by (B, J_1), hence we can assume $p_0\varepsilon_0$: Spec $(A/I)\rightleftarrows$ Spec (B/J). Furthermore there exists an embedding $X\subseteqq \mathbf{P}^m\times$ Spec (B), an invertible B-module L and $m+1$ sections $\eta_0, \ldots, \eta_m$ of L generating L, such that the corresponding point η: Spec $(B)\to\mathbf{P}^m\times$ Spec (B) coincides on Spec (B/J) with the given morphism ε_0. Let the projective closure of X in $\mathbf{P}^m\times$ Spec (B) be defined by the family of forms

$$F_\alpha(U_0, \ldots, U_m)\in B[U_0, \ldots, U_m]$$

and consider indeterminates Y_{ij}, $i, j=0, \ldots, m$ and in the algebra $B\{\{Y_{00}, Y_{01}, \ldots, Y_{mm}\}\}=B\{\{Y\}\}$ the I-adic closure K of the ideal generated by the functions corresponding to

$$F_\alpha\left(\eta_0+\sum_{j=0}^m Y_{0j}\eta_j, \ldots, \eta_m+\sum_{j=0}^m Y_{mj}\eta_j\right)\otimes L^{\otimes -d_\alpha}\subseteqq B\{\{Y\}\}, \quad (d_\alpha=\deg F_\alpha).$$

From lemma 2 we infere that the algebra $C=B\{\{Y\}\}/K$ is formally smooth over (A, I), hence by Corollary 1 the homomorphism $C\to A/I$ given by $(p\circ\varepsilon_0)^*: B\to A/I$, and $Y_{ij}\mapsto 0$ extends to a homomorphism $C\to A$, say by $Y_{ij}\mapsto y_{ij}$, $\varphi: B\to A$. Then, by

$$\left(\varphi,\ \varphi^*\eta_0+\sum_{j=0}^m y_{0j}\ \varphi^*\eta_j : \ldots : \varphi^*\eta_m+\sum_{j=0}^m y_{mj}\varphi^*\eta_j\right),$$

we get a section of X over Spec (A).

Our next aim is to formulate and to prove the so-called NEWTON lemma.

If $\varphi=(\varphi_1, \ldots, \varphi_k)$, $\varphi\in A\{\{T\}\}[U]$ and an A-morphisme $A\{\{T\}\}[U]\to A$, $T_i\mapsto$ $\mapsto t_i$, $i=1, \ldots, n$, $U_j\mapsto u_j$, $j=1, \ldots, m$ is given in the case of $k\leqq n+m$, we define the following ideal in A:

$$C(\varphi, t, u)=\text{the ideal generated by the } (k\times k)\text{-minors of } (\partial\varphi_i/\partial T_j(t, u), \partial\varphi_i/\partial U_h(t, u)) \text{ and by } \varphi_1(t, u), \ldots, \varphi_k(t, u)\,.$$

If $Z\subseteqq$ Spec $(A\{\{T\}\}[U])$ is the set of zeros of φ, the locus $V(C(\varphi, t, u))$ consist of all points of Spec (A), over which Z is not a smooth complete intersection of codimension k in Spec $(A\{\{T\}\}[U])$.

Theorem 3. (NEWTON's lemma, preliminary version). *Assume that* $\varphi = (\varphi_1, \ldots, \varphi_k)$, $\varphi_i \in A\{\{T\}\}[U]$, $T = (T_1, \ldots, T_n)$, $U = (U_1, \ldots, U_m)$, *and let* H *and* I_1 *be ideals in* A *such that* $I_1 \subseteq I$ *and* A/I_1 *is* I*-adic separated. If the system of equations*

$$\varphi(T, U) = 0$$

has a solution (t^0, u^0), $t_i^0 \in I$, $u_j^0 \in A$ modulo $H^2 I_1$ *such that the ideal in* A, *generated by the* $(k \times k)$*-minors of* $(\partial\varphi_i/\partial T_j(t^0, u^0), \partial\varphi_i/\partial U_h(t^0, u^0))$ *contains the ideal* H, *then it has a solution* (t, u) *in* A *such that* $t \equiv t^0 \bmod I_1 H$, $u \equiv u^0 \bmod I_1 H$.

We can again consider a slightly more general situation: Given a morphism $(A, I) \to (B, J)$ in $\mathbf{H}$, assume $p : X \to \operatorname{Spec}(B)$ to be a quasiprojective morphism and $X \to \operatorname{Spec}(A)$ to be formally smooth. Consider a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank and a homomorphism $\varphi : \mathcal{E} \to \mathcal{O}_X$, then φ defines a closed subschema Z of X by $\mathcal{O}_Z = \operatorname{coker}(\varphi)$. By $\Omega^1_{X/A}$ we denote the universal I-adic separated differential module and by $d : \mathcal{O}_X \to \Omega^1_{X/A}$ the universal derivation (for derivations over A with values in I-adic separated quasicoherent $\mathcal{O}_X$-modules).

The universal I-adic separated derivation $d : \mathcal{O}_X \to \Omega^1_{X/A}$ exists and can be described as follows: If $X = \operatorname{Spec}(A\{\{T\}\}[U]/(F_1, \ldots, F_m))$, then $\Omega^1_{X/A}$ corresponds to the module

$$\bigotimes_{j=1}^{n} \mathcal{O}_X dT_j \oplus \bigoplus_{h=1}^{m} \mathcal{O}_X dU_h / (dF_1, \ldots, dF_m),$$

where

$$dF_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial T_j} dT_j + \sum_{h=1}^{m} \frac{\partial F_i}{\partial U_h} dU_h,$$

and

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial T_j} dT_j + \sum_{h=1}^{m} \frac{\partial f}{\partial U_h} dU_h.$$

The derivation d induces an $\mathcal{O}_Z$-linear map

$$\mathcal{E} \otimes \mathcal{O}_Z \to \Omega^1_{X/A} \otimes \mathcal{O}_Z$$

(by $e \otimes 1 \mapsto d\varphi(e) \otimes 1$), and therefore an $\mathcal{O}_Z$-linear map

$$\det(\mathcal{E} \otimes \mathcal{O}_Z \to \Omega^{rk(\mathcal{E})}_{X/A}(\mathcal{E}) \otimes \mathcal{O}_Z$$

(where $\Omega^r_{X/A}$ is defined as $\Lambda^r \Omega^1_{X/A}$). If $s_0 : \operatorname{Spec}(A) \to X$ is a section and if N is the ideal in A such that $\operatorname{Spec}(A/N) = s_0^{-1}(Z)$, we restrict this map to $\operatorname{Spec}(A/N)$. It's dual defines a homomorphism

$$\operatorname{Hom}_A(s_0^*(\Omega^{rk(\mathcal{E})}_{X/A}) \otimes \mathcal{O}_Z), \quad s_0^*(\det(\mathcal{E}) \otimes \mathcal{O}_Z)) \to A/N.$$

Then we define the ideal $C(\varphi, s_0) \subseteq A$ by $C(\varphi, s_0)/N$ = image of this homomorphism. In the special case $X = \operatorname{Spec}(A\{\{T\}\}[U])$, $\mathcal{E} = A\{\{T\}\}[U]^k$ and $\varphi = (\varphi_1, \ldots, \varphi_k)$ it coincides with the ideal defined above. We can define $C(\varphi, s_0)$ in an alternative way as follows: Consider the sheaf $\tilde{A}$ as an $\mathcal{O}_X$-module via the section s_0 and the map induced by φ

$$\operatorname{Der}_A(\mathcal{O}_X, \tilde{A}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \tilde{A}/N\tilde{A})$$
$$D \mapsto D \circ \varphi \bmod N\tilde{A}.$$

If $rk(\mathcal{E})=r$, $\tilde{A}C(\varphi, s_0)$ is generated by N and the elements $\det(D_i(\varphi(e_j)))$, where $(D_1, \ldots, D_r)$ runs through the set $\mathrm{Der}_A(\mathcal{O}_X, \tilde{A})^r$ and $(e_1, \ldots, e_r)$ trough the (local) sections of $\mathcal{E}^r$,

i.e. $C(\varphi, s_0)/N=$ image of

$$\Lambda^r \mathrm{Der}_A(\mathcal{O}_X, \tilde{A})\otimes(\Lambda^r \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \tilde{A}/N\tilde{A}))^{-1}\to A/N\,.$$

Theorem 3' (Newton's Lemma). *Let $(A, I)\to(B, J)$ be a morphism in* **H**, *$X \xrightarrow{p}$ Spec (B) a quasiprojective schema over B and s_0: Spec $(A)\to X$ a section. Assume X to be formally smooth over A in the points of s_0 (Spec(A)) and $p\circ s_0(V(I))\subseteq V(J)$. Let $\mathcal{E}\xrightarrow{\varphi}\mathcal{O}_X$ be a morphism of a vector bundle of finite rank over X, $Z\subseteq X$ the schema of zeros of φ, and H and $I_1\subseteq I$ ideals of A such that $(A/I_1, I/I_1)\in$* **H**.

If

(a) $s_0(\mathrm{Spec}\,(A/H^2I_1))\subseteq Z$

(b) $C(\varphi, s_0)\supseteq H$,

there exists a section s : Spec $(A)\to Z$, *such that*

$$s=s_0 \quad \textit{on} \quad \mathrm{Spec}\,(A/HI_1)\,.$$

Proof. Step I. Reduction to the affine case. By lemma 1 we can assume that $A/I\cong B/J$. If K denotes the kernel of $(p\circ s_0)^*: B\to A$, the pair (B, K) is Henselian (since $K\subseteq J$). Hence we can assume that $X\subseteq \mathbf{P}^m_B$ and there exists a section η : Spec $(B)\to\mathbf{P}^m_B$ given by $(L, \eta_0, \ldots, \eta_m)$ (L an invertible module over B generated by $\eta_0, \ldots, \eta_m$) which coincides on $(p\circ s_0)$ (Spec (A)) with the given section s_0. We consider again the equations $F_\alpha(U_0, \ldots, U_m)=0$ defining the projective closure of X in $\mathbf{P}^m\times$ Spec (B), the algebra $B\{\{Y\}\}$, $Y=(Y_{ij})$, $ij=0, \ldots, m$, the I-adic closure K of the ideal generated by the functions corresponding to

$$F_\alpha\left(\eta_0+\sum_{j=0}^{m} Y_{0j}\eta_j, \ldots, \eta_m+\sum_{j=0}^{m} Y_{mj}\eta_j\right)\otimes L^{-d_\alpha}\subseteq B\{\{Y\}\}$$

($d_\alpha=\deg F_\alpha$) and the algebra $C=B\{\{Y\}\}/K$. By

$$\left(\eta_0+\sum_{j=1}^{m} Y_{0j}\eta_j : \ldots : \eta_m+\sum_{j=1}^{m} Y_{mj}\eta_j\right)$$

a B-morphisme ε : Spec $(C)\to X$ is defined, and by $Y_{ij}\mapsto 0$ and $p\circ s_0$ we get a section t_0 of Spec $(C)\to$ Spec (A). By lemma 2 we see that $(C, J\{\{Y\}\}/K)$ is formally smooth over A.

We consider the homomorphism of sheaves $\varepsilon^*\mathcal{E}\xrightarrow{\varepsilon^*(\varphi)}\mathcal{O}_{\mathrm{Spec}(C)}$ and we shall proof that $C(\varepsilon^*(\varphi), t_0)=C(\varphi, s_0)$. Since $\varepsilon^{-1}Z$ is the schema of zeros of $\varepsilon^*(\varphi)$, the assumptions (a), (b) are satisfied for Spec (C), $\varepsilon^*(\varphi)$ and t_0 then. If we prove the existence of a section t : Spec $(A)\to\varepsilon^{-1}(Z)$ satisfying $t=t_0$ on Spec (A/HI_1), the section $s=\varepsilon\circ t$: Spec $(A)\to Z$ has the required property. Hence we have to prove $C(\varepsilon^*(\varphi), t_0)=C(\varphi, s_0)$ and then the theorem in the special case where $X=$ Spec (C).

Proof of $C(\varepsilon^*(\varphi), t_0)=C(\varphi, s_0)$: Since $t_0^{-1}\varepsilon^{-1}Z=s_0^{-1}Z$, these subschemas of Spec (A) are defined by the same ideal N. Furthermore we have a canonical restriction map $\mathrm{Der}_A(C, A)\to\mathrm{Der}_A(\mathcal{O}_X, \tilde{A})$. By lemma 2 this map is surjective, therefore

$$C(\varepsilon^*(\varphi),\, t_0) = C(\varphi,\, s_0)\,.$$

Hence we have reduced the proof to the case $X = \operatorname{Spec}(A\{\{F\}\})$, F a projective A-module of finite type and such that s_0 corresponds to $A\{\{F\}\} \to A$, $F \to 0$ (by Theorem 1 and Corollary 1).

Step II. Reduction of the case $X = \operatorname{Spec}(A\{\{F\}\})$, $H^2 I_1 + FA\{\{F\}\} \supseteqq \text{Image}$ of $(E \xrightarrow{\varphi} A\{\{F\}\})$ (where $E \xrightarrow{\varphi} A\{\{F\}\}$ is the homomorphism of $A\{\{F\}\}$-modules corresponding to $\mathcal{E} \to \mathcal{O}_X$) to $A\{\{F\}\} = A\{\{T_1, \ldots, T_n\}\}$. We can reduce the proof to the case where F is free, i.e. $X = \operatorname{Spec} A\{\{T_1, \ldots, T_n\}\}$, as follows: The module E can be written as $E = E_0 \otimes_A A\{\{F\}\}$, where E_0 is the projective A-module obtained from E by reduction mod $FA\{\{F\}\}$. An isomorphism $E_0 \otimes_A A\{\{F\}\} \to E$ is obtained by lifting the idendity of E_0 to an A-linear map $j : E_0 \to E$ (observe that E_0 is projective) and by $x \otimes f \mapsto fj(x)$ $(x \in E_0,\, f \in A\{\{F\}\})$. This map is reduced to the identity mod $FA\{\{F\}\}$ and since $FA\{\{F\}\}$ is contained in the Jacobson radical of $A\{\{F\}\}$ and E is of finite type, it is surjective (by Nakayama's lemma) and therefore bijective (since E is projective). Therefore $\varphi : E \to A\{\{F\}\}$ is determined by an A-linear map $\psi : E_0 \to H^2 I_1 + FA\{\{F\}\}$. There exists a projective A-module F' such that $F \oplus F' = AT_1 \oplus \ldots \oplus AT_n$ is free. We can lift ψ to an A-linear map $\psi' : E_0 \to H^2 I_1 + \sum_{\nu=1}^{n} T_\nu A\{\{T\}\}$ and extend it by the embedding $i : F' \to A\{\{T\}\}$ to an A-linear map

$$\psi'' = \begin{pmatrix} \psi' & 0 \\ 0 & i \end{pmatrix} : E_0 \oplus F' \to H^2 I_1 + \sum_{\nu=1}^{n} T_\nu A\{\{T\}\}\,.$$

Any A-derivation $D : A\{\{F\}\} \to A$ can be extended to an A-derivation $\bar{D} : A\{\{T\}\} \to A$, hence $C(\varphi, 0) = C(\psi'', 0)$.

Step III. Proof in the case of

$$X = \operatorname{Spec} A\{\{T_1, \ldots, T_n\}\}$$

$$\varphi : E_0 \to H^2 I_1 + \sum_{\nu=1}^{n} T_\nu A\{\{T\}\}$$

(E_0 a projective A-module). We have to determine elements $t_\nu \in HI_1 \subseteqq I$ such that under the map $A\{\{T\}\} \to A$, $T_\nu \mapsto t_\nu$ the homomorphism φ is mapped to 0. For any map $\varphi : E_0 \to A\{\{T\}\}$ and $t_\nu \in I$ let us denote by $\varphi(t_1, \ldots, t_n) : E_0 \to A$ the composition of φ with the A-homomorphism $A\{\{T\}\} \to A$, $T_\nu \mapsto t_\nu$. We can write

$$\varphi = \sum_{i,j=1}^{r} X_i X_j \psi_{ij} + \sum_{\nu=1}^{n} T_\nu \varphi_\nu + \sum_{\nu,\mu=1}^{n} T_\nu T_\mu \varphi_{\nu\mu} \tag{1}$$

$$\psi_{ij} : E_0 \to I_1, \quad \varphi_\nu : E_0 \to A$$

$\varphi_{\nu\mu} : E_0 \to A\{\{T\}\}$, where $x_1, \ldots, x_r$ are suitable elements of H. $\Big($This is possible since $\operatorname{Im}(\varphi) \subseteqq H^2 I_1 + \sum_{\nu=1}^{n} T_\nu A\{\{T\}\} = H^2 I_1 + \sum_{\nu=1}^{n} T_\nu A + \sum_{\nu,\mu=1}^{n} T_\nu T_\mu A\{\{T\}\}\Big)$. Then we try to find elements t_ν in the form

$$(2)\qquad t_\nu = t_\nu(u) = \sum_{f=1}^{r} x_\varrho u_{\nu\varrho}$$

with elements $u_{\nu\varrho} \in I_1 \subseteqq I$. The ideal $C(\varphi, 0)$ is the ideal generated by the image N of

$$\sum_{i,j=1}^{r} x_i x_j \psi_{ij}$$

and the elements

$$\begin{vmatrix} \varphi_{\nu_1}(w_1) \dots \varphi_{\nu_1}(w_k) \\ \dots \\ \varphi_{\nu_k}(w_1) \dots \varphi_{\nu_k}(w_k) \end{vmatrix} = \langle \varphi_{\nu_1} \wedge \dots \wedge \varphi_{\nu_k}, w_1 \wedge \dots \wedge w_k \rangle$$

if $k = \mathrm{rk}(E_0)$, where $(w_1, \dots, w_k)$ runs through the set E_0^k. Let Δ denote the ideal generated by the elements $\langle \varphi_{\nu_1} \wedge \dots \wedge \varphi_{\nu_k}, w_1 \wedge \dots \wedge w_k \rangle$, we want to show $\Delta = C(\varphi, 0)$. By assumption (a) we have that $N \subseteqq H^2 I_1$ and by (b) that $N + \Delta \supseteqq H$, hence $N \subseteqq (N + \Delta)\, H I_1$.

Therefore the idendity $N + \Delta = NHI_1 + \Delta$ holds. Since N is of finite type, we infere from this idendity (by NAKAYAMA's lemma) that $C(\varphi, 0) = \Delta + N = \Delta$. Since the elements $x_j \in H \subseteqq C(\varphi, 0) = \Delta$ are in Δ, we can find for each index j an A-linear map

$$\gamma_j : AT_1 \oplus \dots \oplus AT_n \to E_0$$

such that

$$(3)\qquad \gamma_j \circ \left(\sum_{\nu=1}^{n} T_\nu \varphi_\nu \right) = x_j\, id_{E_0}$$

(by lemma 3). Substituting (2) into (1) we get

$$\varphi(t(u)) = \sum_{i,j=1}^{r} x_i x_j \psi_{ij} + \sum_{\nu=1}^{n} \sum_{j=1}^{r} x_j u_{\nu j} \varphi_\nu + \sum_{\nu,\mu=1}^{n} \sum_{i,j=1}^{r} x_i x_j u_{\nu i} u_{\mu j} \varphi_{\nu\mu}(t(u)) .$$

Using (3) we can write this as

$$\varphi(t(u)) = \sum_{i=1}^{r} x_i \sum_{\varkappa=1}^{n} \left[\sum_j \psi_{ij} \gamma_j(T_\varkappa) + u_{\varkappa i} + \sum_{\nu,\mu} \sum_j u_{\nu i} u_{\mu j} \varphi_{\nu\mu}(t(u))\, \gamma_j(T_\varkappa) \right] \varphi_\varkappa .$$

Hence it is sufficient to determine the elements $u_{\nu i}$ such that the terms in the square brackets vanish, i.e. we have to consider equations of the type

$$(4)\qquad \begin{aligned} & a_{\varkappa i} + U_{\varkappa i} + \sum_{\nu,\mu,j} U_{\nu i} U_{\mu j} h_{\varkappa j \nu\mu}(U) = 0 \\ & \varkappa = 1, \dots, n, \quad i = 1, \dots, r \\ & a_{\varkappa i} \in I_1, \quad h_{\varkappa j\nu\mu}(U) \in A\{\{U\}\} . \end{aligned}$$

The JACOBIan matrix of this system in $U = 0$ is the unit matrix, and for $U = 0$ the equations vanish modulo I_1, hence by theorem 2 they can be solved by elements $u_{\varkappa i} \in I$. Since $a_{\varkappa i} \in I_1$, we infere from (4) that $u_{\varkappa i} \in I_1 + (\sum_{\nu,j} A u_{\nu j})^2$ and since I_1 is I-adic closed, hence also closed with respect to the $(\sum_{\nu,j} A u_{\nu j})$-adic topology, this implies $u_{\varkappa i} \in I_1$ q.e.d.

4. Generalization of ELKIK's theorem

In this section we consider a W-category where all pairs (A, I) are NOETHERian. We consider $A\{\{T\}\}$ $T=(T_1, \ldots, T_n)$ and an algebraic schema $P \to \operatorname{Spec} A\{\{T\}\}$ which is smooth over $A\{\{T\}\}$.

For any closed subschema $X \subset P$ defined by a sheaf of ideals $K \subset \mathcal{O}_p$ we define the following sheaf of ideals $\mathscr{E}(X, P)=\mathscr{E}$: For $U \subset P$ open and affine, $\mathscr{E}(U)=\sum_f [I(f): K(U)]\, \Delta(f)+K(U)$, where f runs through the set of all tuples $(f_1, \ldots, f_p) \in K(U)^p$, $p=1, 2, \ldots$, $I(f)$ denotes the ideal $\sum_{j=1}^{p} \mathcal{O}_p(u) f_j$ and $\Delta(f)$ the ideal in $\mathcal{O}_p(U)$, image of the map $\Lambda^p \operatorname{Der}_A(\mathcal{O}_p(U), \mathcal{O}_p(U)) \to \mathcal{O}_p(U)=\Lambda^p \mathcal{O}_p(U)^p$ which is induced by the map

$$\operatorname{Der}_A(\mathcal{O}_p(U), \mathcal{O}_p(U)) \to \mathcal{O}_p(U)^p$$
$$\vartheta \mapsto (\vartheta(f_1), \ldots, \vartheta(f_p)).$$

Note the following properties of $\mathscr{E}$:

(A) If $x \in X$, then $\mathscr{E}_x=\mathcal{O}_{p,x}$ if and only if $K_x=I(f)$ for a suitable p-tuple $f \in K_x^p$ satisfying $\Delta(f)=\mathcal{O}_{p,x}$.

(B) If K_x is generated by a regular sequence $f=(f_1, \ldots, f_p)$ in $\mathcal{O}_{p,x}$, the ideals $\mathscr{E}_x$ and $\Delta(f)+K_x \subseteqq \mathscr{E}_x$ have the same set of zeros in $\mathcal{O}_{p,x}$.

Now the formulation of ELKIN's theorem is

Theorem 5. *If* $(A, I) \in \mathbf{H}$, *there exists a function* $d: \mathbf{N}^3 \to \mathbf{N}$, $d(a, r, c) > \max(r, c)$, *with the following property. Assume* $P=\mathbf{P}^m \times \operatorname{Spec} A\{\{T\}\}$ *and* $X \subset P$ *to be a (quasi)projective subschema and* $\mathfrak{X} \subset \mathcal{O}_p$ a *quasicoherent sheaf of ideals, such that for a suitable integer a* $\mathfrak{X}^a \mathcal{O}_p \subseteqq \mathscr{E}(X, P)$ *holds. Then, for any A-morphism*

$$s_0: \operatorname{Spec}(A/I^d) \to X, \quad d=d(a, r, c)$$

such that (if $V(\mathfrak{X})$ *denotes the closed subschema defined by the ideal* $\mathfrak{X}$)

$$s_0(\operatorname{Spec}(A/I^r)) \supseteqq V(\mathfrak{X}) \cap X,$$

there exists a section $s: \operatorname{Spec}(A) \to X$ *such that* $s=s_0$ *on* $\operatorname{Spec}(A/_I c)$.

By lemma 1 and lemma 2 the proof is easily reduced to the case $X \subset \operatorname{Spec}(A\{\{T\}\})=P$ (the ideal $\mathfrak{X}$ has to be replaced by its inverse image in lemma 2). In this case, if $T=(T_1', \ldots, T_n)$, to give an A-morphism $s_0: \operatorname{Spec}(A/I^d) \to X$ is the same as to give an n-tuple $t^0=(t_1^0, \ldots, t_n^0) \in I^{\oplus n}$ such that $F(t^0) \equiv 0 \bmod I^d$, where $F=F(T)$ denotes a tuple of functions from $A\{\{T\}\}$ which generates the ideal of X. If $H \subset A\{\{T\}\}$ denotes the given ideal, the condition $s_0(\operatorname{Spec}(A/I^r)) \subseteqq X \cap V(\mathfrak{X})$ means in this case that $I^r \subseteqq H(t^0)$ (since $d > r$).

In the following we consider the pair (A, I), a q-tuple $F \in A\{\{T\}\}^q$, $T=(T_1, \ldots, T_n)$ and the ideal E corresponding to the embedding $X=V(F) \subset P=\operatorname{Spec} A\{\{T\}\}$ as defined above.

Lemma 4. *Assumed there are given* $t \in I^{\oplus n}$ $x \in I$, *and an ideal*, $N \subseteqq A$, *and the following conditions are satisfied*

(i) $F(t) \equiv 0 \bmod x^s N$ *for an integer* $s > 0$

(ii) *$x^r \in E(t)$ for an integer $r \geqq 0$*
(iii) *$s > r$ and $0 : x^{s-r} = 0 : x^{s-r+1}$.*
In this case there exists an $y \in (x^{s-r}N)^{\oplus n}$ satisfying $F(t+y) \equiv 0 \mod x^{2(s-r)}N$.

Remarks. 1) The assertion is of course trivial if $2(s-r) \leqq s$, i.e. if $s \leqq 2r$, since we can take $y=0$ in this case. However, if $s \geqq 2r+1$, the vector $t+y$ provides a better approximation of a solution of the equations $F=0$ then the vector t.

2) If $x \in I$ is not nilpotent, we can always find an integer k such that $0 : x^k = 0 : x^{k+1} = \ldots$ Hence if in the lemma $s \geqq \max(2r+1, r+k)$, we can find a sequence $t = t^{(0)}, t^{(1)}, t^{(2)}, \ldots, t^{(v)}, \ldots$ of vectors satisfying $t^{(v+1)} \equiv t^{(v)} \mod x^{r+v+1}N$

$$F(t^{(v)}) \equiv 0 \mod x^{s+v}N$$

(observe that the condition $x^r \in E(t^{(v)})$ and $t^{(v+1)} \equiv t^{(v)} \mod x^{r+v+1}N$, $F(t^{(v+1)}) \equiv 0 \mod$, $x^{r+v+1}N$ implies $x^r \in E(t^{(v+1)})$).

Proof of the lemma: If $\mathfrak{J}(F, t)$ denotes the matrix with the rows $\frac{\partial F}{\partial T_j}(t)$, we can write $F(t+y) \equiv F(t) + y\mathfrak{J}(F, t) \mod \sum_{n,j} y_i y_j A$ for any $y = (y_1, \ldots, y_n) \in I^{\oplus n}$. Hence we have to determine $y \in (x^{s-r}N)^{\oplus n}$ in such a way that

$$y\mathfrak{J}(F, t) \equiv -F(t) \mod x^{2(s-r)}N\,. \tag{1}$$

It is sufficient to solve the congruence

$$z\mathfrak{J}(F, t) \equiv -x^r F(t) \mod x^{2s}N, \quad z \in (x^s N)^{\oplus n}. \tag{2}$$

If z is a solution of the congruence (2), we can write

$$z = x^r y,\ y \in (x^{s-r}N)^{\oplus n}, \quad \text{and} \quad x^r[y\,(F, t) + F(t)] = x^r(x^{2s-r}v)$$

for a suitable vector $v \in N^{\oplus n}$, *i.e.*

$$y\mathfrak{J}(F, t) + F(t) - x^{2s-r}v \equiv 0 \mod x^{s-r}A \cap (0 : x^r)\,.$$

But from condition (iii) we infer

$$x^{s-r}A \cap 0 : x^r \subseteqq x^{s-r}A \cap (0 : x^{s-r}) = 0$$

hence $y\mathfrak{J}(F, t) \equiv -F(t) \mod x^{2s-r}N$.

We consider now the congruence (2). If $x^r = x_1 + y_2$ and if the vectors $z_i \in (x^s N)^{\oplus n}$ are solutions of the congruences $z_i\mathfrak{J}(F, t) \equiv -x_i F(t) \mod x^{2s}N$, $i = 1, 2$, the vector $z = z_1 + z_2$ is a solution from (2). Now, since $x^r \in E(t)$, it is a finite sum of an element $x_1 \in I(F)\,(t) \subseteqq x^r N$ and of elements $\delta(t)\,h(t)$, where δ is determined by a p-tuple $f \in I(F)^{\oplus n}$ as $\delta = \det\left(\frac{\partial f}{\partial T_{i_1}}, \ldots, \frac{\partial f}{\partial T_{i_p}}\right)$, and where h is an element of $I(f) : I(F)$. For x_1 we can take $z_1 = 0$ to solve the congruence $z_1\mathfrak{J}(F, t) \equiv -x_1 F(t) \mod x^{2s}N$.

On the other hand consider elements $\delta(t)\,h(t)$, assume for example that $\delta = \det\left(\frac{\partial f}{\partial T_1}, \ldots, \frac{\partial f}{\partial T_p}\right)$. There exists a $(p \times q)$-matrix ψ over $A\{\{T\}\}$ such that

$$hF = f\psi \tag{3}$$

and a $(p\times n)$-matrix Γ over A such that

(4) $\Gamma\mathfrak{J}(f,t)=\delta(t)\,I_p \quad (I_p(p\times p)\text{-unite matrix})$

(since $\delta(t)$ is a $(p\times p)$-minor of $\mathfrak{J}(f,t)$). From (3) we infer (since $F(s)\equiv 0 \bmod x^sN$)

$$h(t)\,\mathfrak{J}(F,t)\equiv\mathfrak{J}(f,t)\,\psi(t) \bmod x^sN\,,$$

hence by (4)

$$h(t)\,\Gamma\mathfrak{J}(F,t)\equiv\delta(t)\,\psi(t) \bmod x^sN$$

and by (3)

$$(h(t)\,f(t)\,\Gamma)\,\mathfrak{J}(F,t)\equiv\delta(t)\,h(t)\,F(t) \bmod x^{2s}N\,.$$

Therefore the vector $z=h(t)\,f(t)\,\Gamma\in(x^sN)^{\oplus n}$ solves the congruence

$$z\mathfrak{J}(F,t)\equiv\delta(t)\,h(t)\,F(t) \bmod x^{2s}N\,, \qquad \text{q.e.d.}$$

Proof of theorem 5. We proceed by induction on dim (A). If all elements of I are nilpotent, we can take the function $d(a,r,c)=\max(r,c,v)+1$ if $I^v=0$. Assume that $x\in I$ is not nilpotent, determine k such that $0:x^k=0:x^{k+1}=\ldots$ and for each integer s an integer $c(s)$ such that $I^{c+c(s)}\cap x^sA\subseteq I^cx^s$ (lemma of ARTIN-REES).

Define $s(a,r)=\max\,(2ar+1,ar+1)$ and $c(a,\ r)=\max\,\{c(s(a,r)),r+1\}$. Because of dim $A/x^sA<$ dim A we can assume that for each s there exists a function $\mathbf{N}^3\to\mathbf{N}$ for $(A/x^c\,A,\ I/x^c\,A)$, which we will denote by $d(s,a,r,c)$.

Then we define

$$d(a,r,c)=d(s(a,r),a,r,c+c(a,r))$$

and we show that it satisfies the assertion of the theorem. To do this assume $F\in A\{\{T\}\}^q$ and $t^0\in I^{\oplus n}$ to be given and satisfying $F(t^0)\equiv 0 \bmod I^d$ and $I^r\subseteq H(t^0)$, where $d=d(a,r,c)$. By induction there exists a vector t such that

(1) $F(t)\equiv 0 \bmod x^sA,\ s=s(a,r)$

(2) $t\equiv t^0 \bmod I^{c+c(a,r)}+x^sA\,.$

Changing t we can assume that

(2′) $t\equiv t^0 \bmod I^{c+c(a,r)}\,.$

From (1) and (2′) we infer (since $F(t^0)\equiv 0 \bmod I^d$ and $c(a,r)\geq c(s(a,r))$)

(3) $F(t)\equiv 0 \bmod x^sI^c,\quad s=s(a,r)$

and

(4) $I^r\subseteq H(t),\quad \text{hence}\quad I^{ar}\subseteq E(t)\,.$

We can therefore apply lemma 4 (with $N=I^c$ and r replaced by $a\cdot r$) to determine a sequence

(5) $t,t^1,t^2,\ldots,t^\nu,\ldots,F(t^\nu)\equiv 0 \bmod x^{s+\nu}I^c$

as in remark 2 (observe $s\geq\max\,(2ar+1,\ ar+k)$). We can write $x^r=h(t)$, $h\in H$, since $x^r\in I^r\subseteq H(t)$.

Case 1. Assume that the schema X of zeros of F on the open set $X_h=\{x\in X,\ h\neq 0 \text{ in } x\}$ is a complete intersection, i.e. defined by a regular sequence $(f_1,\ldots,f_p)$.

By the property (B) of the ideal E we infer

(5) $\qquad h^\mu \equiv 0 \bmod \Delta(f)+I(F) \quad$ and $\quad (h^\mu+g)\, I(F) \subseteqq I(f)$

for $\mu \ll 0$ and for a suitable $g \in I(F)$. Choose v such that $s(a, r)+v = \max(2ar+1, ar+k)+v \geqq 2\mu r+1$ and t^v in (5), then

(6) $\qquad F(t^v) \equiv 0 \bmod x^{2\mu r+1} I^c$

$$t^v \equiv t \bmod x^{r+1} I^c .$$

These congruences imply

$$h(t^v) \equiv h(t) \bmod x^{r+1} I^c$$

hence $Ah(t^v) = x^r A$ and $Ah^\mu(t^v) = x^{r\mu} A$. Therefore the congruences (5), (6) yields

$$x^{r\mu} \in \Delta(f)\,(t_v),\ f(t_v) \equiv 0 \bmod x^{2r\mu+1} I^c ,$$

and we can apply Theorem 3. By this theorem there exists a solution t of the equations $f(t)=0$ such that

(7) $\qquad t \equiv t^v \bmod x^{r\mu+1} I^c .$

We claim that t is also a solution of $F(t)=0$. By (6) and (7) we get $A(h^\mu(t)+g(t)) = x^{r\mu} A$, and $F(t) \equiv 0 \bmod x^{r\mu+1} I^c$. By (5) it consequently follows that

$$x^{\mu r} F(t) = 0 ,$$

hence $F(t) \equiv 0 \bmod (0 : x^{\mu r}) \cap x^{r\mu+1} I^c$. Since we can choose μ arbitrarily great, we can assume that $r\mu+1 \geqq k$, in this case $(0 : x^{\mu r}) \cap x^{r\mu+1} I^c \subseteqq (0 : x^k) \cap x^k A = 0$ holds, hence $F(t)=0$.

Case 2. The general case will be reduced to case 1. Let $G_\mu(T, Z) \in A\{\{T\}\}\, Z_1 \oplus \oplus \ldots \oplus A\{\{T\}\}\, Z_q$ generate the module of relations of F mod $I(F)^2$. Replace $A\{\{T\}\}$ by $A\,\{\{T, T', Z\}\}$ $T' = (T'_1, \ldots, T'_n)$, F by $F' = (F, T'_1, \ldots, T'_n, G_1, \ldots, G_m)$ and H by $H' = HA\{\{T,\ T',\ Z\}\} + \sum_v T'_v\ A\{\{T,\ T',\ Z\}\} + \sum_\mu Z_\mu A\{\{T,\ T',\ Z\}\}$.

If $F(t^0 \equiv)0 \bmod I^d$, replace t^0 by $t'^0 = (t^0,\ 0,\ F(t^0))$, then $F'(t'^0) \equiv 0 \bmod I^d$. If furthermore $H(t^0) \supseteqq I^r$, $H'(t^0) \supseteqq I^r$ follows (for $r \leqq d$), and if $t' + t'^0 \bmod I^c$ and $F'(t') = 0$, the first n components of t' satisfy $t \equiv t^0 \bmod I^c$ and $F(t)=0$. If E' is the ideal corresponding to the embedding $X' = V(F') \subset P' = \operatorname{Spec} A\{\{T,\ T',\ Z\}\}$, then $EA\{\{T,\ T',\ Z\}\} \subseteqq E'$. Therefore we can replace F by F'.

We consider the schemas $S = \operatorname{Spec}(A)$ and

$$\begin{array}{l} X\ = \operatorname{Spec}(A\{\{T\}\}/I(F)) \subset Y = \operatorname{Spec}(A\{\{T\}\}) \subset Z = \operatorname{Spec}(A\{\{T, T'\}\}) \\ {\scriptstyle\pi}\uparrow \\ X' = \operatorname{Spec}(A\{\{T, Z\}\}/I(F, G)) \subset Y' = \operatorname{Spec}(A\{\{T, Z\}\}) \subset Z' \\ \quad = \operatorname{Spec}(A\{\{T,\ T',\ Z\}\}) \end{array}$$

(where π denotes the projection).

We shall show that any affine open set $U' \subseteqq X' - V(H')$ is a complete intersection of $n+q$ hypersurfaces in some open subschema of Z'.

The open set $X' - V(H')$ is mapped into $X - V(H)$ under the map π. Therefore, by property (A) and the ideals E and E; respectively, if we consider the universal separated differential modules (denoted by Ω^1) and the conormal sheaves (denoted by N) restricted to $X' - V(H')$, we have the following exact sequences of locally free sheaves

(1) $0 \to N_{Y'|Z'} \otimes \mathcal{O}_{X'} \to N_{X'|Z'} \to N_{X'|Y'} \to 0$

(2) $0 \to \pi^* N_{X|Y} \to \pi^* \Omega^1_{Y|S} \to \pi^* \Omega^1_{X|S} \to 0$

(3) $0 \to \pi^* \Omega^1_{X|S} \to \Omega^1_{X'|S} \to \Omega^1_{X'|X} \to 0$

(4) $0 \to N_{X'|Y'} \to \Omega^1_{Y'|S} \otimes \mathcal{O}_{X'} \to \Omega^1_{X'|S} \to 0$

and

$$N_{Y'|Z'} \otimes \mathcal{O}_{X'} \cong \pi^* \Omega^1_{Y|S}$$
$$\pi^* N_{X|Y} \cong \Omega^1_{X'|S}.$$

5. The I-adic completion of W-categories

Let us start with a W-category $\mathbf{H}$ and a pair $(A, I) \in \mathbf{H}$. We will construct a W-category $\mathbf{H}_{A'}$ over the I-adic completion A' of A which is in a certain sense minimal.

We will need this construction for the proof of the approximation theorem for W-categories. It is exactly at this stage where the theory of W-categories is still a little bit complicate because we where not able to prove that this construction preserves the property of being NOETHERian. That is why we has to develop the whole theory of W-categories in the non-NOETHERian case too.

For example, if we consider the construction in the category of HENSELian algebras of finite type, the property of being NOETHERian will be presevered. However, if we consider the category of analytic $\mathbf{C}$-algebras, and if $A \to B$ is a morphism of $\mathbf{C}$-algebras, we have to consider algebras of the type

$$B_{A'} = \bigcup B\{u_1, \ldots, u_m\} \subset B' ,$$

where $(u_1, \ldots, u_m)$ runs through the set of all finite sequences in $\mathfrak{m}_{B'}$ and where $B\{u_1, \ldots, u_m\}$ is the image of the free algebra $B\{U_1, \ldots, U_m\}$ in B' under the B-morphism defined by $U_i \mapsto u_i$. We do not know if $B_{A'}$ is NOETHERian.

The general construction runs as follows: Let $(B, \mathfrak{I})$ be an (A, I)-algebra of $\mathbf{H}$ and $\mathfrak{S}$ be the set of finite subsets of the image of IA' in $\mathfrak{I}B'$ (B' the I-adic completion of B). For a $\mathfrak{s} = (s_1, \ldots, s_N) \in \mathfrak{S}$ we define $B_\mathfrak{s}$ to be the image of $B\{\{T_1, \ldots, T_N\}\}$ in B' via the B-homomorphism $T_i \mapsto s_i$.

Definition. $B_{A'} = \bigcup_{\mathfrak{s} \in \mathfrak{S}} B_\mathfrak{s}$.

It is clear that $(B_{A'}, \mathfrak{I}B_{A'})$ is a HENSELian (A', IA')-algebra and the functor of (A, I)-algebras $(B, \mathfrak{I}) \mapsto (B_{A'}, \mathfrak{I}B_{A'})$ is a functor of the subcategory $\mathbf{H}_A$ of all pairs $(B, \mathfrak{I})$ over (A, I) of $\mathbf{H}$ into the category of HENSELian pairs over $(\hat{A}, I\hat{A})$.

Lemma 5. *The canonical morphism $B/I^qB \to B_{A'}/I^qB_{A'}$, $q = 1, 2, \ldots$, is an isomorphism and B is the $\mathfrak{I}B_{A'}$-adic completion of $B_{A'}$.*

Proof. We know that $IB_{A'} = \bigcup_{\mathfrak{s} \in \mathfrak{S}} (IB_{\mathfrak{s}} + \sum_{s_v \in \mathfrak{s}} s_v B_{\mathfrak{s}})$, so the canonical morphism

$$B/IB \to B_{A'}/IB_{A'} = \varinjlim_{\mathfrak{s}} B_{\mathfrak{s}}/(IB_{\mathfrak{s}} + \sum_{s_v \in \mathfrak{s}} s_v B_{\mathfrak{s}})$$

is surjective, i.e. $B_{A'} = B + IB_{A'}$. But this also means that $B_{A'} = B + I^q B_{A'}$, $q \geqq 1$. Since $I^q B' \cap B = I^q B$, the morphism $B/I^q B \to B_{A'}/I^q B_{A'}$ is also injective. This also gives us the isomorphism $B/\mathfrak{J}^q \to B_{A'}/\mathfrak{J}^q B_{A'}$ and the lemma is proved.

Definition/Proposition 2. *Let* $\mathbf{H}_{A'}$ *be the category of all Henselian pairs* $(\bar{B}, \bar{\mathfrak{J}})$ *being finite over a pair* $(B_{A'}, \mathfrak{J}B_{A'})$ *for an* (A, I)*-algebra* $(B, \mathfrak{J}) \in \mathbf{H}$ *such that* $\bigcap_{v=0}^{\infty} I^v \bar{B} = 0$ *and* $A' \to \bar{B}/\bar{\mathfrak{J}}$ *is surjective. Then* $\mathbf{H}_{A'}$ *is a W-category and* $(B, \mathfrak{J}) \hookrightarrow (B_{A'}, \mathfrak{J}B_{A'})$ *is a functor* $\mathbf{H}_A \to \mathbf{H}_{A'}$.

Proof. $(W\ 0)$ and $(W\ 2)$ are fulfilled by definition. To prove $(W\ 3)$ we choose a finite sequence $T = (T_1, \ldots, T_N)$ of indeterminates and $(\bar{B}, \bar{\mathfrak{J}}) \in \mathbf{H}_{A'}$. We may suppose that $(\bar{B}, \bar{\mathfrak{J}})$ is the quotient of a $(B_{A'}, \mathfrak{J}B_{A'})$ of $\mathbf{H}_{A'}$ with kernel N (if $(B, \bar{\mathfrak{J}})$ is finite over $(B_{A'}, \mathfrak{J}B_{A'})$, then there is a surjective map $B\{\{X\}\}_{A'} \to \bar{B}$ for some $(X_1, \ldots, X_l) = X$. We will show that $B\{\{T\}\}_{A'}/\bar{N}$ with $\bar{N} = \bigcap_{q=1}^{\infty} (NB\{\{T\}\}_{A'} + I^q B\{\{T\}\}_{A'})$ is the free pair $\bar{B}\{\{T\}\}$. Let $(\bar{C}, \bar{K})$ be a $(\bar{B}, \bar{\mathfrak{J}})$-algebra of $\mathbf{H}_{A'}$ and $\bar{t}_1, \ldots, \bar{t}_N \in \bar{K}$. We have to show that there exists exactly one $(\bar{B}, \bar{\mathfrak{J}})$-morphism $f : B\{\{T\}\}_{A'}/\bar{N} \to \bar{C}$ with $f(T_i) = \bar{t}_i$. We choose an (A, I)-algebra $(C, K) \in \mathbf{H}$ such that $(\bar{C}, \bar{K})$ is the quotient of $(C_{A'}, KC_{A'})$ with kernel N'. Now $\bar{C} = \bigcup_{\mathfrak{s} \in \mathfrak{S}} C_{\mathfrak{s}}/N' \cap C_{\mathfrak{s}}$ and so the $\bar{t}_1, \ldots, \bar{t}_N$ are already in some $C_{\mathfrak{s}}/N' \cap C_{\mathfrak{s}}$.

By construction $C_{\mathfrak{J}}/N' \cap C_{\mathfrak{J}} \in \mathbf{H}$ holds and this is a $(B, \mathfrak{J})$-algebra. We obtain a unique $(B, \mathfrak{J})$-morphism $f_0 : B\{\{T\}\} \to C_{\mathfrak{s}}/N' \cap C_{\mathfrak{s}}$ with $T_v \mapsto t_v$ and the following commutative diagram.

$$\begin{array}{ccc} B\{\{T\}\} & \to & \bar{C} \\ \downarrow & & \downarrow \\ B'[[T]] & \to & \bar{C}' . \end{array}$$

For this reason we can lift f_0 to a $(\bar{B}, \bar{\mathfrak{J}})$-morphism $B\{\{T\}\}_{A'} \to \bar{C}$ annulling $\bar{N}$, i.e. $B\{\{T\}\}_{A'}/\bar{N} = \bar{B}\{\{T\}\}$ is free in $\mathbf{H}_{A'}$.

Now we have to prove that the canonical morphisms

$$\bar{B}\{\{T\}\}/(T_1, \ldots, T_N)^v \to \bar{B}[T]/(T_1, \ldots, T_N)^v$$

are injective, or the canonical morphisms

$$\Phi_v : \bar{B}[T] \to \bar{B}\{\{T\}\}/(T_1, \ldots, T_N)^v$$

are surjective.

However, the canonical morphism $\varinjlim_{\mathfrak{J}} B_{\mathfrak{J}}\{\{T\}\} \to \bar{B}\{\{T\}\}$ being surjective we obtain a commutative diagram

$$\begin{array}{ccc} \bar{B}[T] & \xrightarrow{\Phi_v} & \bar{B}\{\{T\}\}/(T_1, \ldots, T_N)^v \\ \downarrow & & \downarrow \\ \varinjlim_{\mathfrak{s}} B_{\mathfrak{s}}[T] & \twoheadrightarrow & \varinjlim_{\mathfrak{s}} B_{\mathfrak{s}}\{\{T\}\}/(T_1, \ldots, T_N)^v \end{array}$$

Then Φ_v is also surjective.

To prove (W 1) let $(\bar{B}, \bar{\mathfrak{I}}) \to (\bar{B}'', \bar{\mathfrak{I}}'')$ be a morphism in $\mathbf{H}_{A'}$ and $\bar{K} \subseteqq \bar{B}''$ a closed ideal (with respect to the $\bar{\mathfrak{I}}$-adic topology) such that $\bar{B}''/\bar{K} + \mathfrak{I}\bar{B}''$ is $\bar{B}$-finite. We will prove that $\bar{B}''/\bar{K}$ is $\bar{B}$-finite. Now $(\bar{B}, \bar{\mathfrak{I}})$ is finite over some $(B_{A'}, \mathfrak{I}B_{A'})$ for a $(B, \mathfrak{I}) \in \mathbf{H}$ and $(B, \mathfrak{I})$ is the quotient of some $A\{\{T\}\}$ in $\mathbf{H}$. So we may suppose that $(\bar{B}, \bar{\mathfrak{I}})$ is the free object $A'\{\{T\}\}$ in $\mathbf{H}_{A'}$. On the other hand $(\bar{B}'', \bar{\mathfrak{I}}'')$ is the quotient of some pair $(B''_{A'}, \mathfrak{I}''_{A'})$ for a suitable pair $(B'', I'') \in \mathbf{H}$. Since $\bar{B}$ is free, we may assume that $(\bar{B}'', \mathfrak{I}'') = (B''_{A'}, \mathfrak{I}''B''_{A'})$. Let $\bar{K} + IB''_{A'} = f_1 B''_{A'} + \ldots + f_r B''_{A'} + IB''_{A'}$, $f_1, \ldots, {}_r \in \bar{K}$ ($B''_{A'}/IB''_{A'} = B''/IB''$ is noetherian). We consider the free algebra $A'\{\{T, T'\}\}$, $T' = (T'_1, \ldots, T'_r)$, and choose an $A'\{\{T\}\}$-morphism $A'\{\{T, T'\}\} \to B''_{A'}$, $T'_i \mapsto f_i$. Now the algebra $B''_{A'}/(IA)\{\{T, T'\}\} B''_{A'} = B''_{A'}/\bar{K} + (IA')\{\{T\}\}$ is $A'\{\{T, T'\}\}$-finite and it is sufficient to show that the algebra $B''_{A'}$ is $A'\{\{T, T'\}\}$-finite.

Thus we are in the following situation: Let $(B''_{A'}, \mathfrak{I}''B''_{A'})$ be an $A'\{\{T\}\}$ algebra such that $B''_{A'}/(IA')\{\{T\}\} B''_{A'}$ is $A'\{\{T\}\}$-finite. We have to show that $B''_{A'}$ is $A'\{\{T\}\}$-finite. Since $B''_{A'}/IB''_{A'} = B''/IB'' \in \mathbf{H}$, we infer that $B''_{A'}/IB''_{A'}$ is $A'\{\{T\}\}$-finite, too. We choose a set of generators modulo $IB''_{A'}$ $W_1, \ldots, W_q \in B''_{A'}$ and consider for suitable $\mathfrak{s} = (s_1, \ldots, s_r)$ (such that $W_1, \ldots, W_q$ and the images of $T_1, \ldots, T_N$ in $B'_{A'}$ are in $B''_{\mathfrak{s}}$) the algebra-morphism. $A\{\{T\}\}_{\mathfrak{s}} \to B''_{\mathfrak{s}}$. It is clear that $B''_{\mathfrak{s}}/IB''_{\mathfrak{s}} + \sum_{v=1}^{r} s_v B''_{\mathfrak{s}}$ is $A\{\{T\}\}_{\mathfrak{s}}$-finite generated by $W_1, \ldots, W_q$. But the algebras $B''_{\mathfrak{s}}$, $A\{\{T\}\}_{\mathfrak{s}}$ are contained in $\mathbf{H}$ and consequently $B''_{\mathfrak{s}} = \sum_{i=1}^{q} A\{\{T\}\}_{\mathfrak{s}} W_i$. This implies $B''_{B'} = \sum_{i=1}^{q} A'\{\{T\}\} W_i$. Proposition 2. is proved.

6. Generalization of Artin's theorem

In this chapter we will prove the famous approximation theorem of M. Artin (cf. [2], [3]) for semi-excellent Weierstrass categories over a field or an excellent discrete valuation ring. In this way we give a common proof for the analytic and the algebraic case (cf. examples of excellent Weierstrass categories in chapter 0).

One of the basic tools to prove the approximation theorem is the following lemma:

Approximation principle. Let $\mathbf{H}$ *be a* W*-category,* $(A, I) \to (B, \mathfrak{I}) \to (C, K)$ *morphisms in* $\mathbf{H}$ *and let* A *be a* Noetherian *and* I*-adic complete ring. Let* $X \xrightarrow{p} \operatorname{Spec} C$ *be* *a quasiprojective B-scheme and s_0:* Spec $B' \to X$ *a formal section of the I-adic completion B' of B such that X is formally smooth in s_0* (Spec B') *and $ps_0(V(\mathfrak{I}))$. Let $\mathfrak{E} \xrightarrow{\varphi} \mathcal{O}_x$ be a morphism of a vector bundle of finite rank over X, $Z \subseteqq X$ the scheme of zeros of φ and $C(\varphi, s_0)$ be defined as in theorem* **3**. *If*

(a) s_0 (Spec B')$\subseteqq Z$

(b) *$B'/C(\varphi, s_0)$ is finite over A*

then there exist a $(B, \mathfrak{I})$-algebra $(D, L) \in$ **H**, *a quasiprojective D-scheme Y being smooth over B a B- morphisms: $Y \to Z$, and a formal section $\bar{t}$:* Spec $B' \to Y$ *such that $s\bar{t} = s_0$.*

As in the proof of Theorem 3 resp. ELKIK's theorem we can reduce the proof to the following.

Lemma 6. *With the same* **H**, *(A, I), $(B, \mathfrak{I})$ as in 6.1. Let $F = (F_1, \ldots, F_m) = 0$ be a system of equations, $F_i \in B\{\{T\}\}[T']$, $T = (T_1, \ldots, T_N)$, $T' = (T'_1, \ldots, T'_N)$, $N + N' \geqq m$ and $(\bar{t}, \bar{t}')$ a formal solution of $F = 0$, $\bar{t} \in \mathfrak{I}B'^N$, $\bar{t}' \in B'^{N'}$, such that for the ideal $\Delta_m(F, \bar{t}, \bar{t}')$ generated by the $(m \times m)$-minors of the matrix $\partial(F_1, \ldots, F_m)/\partial(T, T')(\bar{t}, \bar{t}')$ in B' the A-algebra $B'/\Delta_m(F, \bar{t}, \bar{t}')$ is finite.*

Then there exists a free B-algebra $B\{\{Z\}\}$ in **H**, *$Z = (Z_1, \ldots, Z_q)$, $t(Z) \in \mathfrak{I}\{\{Z\}\}$ $B\{\{Z\}\}^{N'}$, $t'(Z) \in B\{\{Z\}\}^{N'}$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_q) \in \mathfrak{I}B'$ such that $F(t(z), t'(z)) = 0$ and $t(\bar{z}) = \bar{t}$, $t'(\bar{z}) = \bar{t}'$.*

Moreover, if $K \subseteqq \mathfrak{I}$ is a finitely generated ideal in B such that B/K is NOETHERIAN *and $(\bar{t}, \bar{t}'$* mod *$KB') \in B/K$, then one can choose $\bar{z}$ to be from KB'^q.*

Proof. Let $K \subseteqq \mathfrak{I}$ be an ideal such that B/K is noetherian and $(\bar{t}, \bar{t}'$ mod $KB') \in \in B/K$. We choose $h_1, \ldots, h_a \in K$ such that $(\bar{t}, \bar{t}') = (t^0, t'^0) + \bar{M}h$, $h = \begin{pmatrix} h_1 \\ \vdots \\ h_a \end{pmatrix}$, (t^0, t'^0) $B^{N+N'}$ and $\bar{M}$ a $N + N' \times a$-matrix over B'. Now $B' = B + \Delta_0^2 B'$, $\Delta_0 = \Delta_m(F, \bar{t}, \bar{t}') \cap B$: $B'/\Delta_m(F, \bar{t}, \bar{t}')$ is a finite A-modul and a finite B-modul too. $B/\mathfrak{I} \to \to B/\mathfrak{I} \otimes_B B'/\Delta_m(F, \bar{t}, \bar{t}')$ is surjective and by the lemma of NAKAYAMA $B \to B'/\Delta_m(F, \bar{t}, \bar{t}')$ is also surjective, i.e. $B/\Delta_0 \xrightarrow{\sim} B'/\Delta_m(F, \bar{t}, \bar{t}')$. Especially B/Δ_0 is NOETHERian and complete with respect to the I-adic topology, i.e. $B/\Delta_0 \cong B'/\Delta_0 B'$. This means $\Delta_0 B' = \Delta_m(F, \bar{t}, \bar{t}')$ and $B' = B + \Delta_0 B'$. Especially we get $B' = B + \Delta_0^2 B'$. Using this fact we can write $\bar{M} = M + \sum d_i \bar{M}$, $d_i \in \Delta_0^2$, M a $(N + N') \times a$-matrix over B, M_i $(N + N') \times a$-matrices over B'. So we get $(\bar{t}, \bar{t}') = (t^0, t'^0) + Mh + \sum d_i (\bar{M}_i h)$. Now let $B\{\{Z\}\} \in$ **H**, $Z = (Z_i)_i$, $Z_j = (Z_{j1}, \ldots, Z_{j\,N+N'})$, be the free B-algebra.

We will consider the system of equations $G(T, T') = F((T, T') + \sum d_i Z_i) = 0$. The idea is to apply NEWTON's lemma to this system and to $(\bar{t}, \bar{t}') = (t^0, t'^0) + Mh$. First we will show that $\Delta_m(F, \bar{t}, \bar{t}') = \Delta_0$ and $\Delta_m(G, \bar{t}, \bar{t}') = \Delta_0 B\{\{Z\}\}$.

Obviously $(\bar{t}, \bar{t}') \equiv (\bar{t}, \bar{t}')$ mod $\Delta_0^2\, KB$ implies

$$\Delta_m(F, \bar{t}, \bar{t}')B' \subseteqq \Delta_m(F, \bar{t}, \bar{t}') = \Delta_0 B' \subseteqq \Delta_m(F, \bar{t}, \bar{t}')\, B' + \Delta_0^2 KB',$$

i.e. $\Delta_m(F, \bar{t}, \bar{t}') \subseteqq \Delta_0 \subseteqq \Delta_m(F, \bar{t}, \bar{t}') + \Delta_0^2 K$ (because $B/\Delta_2^0\, K$ is A finite we have $HB' \cap \cap B = H$ for all ideals $H \supseteqq \Delta_0^2 K$), but this means $\Delta_m(F, \bar{t}, \bar{t}') = \Delta_0$. Now we know

$(\tilde{t}, \tilde{t}') \equiv (\bar{t}, \bar{t}') + \sum d_i Z_i \bmod \Delta_0^2 \sum_{i,v} B\{\{Z\}\}\ Z_{iv}$ and similarly obtain $\Delta_m(G, \tilde{t}, \tilde{t}') = = \Delta_0 B\{\{Z\}\}$ (note that $\partial G/\partial(T, T')\ (\tilde{t}, \tilde{t}') = \partial F/\partial(T, T')\ ((\tilde{t}, \tilde{t}') + \sum d_i Z_i)$). Now $G(\tilde{t}, \tilde{t}') = F((\tilde{t}, \tilde{t}') + \sum d_i Z_i) \equiv F(\bar{t}, \bar{t}') \bmod \Delta_0^2 \sum_{i,v} Z_{iv} B\{\{Z\}\}$ and $0 = F(\bar{t}, \bar{t}') \equiv F(\bar{t}, \bar{t}')$ mod $\Delta_0^2 KB'$, i.e.

$$G(\tilde{t}, \tilde{t}') \equiv 0 \bmod \Delta_0^2 (KB\{\{Z\}\} + \sum_{i,v} Z_{iv} B\{\{Z\}\})\ .$$

We can apply NEWTON's lemma and get a solution $(\tilde{\tilde{t}}, \tilde{\tilde{t}}') \in B\{\{Z\}\}^{N+N'}$ of the system of equations $G(T, T') = 0$. Then $(t(Z), t'(Z)) = (\tilde{\tilde{t}}, \tilde{\tilde{t}}') + \sum d_i Z_i$ is a solution for $F(T, T') = 0$.

ARTIN's theorem. *Let* **H** *be a semiexcellent* WEIERSTRASS *category over an excellent discrete valuation ring* R, $(A, I) \to (B, \mathfrak{I})$ *a morphism in* **H** *and let* $X \xrightarrow{p} \operatorname{Spec} B$ *be a quasiprojective B-scheme. If* $\bar{s} : \operatorname{Spec} A' \to X$ *is a formal section of the I-adic completion* A' *of* A, *then there exists a free* (A, I)*-algebra* $(C, K) \in$ **H**, *an A-algebra-morphism* $s : \operatorname{Spec} C \to X$ *and a formal section* $\bar{s}_0 : \operatorname{Spec} A' \to \operatorname{Spec} C$ *such that* $s \circ \bar{s}_0 = \bar{s}$. The proof of this theorem divides into several steps:

Step I. *Reduction to the case that A is regular.*

Theorem 6. *Let* **H** *be a semiexcellent* WEIERSTRASS *category over an excellent discrete valuation ring R with prime element* π. *Let* $F = (F_1, \ldots, F_m) = 0$ *be a system of equations,*

$$F_i \in R\{\{X, Y\}\}\ [T], \quad X = (X_1, \ldots, X_n), \quad Y = (Y_1, \ldots, Y_N)\ ,$$

$T = (T_1, \ldots, T_{N'})$ *and* $(\bar{y}, \bar{t})$ *a formal solution of* $F = 0$,

$$\bar{y} \in (\pi, X)\ R'[[X]]^N, \quad \bar{t} \in R'[[X]]^{N'}\ .$$

Then there exist $y(Z) \in (\pi, X, Z)\ R\{\{X, Z\}\}^N$, $t(Z) \in R\{\{X, Z\}\}^{N'}$ *for a suitable* $Z = (Z_1, \ldots, Z_l)$ *and* $\bar{z} \in (\pi, X)\ R'[[X]]^l$ *such that* $F(y(Z), t(Z)) = 0$ *and* $y(\bar{z}) = \bar{y}$, $t(\bar{z}) = \bar{t}$. In this step we will show that Theorem 6 implies ARTIN's theorem. We may start with a system $(F_1^0, \ldots, F_r^0) = F^0 = 0$ of equations, $F_i^0 \in A\{\{Y\}\}\ [T]$, $A \in$ **H** and a formal solution $(\bar{y}^0, \bar{t}^0) \in A^{N+N'}$ of $F^0 = 0$.

Let $A = R\{\{X\}\}/K$ and let K be generated by $b_1, \ldots, b_h$. We choose $F_i \in R\{\{X, Y\}\}\ [T]$, $(\bar{y}, \bar{t}) \in R'[[X]]^{N+N'}$ such that $F_i \bmod KR\{\{X, Y\}\}\ [T] = F_i^0$ and $(\bar{y}, \bar{t}) \bmod KR'[[X]]^{N+N'} = (\bar{y}^0, \bar{t}^0)$.

Then $F_i(\bar{y}, \bar{t}) = \sum_{v=1}^{h} b_v \bar{w}_{vi}$ for suitable $\bar{w}_{vi} \in R'[[X]]$. We apply now Theorem 6 to the system $F_i - \sum_{v=1}^{\mu} b_v W_{vi} \in R\{\{X, Y\}\}\ [T, W]$ and get $W_{vi}(Z)$, $y(Z)$, $t(Z)$ from $R\{\{X, Z\}\}$ for a suitable $Z = (Z_1, \ldots, Z_l)$ and a $\bar{z} \in R'[[X]]^l$ such that $y(\bar{z}) = \bar{y}$, $t(\bar{z}) = \bar{t}$ and $W_{vi}(\bar{z}) = \bar{w}_{vi}$ and $F_i(y(Z), t(Z)) = \sum_{v=1}^{\mu} b_v W_{vi}(Z)$. Now $(y^0(Z), t^0(Z)) = = (y(Z), t(Z)) \bmod KR\{\{X, Z\}\} \in A\{\{Z\}\}$ is the required solution of $F^0 = 0$.

Step II. *Reduction to the case that the* $F_1, \ldots, F_m$ *of* 6. *generate the kernel I of*

$$R\{\{X, Y\}\}\ [T] \to R'[[X]] \quad and \quad I \cap R\{\{X, Y\}\} = (0)\ .$$

To prove Theorem 6. it is clear that we may suppose that $F_1, \ldots, F_m$ generate the kernel I of the morphism $\delta: R\{\{X, Y\}\}[T] \to R'[[X]]$ defined by $Y \mapsto \bar{y}$, $T \mapsto \bar{t}$ (otherwise we may add some equations). If $I \cap R\{\{X, Y\}\} = (0)$, we finished. In the other case we choose an automorphism $\varphi: R\{\{X, Y\}\}[T] \to R\{\{X, Y\}\}[T]$ with the following properties:

(i) $\varphi/R\{\{X, Y\}\} \equiv \mathrm{id}_{R\{\{X,Y\}\}} \bmod (X, Y)^2 R\{\{X, Y\}\}$

(ii) $\varphi(R\{\{X, Y\}\} \subseteq R\{\{X, Y\}\}$

(iii) $\varphi(I)$ is generated by $G_1, \ldots, G_m \in R\{\{X, Y_1, \ldots, Y_t\}\}[Y_{t+1}, \ldots, Y_N, T]$ and

$(G_1, \ldots, G_m) R\{\{X, Y_1, \ldots, Y_t\}\}[Y_{t+1}, \ldots, Y_N, T] \cap R\{\{X, Y_1, \ldots, Y_t\}\} = 0.$

(We may choose φ to be the composition of automorphisms of the type $X_i \mapsto X_i + Y_k^{n_{ki}}$, $Y_i \to Y_i + Y_k^{m_{ki}}$ and apply the Preparation Theorem to get the $G_1, \ldots, G_m$).

Now $\delta \circ \varphi^{-1}: R\{\{X, Y\}\}[T] \to R'[[X]]$ is a morphism mapping $\varphi(I)$ to zero, but in general $\delta \circ \varphi^{-1}$ is not an $R\{\{X\}\}$-morphism. Because of (i) and (ii) $\delta \circ \varphi^{-1}/R\{\{X\}\}$ induces an R-automorphism of $R'[[X]]$. Let ψ be the inverse of this automorphism, then $\psi \circ \delta \circ \varphi^{-1}: R\{\{X, Y\}\}[T] \to R'[[X]]$ is an $R\{\{X\}\}$-morphism mapping $\varphi(I)$ to zero, i.e. if we denote $\tilde{y} = \psi \circ \delta \circ \varphi^{-1}(Y)$, $\tilde{t} = \psi \circ \delta \circ \varphi^{-1}(T)$, then $G_i(\tilde{y}, \tilde{t}) = 0$, $i = 1, \ldots, m$.

Now let us suppose that Theorem 6 holds for the G_i. Then, for a suitable $Z = (Z_1, \ldots, Z_s)$, there exist $\tilde{y}(Z), \tilde{t}(Z) \in R\{\{X, Z\}\}^{N+N'}$, and $\tilde{z} \in R'[[X]]^s$ such that $G_i(\tilde{y}(Z), \tilde{t}(Z)) = 0$ and $\tilde{y}(\tilde{z}) = \tilde{y}$, $\tilde{t}(\tilde{z}) = \tilde{t}$. Let λ denote the $R\{\{X, Z\}\}$-morphism $R\{\{X, Z, Y\}\}[T] \to R\{\{X, Z\}\}$ defined by $\tilde{y}(Z), \tilde{t}(Z)$ and $\mu: R\{\{X, Z\}\} \to R'[[X]]$ the $R\{\{X\}\}$-morphism defined by $\tilde{z}$, then $\mu \circ \lambda/R\{\{X, Y\}\}[T] = \psi \circ \delta \circ \varphi^{-1}$. Let us denote the canonical prolongation of φ to $R\{\{X, Z, Y\}\}[T]$ by φ too, then $\lambda \circ \varphi$ (especially $\mu \circ \lambda \circ \varphi$) maps I to zero, but in general $\lambda \circ \varphi/R\{\{X, Z\}\}$ is not the identity (note $\lambda \circ \varphi(Z) = Z$), especially $\mu \circ \lambda \circ \varphi/R\{\{X\}\}$ may not be the canonical injection. Because of (i) and (ii) $\lambda \circ \varphi/R\{\{X, Z\}\}$ is an R-automorphism (and $\mu \circ \lambda \circ \varphi/R\{\{X\}\}$ induces an R-automorphism of $R'[[X]]$). Let σ be the inverse of $\lambda \circ \varphi/R\{\{X, Z\}\}$, then $\sigma \circ \lambda \circ \varphi$ is an $R\{\{X, Z\}\}$-morphism of $R\{\{X, Z, Y\}\}[T]$ into $R\{\{X, Z\}\}$ mapping I to zero, i.e. if $y(Z) = \sigma \circ \lambda \circ \varphi(Y)$ and $t(Z) = \sigma \circ \lambda \circ \varphi(T)$, then $F_i(y(Z), t(\mathrm{S})) = 0$, $i = 1, \ldots, m$. If we put $\bar{z} = \psi^{-1}(\tilde{z})$, it is not difficult to see that $y(\bar{z}) = \bar{y}$ and $t(\bar{z}) = \bar{t}$.

Step III. *Reduction to the finiteness condition of the approximation principle* (NERON*'s blowing up*).

We may start with the following situation (with the notations of theorem 6): $I = (F_1, \ldots, F_l)$ is the kernel of the $R\{\{X\}\}$-morphism $R\{\{X, Y\}\}[T] \to R'[[X]]$ defined by $\bar{y}$ and $\bar{t}$ and $I \cap R\{\{X, Y\}\} = 0$.

Lemma 7. *Let $F_1, \ldots, F_m \in I$ be a minimal set of generators of $IR\{\{X, Y\}\}[T]_I$ and $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ the ideal generated by the m-minors of the Jacobian matrix*

$$\partial(F_1, \ldots, F_m)/\partial(Y, T)(\bar{y}, \bar{t}), \quad \textit{then} \quad \Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \neq 0 .$$

Remark. Lemma 7 also holds if we replace the valuation ring R by a field in the assumptions of Lemma 7.

Proof. Let $B=:R\{\{X\}\}$ and without restriction of generality we may suppose that the $\bar{t}_i$ are no units, then we can lift the morphism $B\{\{Y\}\}\,[T]\to B'$ to a morphism $B'[[Y,T]]\to B'$ with the kernel H generated by the $Y_i-\bar{y}_i$, $T_j-\bar{t}_j$. Let us denote $Q=B'[[Y,T]]_H$, $\mathfrak{m}_Q$ the maximal ideal of Q and $\bar{K}$ the residue field, then we have $\mathfrak{m}_Q\cap P=\mathfrak{m}_P$ the maximal ideal of $P:=R\{\{X,Y\}\}\,[T]_I$. Now **H** is semi-excellent and consequentely Q is a formally smooth local P-algebra. Now let us consider the canonical morphism. $\varphi:\operatorname{Hom}_Q(\mathfrak{m}_Q,\bar{K})\to\operatorname{Hom}_P(\mathfrak{m}_P,\bar{K})$ induced by the inclusion $\mathfrak{m}_P\to\mathfrak{m}_Q$. With respect to the base $U_1,\ldots,U_{N+N'}$ of

$$\operatorname{Hom}_Q(\mathfrak{m}_Q,\bar{K}),\; U_i\,(Y_j-\bar{y}_j)=\delta_{ij},\quad U_i\,(T_j-\bar{t}_j)=0\quad\text{if}\quad i\leqq N$$

and $U_i\,(Y_j-\bar{y}_j)=0$, $U_i\,(T_j-\bar{t}_j)=\delta_{ij}$ if $i>N$, and the base $v_1,\ldots,v_m$ of $\operatorname{Hom}_P(\mathfrak{m}_P,K)$, $v_i(F_j)=\delta_{ij}$ the matrix associated to φ is just the Jacobian matrix $\partial(F_1\ldots,F_m)/\partial(Y,T)\,(\bar{y},\bar{t})$. To prove the lemma we have to show that φ is surjective. Let us consider the following commutative diagram:

$$\begin{array}{ccc}\operatorname{Der}_K(Q,\bar{K}) & \longrightarrow & \operatorname{Hom}_Q(\mathfrak{m}_Q,\bar{K})\\ \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varphi}\\ \operatorname{Der}_K(P,\bar{K}) & \xrightarrow[\sigma]{} & \operatorname{Hom}_P(\mathfrak{m}_P,\bar{K})\end{array}$$

given by the restriction maps ($K=Q(B)$ the fraction field of B).

We will prove that ϱ and σ are surjective. Let $v\in\operatorname{Hom}_P(\mathfrak{m}_P,\bar{K})=$ $=\operatorname{Hom}_P(\mathfrak{m}_P/\mathfrak{m}_P^2,\bar{K})$ and $P/\mathfrak{m}_P:=L\hookrightarrow P/\mathfrak{m}_P^2=:\bar{P}$ be an embedding of the coefficient field of Pwhich extends the canonical embedding $K\hookrightarrow\bar{P}$.

Such an embedding $L\hookrightarrow\bar{P}$ exists because L/K is a separable extension ($L\subseteqq Q(B')$ and $Q(B')/Q(B)$ is separable) and $\bar{P}$ is a complete local ring (Theorem of Cohen, cf. *EGA*). Now $\bar{P}\cong L\oplus\mathfrak{m}_P/\mathfrak{m}_P^2$ and clearly the morphism v is the restriction of the derivation ϑ_v, $\vartheta_v\,(x+y):=v(y)$ ($x\in L$, $y\in\mathfrak{m}_P/\mathfrak{m}_P^2$), i.e. σ is surjective.

The morphism ϱ is surjective because of the formal smoothness of the P-algebra Q (cf. *EGA*):

Let $\vartheta:P\to\bar{K}$ be a K-derivation. In order to get a lifting $\vartheta:Q\to\bar{K}$ we consider the ring $E=Q[Y]/(Y^2,Y\mathfrak{m}_Q)=Q\oplus y\bar{K}$ together with the P-algebra structure $p\cdot(q+ay)=pq+(\bar{p}a+\vartheta(p)\,\bar{q})\,y$, $p\in P$, $q\in Q$, $a\in\bar{K}$, $\bar{p}$, $\bar{q}$ the residue classes of p resp. q in $\bar{K}$. Since the canonical morphism $E\to E/yE=Q$ is a P-algebra morphism and because of the formal smoothness of Q, we can lift the identity in $\operatorname{Hom}_P(Q,Q)$ to a morphism

$$\psi':Q\to E'=Q\oplus y\bar{K}:$$

$$\begin{array}{ccc}P & \longrightarrow & Q'\\ \downarrow & \overset{\psi'}{\swarrow} & \Vert\\ E' & \to E'/yE'= & Q'\,.\end{array}$$

Now it is clear that $\psi'(Q)\subseteqq E$ and $\psi'(q)=q+y\tilde{\vartheta}(q)$. It is not difficult to verify that $\tilde{\vartheta}:Q\to\bar{K}$ is a derivation and $\tilde{\vartheta}/P=\vartheta$. The lemma is proved.

To apply the approximation principle we are intrested in having $R'[[X]]/\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ to be a finite $R'[[X_1, \ldots, X_{n-1}]]$-module. The next lemma will arrange this situation.

Lemma 8. *We use the same notations and assumptions as in lemma 7. The B-morphism $B\{\{Y\}\}[T] \to B'$ given by given by $Y_i \mapsto \bar{y}_i$, $T_i \mapsto \bar{t}_i$ extends to a B-morphism $B\{\{Y\}\}[T, Z] \to B'$, $Z = (Z_1, \ldots, Z_s)$, $Z_i \mapsto \bar{z}_i$, with kernel K and a minimal set of generators $G_1, \ldots, G_t$ of $B\{\{Y\}\}[T, Z]_K$ such that the ideal*

$$\Delta_t(G_1, \ldots, G_t, \bar{y}, \bar{t}, \bar{z}) \not\subseteq \pi B .$$

Proof. We know that $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \neq 0$ (lemma 7). If $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \not\subseteq \pi B'$ we have finished. Otherwise let $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \not\subseteq \pi^k B$ $k > 1$ and we will drop this k step by step. Let $l(F_1, \ldots, F_m, \bar{y}, \bar{t}) = \mathrm{ord}_\pi \Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ and $l(I, \bar{y}, \bar{t}) = \min\{l(F_1, \ldots, F_m, \bar{y}, \bar{t}), F_1, \ldots, F_m \in I$ generating $IB\{\{Y\}\}[T]_I\}$.

To prove the lemma it is sufficient to prove the following lemma:

NERON's *π-desingularization: With the notations and assumptions of lemma 7 the B-morphism $B\{\{Y\}\}[T] \to B'$ extends to a B-morphism $B\{\{Y\}\}[T, Z] \to B'$, $Z = (Z_1, \ldots, Z_e)$, $Z_i \mapsto \bar{z}_i$, with kernel K such that $l(K, \bar{y}, \bar{t}, \bar{z}) < l(I, \bar{y}, \bar{t})$.*

To prove this lemma we may suppose that $l(I, \bar{y}, \bar{t}) = k > 0$. Let $F_1, \ldots, F_m \in I$ such that $\mathrm{ord}_\pi \Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) = k$, i.e. there exists an m-minor of the JACOBIan matrix $\partial(F_1, \ldots, F_m)/\partial(Y, T)\,(\bar{y}, \bar{t})$ which is exactly divisible by π^k. Let $K = \{f \in B\{\{Y\}\}[T], \pi / f(\bar{y}, \bar{t})\}$ and $G_1, \ldots, G_e \in B\{\{Y\}\}[T]$ such that $\pi, G_1, \ldots, G_e$ is a minimal set of generators of $B\{\{Y\}\}[T]_K$. The residue classes of the G_i mod π generate $\bar{K}(B\{\{Y\}\}[T]/\pi\, B\{\{Y\}\}[T]_{\bar{K}} =: \mathfrak{m}$, $\bar{K}$ is the kernel of the B/π B-morphism $B/\pi B\,\{\{Y\}\}[T] \to B'/\pi B'$ induced by the residue classes of the $\bar{y}_j$ and $\bar{t}_j$ mod π. Using Lemma 7 (Remark) we obtain that the ideal generated by the e-minors of the Jacobian matrix of the G_i mod π is not in $\mathfrak{m}$, i.e.

(i) $\Delta_e(G_1, \ldots, G_e, \bar{y}, \bar{t}) \not\subseteq \pi B'$.

Furthermore the residue classes of $F_1, \ldots, F_m$ mod π are linearly dependent in $\mathfrak{m}/\mathfrak{m}^2$, i.e. especially

(ii) $I \cap K^2 \not\subseteq K \cdot I$.

If the $F_1, \ldots, F_m$ were linearly independent, we could find $F_{m+1}, \ldots, F_e \in K$ such that $F_1, \ldots, F_e$ mod π be a base of $\mathfrak{m}/\mathfrak{m}^2$. By the above Lemma the ideal generated by the e-minors of the Jacobian of $F_1, \ldots, F_e$ mod π is not in $\mathfrak{m}$. But this is not possible because the ideal generated by the m-minors of the JACOBIan of the $F_1, \ldots, F_m$ mod π is already in $\mathfrak{m}$.

Now let us consider the B-morphism $B\{\{Y\}\}[T, Z] \to B'$, $Z = (Z_1, \ldots, Z_e)$ and $Y_i \mapsto \bar{y}_i$, $T_i \mapsto \bar{t}_i$, $Z_i \mapsto \dfrac{G_i(\bar{y}, \bar{t})}{\pi}$. Let $\mathfrak{J}$ be the kernel of this morphism, then $\pi Z_i - G_i$ and I are in $\mathfrak{J}$ and because of the assumptions at the beginning of (III) we have $\mathfrak{J} \cap B\{\{Y\}\} = 0$ and $ht\mathfrak{J} = m + e$.

We will show that $l(\mathfrak{J}, \bar{y}, \bar{t}, \bar{z}) < k$. We have to choose a "good" minimal set of generators $H_1, \ldots, H_{m+e}$ of $B\{\{Y\}\}[T, Z]_I$ such that $\mathrm{ord}_\pi \Delta_{m+e}(H_1, \ldots, H_{m+e}, \bar{y}, \bar{t}, \bar{z}) < k$. By definition of the G_i we can choose a $H \in B\{\{Y\}\}[T]$ $H \not\in K$ and D_i,

$K_{ij} \in B\{\{Y\}\}\,[T]$ such that

$$HF_i = \pi D_i + \sum_{j=1}^{e} K_{ij} G_j ,$$

i.e.

(1) $$HF_i = \pi \tilde{H}_i + \sum_{j=1}^{e} K_{ij}\,(G_j - \pi Z_j),\quad \tilde{H}_i \in B\{\{Y\}\}\,[T,\,Z].$$

Using (ii) we can find $L_1, \ldots, L_m \in B\{\{Y\}\}\,[T]$, $L_k \notin K$ for some $k \in \{1, \ldots, m\}$ such that $\sum_{i=1}^{m} L_i F_i \in K^2$, i.e.

(2) $$\sum_{i=1}^{m} L_i F_i = \sum_{j=1}^{e} M_j\,(G_j - \pi Z_j) + \pi^2 \tilde{H}$$

for suitable $\tilde{H}$, $M_j \in B\{\{Y\}\}\,[T,\,Z]$.

Now $H_1 := \tilde{H}_1, \ldots, H_{k-1} := \tilde{H}_{k-1}$, $H_k := \tilde{H}$, $H_{k+1} := \tilde{H}_{k+1}, \ldots, H_m := \tilde{H}_m$, $H_{m+1} := \pi Z_1 - G_1, \ldots, H_{m+e} := \pi Z_e - G_e$ generate $\mathfrak{J}B\{\{Y\}\}\,[T,\,Z]_{\mathfrak{J}}$.

Using (1) and (2) we will show that:

(3) $$\operatorname{ord}_\pi \Delta_{m+e}(H_1, \ldots, H_{m+e}, \bar{y}, \bar{t}, \bar{z}) < k\,.$$

Differentiating (1) and (2) we obtain

(4) $$H\partial F_i/\partial(Y,\,T) \equiv \pi\partial H_i/\partial(Y,\,T) + \sum_{j=1}^{e} (K_{ij})\,\partial G_j/\partial(Y,\,T) \bmod \mathfrak{J}$$

$$\frac{\partial H_i}{\partial Z_t} = K_{it} \bmod \mathfrak{J}$$

$$i = 1, \ldots, k-1, k+1, \ldots, m$$

(5) $$\sum_{i=1}^{m} L_i\partial F_i/\partial(Y,\,T) \equiv \sum_{j=1}^{e} M_j\partial G_j/\partial(Y,\,T) + \pi^2\partial H_k/\partial(Y,\,T) \bmod \mathfrak{J}$$

$$\pi\,\frac{\partial H_k}{\partial Z_t} \equiv M_t \bmod \mathfrak{J}$$

(6) $$\partial H_{m+i}/\partial(Y,\,T) = -\partial G_i/\partial(Y,\,T)$$

$$\partial H_{m+i}/\partial Z_j = \pi \cdot \delta_{ij}$$

$$i = 1, \ldots, e\,.$$

Using (4), (5) and (6) we can see that the Jacobian matrix.

(7) $$\partial(\pi H_1, \ldots, \pi H_{k-1}, \pi^2 H_k, \pi H_{k+1}, \ldots, \pi H_m, H_{m+1}, \ldots, H_{m+e})/$$
$$\partial(Y,\,T,\,Z)\,(\bar{y},\,\bar{t},\,\bar{z}) \text{ is equivalent to the matrix}$$

(8) $$\begin{pmatrix} \partial(F)/\partial(Y,\,T)\,(\bar{y},\,\bar{t}) & 0 \\ -\partial(G)/\partial(Y,\,T)\,(\bar{y},\,\bar{t}) & \pi I_e \end{pmatrix},$$

I_e the $e \times e$ unite matrix. By the definition of $l(I,\,\bar{y},\,\bar{t}) = k$ the matrix $\partial(F)/\partial(Y,\,T)$ $(\bar{y},\,\bar{t})$ has a m-minor which is not divisible by π^{k+1}. Then it is clear that the matrix

(8) has an $m+e$-minor not divisible by $\pi^{k+1+d-(d-m)}=\pi^{k+m+1}$, because $\Delta_e(G, \bar{y}, \bar{t}) \subseteqq / \pi B'$ (cf. (i)). Finally we have an $m+e$-minor of (7) not divisible by π^{k+m+1}, i.e. there is an $m+e$-minor of $\partial(H_1, \ldots, H_{m+e})/\partial(Y, T, Z)$ $(\bar{y}, \bar{t}, \bar{z})$ not divisible by π^k, i.e. (3) holds and consequently $l(K, \bar{y}, \bar{t}, \bar{z})<k$. NERON's desingularization step is finished.

Step IV. *Induction on n using the approximation principle.* We may start with the following situation (with the notations of Theorem 6):

$I=(F_1, \ldots, F_e)$ is the kernel of the $R\{\{X\}\}$-morphism $R\{\{X, Y\}\}[T] \to R'[[X]]$ defined by $\bar{y}$ and $\bar{t}$, and

$$\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \subseteqq / \pi R'[[X]] .$$

Clearly it is enough to look for a solution of $F_1=\ldots=F_m=0$ ($F_1, \ldots, F_m$ generate $IR\{\{X, Y\}\}[T]_I$): Let us suppose for a moment that for suitable $Z=(Z_1, \ldots, Z_t)$, $y(Z), t(Z) \in R\{\{X, Z\}\}$ and $\bar{z} \in R'[[X]]^t$

$$F_i(y(Z), t(Z))=0 \quad i=1, \ldots, m \quad \text{and} \quad y(\bar{z})=\bar{y}, \quad t=(\bar{z})=\bar{t} .$$

If $r>m$, then $G \cdot F_r=\sum_{j=1}^{m} H_{rj}F_j$ for a suitable $G \in / I$, i.e.

$$G(y(Z), t(Z)) \cdot F_r(y(Z), t(Z))=0 .$$

Now it is clear that $G \in / I$ implies $G(\bar{y}, \bar{t}) \neq 0$ and especially $G(y(Z), t(Z)) \neq 0$, i.e. $F_r(y(Z), t(Z))=0$.

1. Case $n=0$.

In this case $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ is a unit (for simplicity we may suppose the $\bar{t}_i$ not to be units). Let

$$\det \partial(F_1, \ldots, F_m)/\partial(Y_{r_1}, \ldots, Y_{r_s}, T_{j_{s+1}}, \ldots, T_{j_m})\, (\bar{y}, \bar{t})$$

be a unit. Then we consider the system

$$F_1=0, \ldots, F_m=0, \quad F_{m+1}=: Y_{r_{s+1}}-Z_1=0, \ldots, F_{N+N'}=: T_{j_s}-Z_{N+N'-m}=0$$

defined over $R\{\{Z_1, \ldots, Z_{N+N'-m}, Y\}\}[T]$. We have $F_i(0,0) \equiv 0 \bmod (\pi, Z_1, \ldots, Z_{N+N'-m})$ and $\det(\partial(F_1, \ldots, F_{N+N'})/\partial(Y, T)\,(0,0))=$ unit in $R\{\{Z_1, \ldots, Z_{N+N'-m}\}\}$.

Using the implicit function theorem we get $y(Z), t(Z) \in R\{\{Z_1, \ldots, Z_{N+N'-m}\}\}$ such that

$$\begin{aligned} &F_i(y(Z), t(Z))=0 && i=1, \ldots, m \\ &y_{r_{s+j}}(Z) = Z_j && j=1, \ldots, N'-s \\ &t_{j_k}(Z) = Z_{N'-s+k} && k=1, \ldots, N+s-m . \end{aligned}$$

Using the implicit function theorem for the system

$$F_i(y(Z), t(Z))$$

defined over $R'[[Z]]$ again we get the required $\bar{z} \in R'^{N+N'-m}$ such that $y(\bar{z})=\bar{y}$ and $t(\bar{z})=\bar{t}$ putting $\bar{z}_1=\bar{y}_{r_{s+1}}, \ldots, \bar{z}_{N+N'-m}=\bar{t}_{j_s}$.

2. Case: $n > 0$.

We suppose that the theorem 6 is true for $R\{\{X_1, \ldots, X_{n-1}\}\}$. We may further suppose (after having applied a suitable R'-automorphism of $R'[[X]]$ and having used the preparation theorem) that $R'[[X]]/\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ is a finite $R'[[X_1, \ldots, X_{n-1}]]$-module. Now we may apply the approximation principle lemma 6 to the pairs $(R'[[X_1, \ldots, X_n]], (\pi, X_1, \ldots, X_n))$ and $R\{\{X\}\}_{R'[[X_1,\ldots,X_n]]}$. There exists a free algebra $R\{\{X\}\}_{R'[[X_1,\ldots,X_{n-1}]]}\{\{Z\}\}$,

$$Z = (Z_1, \ldots, Z_s) \quad \text{and} \quad \tilde{y}(Z), \quad \tilde{t}(Z) \in R\{\{X\}\}_{R'[[X_1,\ldots,X_{n-1}]]}\{\{Z\}\}^{N+N'}$$

and $\bar{z} \in R'[[X]]^s$ such that

$$F_i(\tilde{y}(Z), \tilde{t}(Z)) = 0 \quad i = 1, \ldots, m \quad \text{and} \quad \tilde{y}(\bar{z}) = \bar{y}, \tilde{t}(\bar{z}) = \bar{t}\,.$$

Now $R\{\{X\}\}_{R[[X_1,\ldots,X_{n-1}]]}\{\{Z\}\} = R\{\{X, Z\}\}_{R'[[X_1,\ldots,X_{n-1}]]}$ and looking carefully at the construction of $R\{\{X, Z\}\}_{R'[[X_1,\ldots,X_{n-1}]]}$ we can choose a

$$\bar{u} = (\bar{u}_1, \ldots, \bar{u}_k), \quad \bar{u}_i \in (\pi, X_1, \ldots, X_{n-1})\, R'[[X_1, \ldots, X_{n-1}]]\,,$$

such that $(\tilde{y}(Z), \tilde{t}(Z)) \in R\{\{X, Z\}\}_{\bar{u}}^{N+N'}$. Let K be the kernel of the $R\{\{X_1, \ldots, X_{n-1}\}\}$-morphism $R\{\{X_1, \ldots, X_{n-1}, U\}\} \to R\{\{X_1, \ldots, X_{n-1}\}\}_{\bar{u}}$ defined by $U \mapsto \bar{u}$, $u(Z') \in R\{\{X_1, \ldots, X_{n-1}, Z'\}\}$ be a zero of K, $Z' = (Z_1', \ldots, Z_b')$ and $\bar{z}' \in R[[X_1, \ldots, X_{n-1}]]^b$ such that $u(\bar{z}') = \bar{u}$.

We consider the $R\{\{X_1, \ldots, X_{n-1}\}\}$-morphism

$$R\{\{X_1, \ldots, X_{n-1}\}\}_{\bar{u}}\,\{\{X, Z\}\} \to R\{\{X, Z, Z'\}\}$$

defined by $\bar{u} \mapsto u(Z')$, $X \mapsto X$, $Z \mapsto Z$ and denote the image of $\tilde{y}$ resp. $\tilde{t}$ via this morphism by $y(Z, Z')$ resp. $t(Z, Z')$. Then $F_i(y(Z, Z'), t(Z, Z')) = 0$ and $y(\bar{z}, \bar{z}') = \bar{y}$, $t(\bar{z}, \bar{z}') = \bar{t}$. The theorem is proved.

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