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MILNOR Number of Complete Intersections and NEWTON Polygons

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The main result of this paper will be a formula to compute the Milnor number of an isolated complete intersection singularity using the Newton polygon. We were inspired by the articles of KOUCHNIRENKO [4], who gave such a formula for hypersurfaces, and GREUEL and HAMM [2], who proved a similar result for quasihomogeneous complete intersections with slightly different methods. We use the filtrations of KOUCHNIRENKO to generalize the methods of GREUEL and HAMM. Let K be an algebraically closed field of characteristic 0, $f_1, \ldots, f_r \in A := K\langle X_1, \ldots, X_m\rangle$ (the ring of algebraic power series over K in m variables $X_1, \ldots, X_m$), $r<m$, such that $\mathfrak{O}_X := A/(f_1, \ldots, f_r)$ is a $n := m-r$-dimensional K-algebra with isolated singularity. We associate the Newton polygon $\Gamma_{\underline{f}}$ to $\mathfrak{O}_X$ resp. $\underline{f} := (f_1, \ldots, f_r)$: Let $\mathfrak{M}_{\underline{f}} = \bigcup_{i=1}^{r} \operatorname{supp}(f_i)$, $\operatorname{supp}(\sum a_{i_1\ldots i_m} X_1^{i_1}, \ldots, X_m^{i_m}) = \{\underline{i} := (i_1, \ldots, i_m) \in \mathbf{N}^m, a_{\underline{i}} \neq 0\}$, $\Gamma_{\underline{f}}$ is the union of the compact faces of the boundary of the convex envelop $\Gamma(\underline{f})$ of $\mathfrak{M}_{\underline{f}} + \mathbf{N}^m$ in $\mathbf{R}_+^m$ ($\mathbf{R}_+$ the positive real numbers, $\mathbf{N}$ the positive integers).

We can show that similar to [2] and [4] for almost all complete intersections $\underline{f}$ the MILNOR number $\mu(\mathfrak{O}_X)$ cf. [1] is equal to the NEWTON number $\nu(\Gamma_{\underline{f}})$ of the polygon $\Gamma_{\underline{f}}$ cf. [4], i.e. to a suitably counted volume of the polyhedron $\Gamma_-(\underline{f}) := (\mathbf{R}_+^m \setminus \Gamma(\underline{f})) \cup \Gamma_{\underline{f}} \cup \{0\}$ given by $\Gamma_{\underline{f}}$.

1. The NEWTON polygon and the main theorem

We will keep the notations of the introduction. To get a finite volume of the polyhedron $\Gamma_-(\underline{f})$ we have to suppose that the NEWTON polygon intersects all coordinate axes of $\mathbf{R}_+^n$:

1.1. Definition. *The complete intersection $\underline{f}$ is called conveninent if $\mathfrak{M}_{\underline{f}} \cap \{(0, \ldots, i_k, 0, \ldots, 0), i_k \in \mathbf{N}\} \neq \emptyset$, $k = 1, \ldots, m$.*

In the paper of KOUCHNIRENKO [4] the NEWTON number $\nu(\Gamma_{\underline{f}})$ is just

$$\sum_{d=0}^{m} (-1)^{m-d} d!\, V_d,$$

where $V_0 = 1$, V_m is the m-dimensional volume of $\Gamma_-(\underline{f})$ and $V_d\,(d<m)$ is the sum of the d-dimensional volumes of the intersections of $\Gamma_-(\underline{f})$ with all coordinate plans of dimension d. If we have more than one equation, then the coefficients λ_d of the d-dimensional volumes will change and some more information of the $\underline{f}$ will enter (cf. [2]). To make this precise we must look at the NEWTON polygon more carefully.

As in the paper of KOUCHNIRENKO [4] we can associate a homogeneous map $h:\mathbf{R}_+^m \to \mathbf{R}_+$ to our NEWTON polygon such that $h(\Gamma)=1$: Let ψ_d be the set of closed $(d-1)$-dimensional faces of $\Gamma(\underline{f})$ which do not lie on any coordinate plan. Any $\Delta\in\psi_m$ is in a unique hyperplane given by $\Sigma\, P_{\Delta,i}\cdot X_i = M$. We will choose $P_{\Delta,i}$ such that M is the same for all Δ and gdc $(\{P_{\Delta,i}\}_{\Delta\in\psi_m, i=1,\dots,m}, M)=1$.

Then the $P_{\Delta,i}$ and M are unique and

$$h(\underline{x}) = \frac{1}{M}\min\{\langle \underline{x}, \underline{P}_\Delta\rangle, \Delta\in\psi_m\}$$

(here $\underline{x}\in\mathbf{R}_+^m$, $\underline{P}_\Delta = (P_{\Delta,1},\dots,P_{\Delta,m})$ and $\langle\,,\rangle$ is the scalar product).

With h we get a filtration on A given by

$$A_{(q)} := \{f\in A,\ M\cdot h\,(\mathrm{supp}\,(f))\subseteqq q+\mathbf{N}\}\,.$$

It is not difficult to see that

$$A_{(q)}\cdot A_{(r)}\subseteqq A_{(q+r)},\quad \bigcap_{q\geqq 0} A_{(q)} = 0\,. -$$

Let $\bar{A}$ be the graded ring given by this filtration. We can consider $\bar{A}$ to be the polynomial ring $K[X_1,\dots,X_m]$ with the multiplication

$$\underline{X}^{\underline{\nu}}\cdot\underline{X}^{\underline{\mu}} = \begin{cases}\underline{X}^{\underline{\nu}+\underline{\mu}} & \text{if}\quad \underline{\nu},\underline{\mu}\in K(\Delta)\quad\text{for a}\quad \Delta\in\psi_m\\ 0 & \end{cases}$$

(here $K(\Delta)$ is the convex cone defined by Δ, i.e. the union of all lines in $\mathbf{R}_+^m$ through 0 and Δ; $\underline{X}=(X_1,\dots,X_m)$, $\underline{\nu}=(\nu_1,\dots,\nu_m)\in\mathbf{N}^m$ and $\underline{X}^{\underline{\nu}} = X_1^{\nu_1}\cdot\ldots\cdot X_m^{\nu_m}$).

This graded ring $\bar{A}$ will play an important role in giving the connection between the MILNOR number and the NEWTON number. $\bar{A}$ can be described by the rings $A_\Delta = K[\underline{X}^{\underline{\mu}}, \underline{\mu}\in K(\Delta)]$ of the faces Δ.

1.2. Proposition. *Let Δ be a face of Γ, then*

(i) $A_\Delta = \bar{A}/I_\Delta$, I_Δ *= ideal generated by* $\underline{X}^{\underline{\mu}}$, $\underline{\mu}\notin K(\Delta)$,

(ii) A_Δ *is a graded Noetherian Cohen-Macaulay ring of dimension* $d_\Delta = \dim\Delta + 1$,

(iii) *The sequence* $0\to\bar{A}\to\bigoplus_{\Delta\in\psi_m} A_\Delta\to\bigoplus_{\Delta\in\psi_{m-1}} A_\Delta\to\ldots\to\bigoplus_{\Delta\in\psi_1} A_\Delta\to 0$ *given by the restriction maps is an exact sequence of graded* $\bar{A}$*-moduls.*

Proof. (i) is trivial, (ii) cf. [3], (iii) cf. [4].

In order to have good conditions to work at the A_Δ, we must add some conditions on the $\underline{f}$: By $\underline{f}^{(\Gamma)} = (f_1^{(\Gamma)},\dots,f_r^{(\Gamma)})$ we will denote the inertial form of $\underline{f}$ in $\bar{A}$ and call it the principal part of $\underline{f}$, i.e. $f_i = f_i^{(\Gamma)} + f_i'$, $f_i^{(\Gamma)}\in A_{(m_i)}\setminus A_{(m_i+1)}$ and $f_i'\in \bar{A}_{(m_i+1)}$.

The sequence $\underline{M}=\left(\frac{m_1}{M}, \ldots, \frac{m_r}{M}\right)$ is called the weight of $\underline{f}$. Furthermore we will denote the restriction of $\underline{f}^{(\Gamma)}$ to the face Δ by $\underline{f}^{(\Delta)}$, i.e. if $f_i^{(\Gamma)}=\sum_{h(\underline{\nu})=\frac{m_i}{M}} a_{i,\underline{\nu}} \cdot \underline{X}^{\underline{\nu}}$, then

$$f_i^{(\Delta)}=\sum_{\substack{\underline{\nu}\in K(\Delta)\\ h(\underline{\nu})=\frac{m_i}{M}}} a_{i,\underline{\nu}} \cdot \underline{X}^{\underline{\nu}}.$$

For any face Δ of Γ the s-minors of $\left(X_i \dfrac{\partial f_j^{(\Delta)}}{\partial X_i}\right)$, $j=1, \ldots, s$, generate an ideal $\mathfrak{J}(\boldsymbol{f}_s^{(\Delta)})$ in $\boldsymbol{A}_\Delta$ (here $\boldsymbol{f}_s^{(\Delta)}=(f_1^{(\Delta)}, \ldots, f_s^{(\Delta)})$).

1.3. Definition. *The equations $\boldsymbol{f}$ are called non-degenerate, if for any s, $1\leqq s\leqq r$, and any face Δ of Γ, $d_\Delta\geqq s$, the ideal $(\boldsymbol{f}_s^{(\Delta)}, \mathfrak{J}(\boldsymbol{f}_s^{(\Delta)}))$ is $\mathfrak{m}_\Delta$-primary ($\mathfrak{m}_\Delta:=(X_1, \ldots, X_m)\cap \boldsymbol{A}_\Delta$).*

1.4. Remark. *Let $(\boldsymbol{f}_s, \mathfrak{J}(\boldsymbol{f}_s^{(\Delta)}))$ be $\mathfrak{m}_\Delta$-primary, then*
(i) *$\boldsymbol{f}_s^{(\Delta)}$ is a regular sequence in $\boldsymbol{A}_\Delta$,*
(ii) *$\dim\,(\boldsymbol{A}_\Delta/\boldsymbol{f}_t^{(\Delta)}, \mathfrak{J}(\boldsymbol{f}_s^{(\Delta)})))\leqq s-t-1$ for $t<s$.*

We will later show that almost all complete intersections $\boldsymbol{f}$ are non-degenerate, i.e. more precisely, that the nondegenerate $\boldsymbol{f}^{(\Gamma)}$ contain a ZARISKI open dense subset of all possible $\boldsymbol{f}^{(\Gamma)}$.

Now we are able to define the Newton-number of the polygon Γ with respect to the weight $\boldsymbol{M}=(M_1, \ldots, M_r)$ and to state the main theorem.

1.5. Definition. *Let V_m be the m-dimensional volume of $\Gamma_-(\boldsymbol{f})$ and V_d the sum of the d-dimensional volumes of the intersections of $\Gamma_-(\boldsymbol{f})$ with all coordinate planes of dimension d (if $d<m$).*

$$\nu(\Gamma_f, \boldsymbol{M}):=\sum_{d=r}^{m}(-1)^{m-d}\lambda_d V_d+(-1)^{m-r+1}$$

with

$$\lambda_d:=d!\sum_{|\underline{\alpha}|=d-r} M_1^{\alpha_1+1}\ldots M_r^{\alpha_r+1}$$

is called the Newton-number of Γ_f with respect to the weight $\boldsymbol{M}$.

1.6. Remark. *If we have for the special cases*
(i) *$r=1$ and $M_1=1$, then $\lambda_d=d!$ (KOUCHNIRENKO's case),*
(ii) *$M_1=\ldots=M_r=1$, then $\lambda_d=\binom{d-1}{r-1}d!$*

1.7. Theorem. *Let $\mathfrak{O}_X=A/f$ be an isolated complete intersection singularity with the NEWTON polygon Γ_f and the weight $\boldsymbol{M}$ such that $\boldsymbol{f}$ is convenient and $\boldsymbol{M}\in\mathbf{N}^r$, then*

$$\mu(\mathfrak{O}_X)\geqq \nu(\Gamma_f, \boldsymbol{M}).$$

If moreover $\boldsymbol{f}$ is non-degenerate, then equality holds.

The proof of the theorem splits into several steps. The idea is like this: Let

$$\Omega^p_{f_i} := \Omega^p / df_1 \wedge \Omega^{p-1} + \ldots + df_i - \Omega^{p-1}$$

(Ω^p is the free A-module generated by the differentials $dX_1, \ldots, dX_m$) and

$$V^j := \Omega^n_f / d\Omega^{n-1}_f + \sum_{k=1}^{j} f_k \Omega^n_f$$

(d is the differential). Then we know by GREUEL [1] that $\mu(\mathfrak{D}_X) = \dim_K V^r$.

The idea of GREUEL and HAMM (cf. [2]) in the quasihomogeneous case is to compute $\dim_K \operatorname{gr}(V^r)$ in terms of the given weight of f and the degrees of the X_j by the following exact sequences:

$$\text{(A)} \qquad 0 \to \Omega^0_{f_s} \xrightarrow{\wedge df_{s+1}} \Omega^1_{f_s} \to \ldots \xrightarrow{\wedge df_{s+1}} \Omega^p_{f_s} \to \Omega^p_{f_{s+1}} \to 0$$

$$\text{(B)} \qquad 0 \to K[f] \to \Omega^0_f \xrightarrow{d} \Omega^1_f \xrightarrow{d} \ldots \to \Omega^n_f$$

$$\text{(C)} \qquad 0 \to V^s \xrightarrow{f_{s+1}} V^s \to V^{s+1} \to 0 .$$

We will show that the filtration of A induced by the Newton polygon can be extended to a "good" filtration of Ω^p_f, such that the corresponding graded sequences are also exact. Then the POINCARÉ-series of $\operatorname{gr}\Omega^p_f$ can be computed in terms of the NEWTON polygon and one gets $\dim_K \operatorname{gr}(V^r) = \nu(\Gamma_f, M)$ under the assumption of the theorem, f is non-degenerate.

2. A generalized DE-RHAM-lemma and applications

In this chapter we will always suppose f to be a complete intersection, convenient and non-degenerate. Now we are interested in defining a filtration at the Ω^p given by the NEWTON polygon such that this filtration is compatible with the differential d. It is a little bit complicated since the canonical filtration of Ω^p given by

$$\Omega^p_{(q)} := \bigoplus_{i_1 < \cdots < i_p} A_{(q)} dX_{i_1} \wedge \ldots \wedge dX_{i_p}$$

does not have this property. So we have to go this way: Let

$$\dot{\Omega}^p := \bigwedge^p \dot{\Omega}^1 ; - \dot{\Omega}^1 = \bigoplus_{i=1}^{m} A \frac{dX_i}{X_i}$$

the Newton polygon induces a filtration on $\dot{\Omega}^p$ by

$$\dot{\Omega}^p_{(q)} := \bigoplus_{i_1 < \cdots < i_p} A_{(q)} \frac{dX_{i_1} \wedge \ldots \wedge dX_{i_p}}{X_{i_1} \ldots X_{i_p}}$$

which is compatible with the differential $\dot{d}: f \mapsto \sum_i X_i \frac{\partial f}{\partial X_i} \frac{dX_i}{X_i}$. Similarly we define for $0 \leqq s, t \leqq r$:

$$\dot{\Omega}^p_{s,t} := \dot{\Omega}^p / \sum_{i=1}^{s} \dot{d}f_i \wedge \dot{\Omega}^{p-1} + \sum_{j=1}^{t} f_j \dot{\Omega}^p$$

with the induced filtration. The associated graded modules are denoted by $\tilde{\Omega}^p_{s,t} := \mathrm{gr}\,(\dot{\Omega}^p_{s,t})$. $\tilde{\Omega}^p := \tilde{\Omega}^p_{0,0}$ is a free $\bar{A}$-module of rank $\binom{m}{p}$.

Now we consider Ω^p as a filtred submodule of $\dot{\Omega}^p$ generated over K by the

$$\left\{ \underline{X}^{\underline{\mu}} \frac{dX_{i_1} \wedge \ldots \wedge dX_{i_p}}{X_{i_1} \ldots X_{i_p}}, \text{ such that } \mu_{i_1} \geqq 1, \ldots, \mu_{i_p} \geqq 1 \right\}.$$

Let $\bar{\Omega}^p_{s,t} := \mathrm{gr}\left(\Omega^p / \sum_{i=1}^{s} df_i \wedge \Omega^{p-1} + \sum_{j=1}^{t} f_j \Omega^p\right)$, later we will see that $\bar{\Omega}^p_{s,t}$ is a graded $\bar{A}$-submodule of $\tilde{\Omega}^p_{s,t}$. Some difficulty arises because $\bar{\Omega}^p$ is not a free $\bar{A}$-module. Thus our way to get exact graded sequences of the $\bar{\Omega}^p_{s,t}$ will be to look first at the $\tilde{\Omega}^p_{s,t}$ and than to deduce some results about the $\bar{\Omega}^p_{s,t}$.

We will often use the notation: if M is a graded $\bar{A}$-module, then $M_\Delta := M \otimes_{\bar{A}} \boldsymbol{A}_\Delta$, for instance $\tilde{\Omega}^p_\Delta$ is a free $\boldsymbol{A}_\Delta$-module generated over K by $\left\{ \underline{X}^{\underline{\mu}} \frac{dX_{i_1} \wedge \ldots \wedge dX_{i_p}}{X_{i_1} \ldots X_{i_p}};\ \mu \in K(\Delta) \right\}$. Now let $\mathfrak{p}$ be a prime ideal of A_Δ not contained in $V(\underline{f}_t^{(\Delta)}, \mathfrak{J}(\underline{f}_s^{(\Delta)}))$ and $A_t := \bar{A}/(\underline{f}_t)$, then $(*)$ $\tilde{\Omega}^p_{s,t,\Delta} \otimes_{A_\Delta} \boldsymbol{A}_{\Delta,\mathfrak{p}}$ is a free $\boldsymbol{A}_{t,\Delta,\mathfrak{p}}$-module of rank $\binom{m-s}{p}$. Very useful for us is the following

2.1 Lemma. *Let $V \subset X = \mathrm{Spec}\, A$ be a closed subset, M a finitely generated A-module, then the restriction map $\Gamma(X, M) \to \Gamma(X - V, M)$ is bijective (resp. injective) if* $\mathrm{cod}\, hM \geqq \dim V + 2$ (*resp.* $\geqq \dim V + 1$).

Now we can start with

2.2. De Rham-lemma. *If $s \leqq r$, then*

(i) $\mathrm{cod}\, h\, \tilde{\Omega}^p_{s,\Delta} \geqq d_\Delta - p$ *for* $d_\Delta \geqq p + s - 1$;

(ii) $0 \to \tilde{\Omega}^p_s \to \bigoplus_{\Delta \in \Psi_m} \tilde{\Omega}^p_{s,\Delta} \to \ldots \to \bigoplus_{\Delta \in \Psi_{p+s}} \tilde{\Omega}^p_{s,\Delta}$ *is exact;*

(iii) $0 \to \tilde{\Omega}^{p-1}_s \xrightarrow{\wedge \tilde{d}f_s} \tilde{\Omega}^p_{s-1} \to \tilde{\Omega}^p_s \to 0$ *is exact for* $p + s - 1 \leqq m$.[1])

Proof. If $p = 0$ or $s = 0$, (i) and (ii) are fulfilled by (1.2.) and the fact that $\tilde{\Omega}^p_\Delta$ is a free $\boldsymbol{A}_\Delta$-module. Let (i) and (ii) be fulfilled for $(s-1, p)$ and $(s, p-1)$, then we

[1]) $\tilde{d}$ is the differential induced by $\dot{d}$ in the graded modules. Usually we will simply write d instead of $\tilde{d}, \dot{d}, \bar{d}$ ect. if the connection is clear. $\tilde{d}f_i$ depends only on the initial form $f_i^{(\Gamma)}$. But we will also not distinguish between f_i, $f_i^{(\Gamma)}$, $f_i^{(\Delta)}$.

show the lemma for (s, p). We regard the commutative diagramme:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \tilde{\Omega}_s^{p-1} & \xrightarrow{\wedge df_s} & \tilde{\Omega}^p_{s-1} & \to & \tilde{\Omega}^p_s & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{\Delta \in \Psi_m} \tilde{\Omega}^{p-1}_{s,\Delta} & \xrightarrow{\wedge df_s} & \bigoplus \tilde{\Omega}^p_{s-1,\Delta} & \to & \bigoplus \tilde{\Omega}^p_{s,\Delta} & \to 0 \\
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{\Delta \in \Psi_{p+s-1}} \tilde{\Omega}^{p-1}_{s,\Delta} & \xrightarrow{\wedge df_s} & \bigoplus \tilde{\Omega}^p_{s-1,\Delta} & \to & \bigoplus \tilde{\Omega}^p_{s,\Delta} & \to 0 .
\end{array}$$

By induction hypothesis the diagramme has exact first and second columns. Over Spec $A_\Delta - V(\mathfrak{I}(f_s))$ from the second row all is exact because of $(*)$. Now cod $h\tilde{\Omega}^{p-1}_{s,\Delta} \geqq d-p+1 \geqq s \geqq \dim V(\mathfrak{I}(f_s))+1$, hence by (2.1.) all rows are exact. So we get (i), (iii) and also (ii) by looking at the diagramme carefully.

2.3. Corollary. *For $1 \leqq s \leqq r$ and $p+s \leqq m$. The sequence is exact:*

$$0 \to \tilde{\Omega}^0_{s-1} \xrightarrow{\wedge df_s} \tilde{\Omega}^1_{s-1} \to \dots \xrightarrow{\wedge df_s} \tilde{\Omega}^{p+1}_{s-1} \to \tilde{\Omega}^{p+1}_s \to 0 .$$

2.4. Proposition.

(i) *f_s is a regular sequence of $\tilde{\Omega}^p_s$ for $p+s \leqq m$;*

(ii) $0 \to \tilde{\Omega}^p_{s,t} \to \bigoplus_{\Delta \in \Psi_m} \tilde{\Omega}^p_{s,t,\Delta} \to \dots \to \bigoplus_{\Delta \in \Psi_{p+s}} \tilde{\Omega}^p_{s,t,\Delta}$ *is exact for $p+s+2 \leqq m$, $t \leqq s$;*

(iii) $\tilde{\Omega}^{m-s-1}_{s,t} \to \bigoplus_{\Delta \in \Psi_m} \tilde{\Omega}^{m-s-1}_{s,t,\Delta}$ *is injective, $t \leqq s$.*

Proof. For $t=0$ (ii) and (iii) follows by (2.2.). Now we consider again some diagramme:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \tilde{\Omega}^p_{s,t-1} & \xrightarrow{\cdot f_t} & \tilde{\Omega}^p_{s,t-1} & \to & \tilde{\Omega}^p_{s,t} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{\Delta \in \Psi_m} \tilde{\Omega}^p_{s,t-1,\Delta} & \xrightarrow{\cdot f_t} & \bigoplus \tilde{\Omega}^p_{s,t-1,\Delta} & \to & \bigoplus \tilde{\Omega}^p_{s,t,\Delta} & \to 0 \\
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{\Delta \in \Psi_{p+s}} \tilde{\Omega}^p_{s,t-1,\Delta} & \xrightarrow{\cdot f_t} & \bigoplus \tilde{\Omega}^p_{s,t-1,\Delta} & \to & \bigoplus \tilde{\Omega}^p_{s,t,\Delta} & \to 0 .
\end{array}$$

Because f is non-degenerate from (x), (2.1) and the induction hypothesis for $t-1$ we get that all sequences are exact besides the last map of the third column. But if $p+s+2 \leqq m$ and $\omega \in \bigoplus_{\Delta \in \Psi_{p+s+1}} \tilde{\Omega}^p_{s,t,\Delta}$ is mapped to 0, then ω has a preimage over all points of Spec A_Δ outside $V(f_{t-1}, \mathfrak{I}(f_s))$ which are induced by some $\varphi \in$

$\in \bigoplus_{\Delta\in\Psi_{p+s+2}} \tilde{\Omega}^p_{s,t,\Delta}$. There exist such a φ because of $\operatorname{cod} h\tilde{\Omega}^p_{s,t,\Delta} \geqq d-p-t=s-t+2 \geqq$ $\geqq \dim V(f_{t-1}, \mathfrak{J}(f_s))+2$ and (2.1.). The image of φ coincides with ω in $\bigoplus_{\Delta\in\Psi_{s+p+1}} \tilde{\Omega}^p_{s,t,\Delta}$ by the same reason.

In order to proof exactness of the relative DE-RHAM-complex for $\tilde{\Omega}^{\cdot}_{s,t}$ we first look at the restrictions to some Δ.

2.5. Proposition. *For $t \leqq s$ and any face Δ, $0-\mathfrak{K}_{s,t,\Delta} \to \tilde{\Omega}^0_{s,t,\Delta} \xrightarrow{d} \dots \to \tilde{\Omega}^{d_\Delta - s}_{s,t,\Delta}$ is exact. $\mathfrak{K}_{s,t} := K[f^{(\Gamma)}_{t+1}, \dots, f^{(\Gamma)}_s] \subseteq \bar{A}$, with induced grading.*

Proof. *step* 1: $s=t$ and localized at some $\mathfrak{p} \notin V(f_s, \mathfrak{J}(f_s))$. For $s=t=0$ the complex $(\tilde{\Omega}^{\cdot}, \tilde{d})$ is exact and strict, therefor the graded complex is also exact and the kernel of the first map is just K.

We use induction on s. For $1 \leqq p \leqq d_\Delta - s$ let $\bar{\omega} \in \tilde{\Omega}^{p-1}_{s,s,\Delta}$ be homogeneous of degree q, such that $d\bar{\omega}=0$. Let $\omega \in \tilde{\Omega}^{p-1}_{s-1,s-1,\Delta}$ be a homogeneous representative of $\bar{\omega}$, then there are $\varphi_1 \in \tilde{\Omega}^{p-1}_{s-1,s-1,\Delta}$ and $\psi_1 \in \tilde{\Omega}^p_{s-1,s-1,\Delta}$ both of degree $q-m_s$, such that

$$d\omega = df_s \wedge \varphi_1 + f_s\psi_1 = d(f_s\varphi_1) + f_s\,(\psi_1 - d\varphi_1),$$

hence

$$0 = df_s \wedge (\psi_1 - d\varphi_1) + f_s d\psi_1 .$$

Now $\tilde{\Omega}^{p+1}_{s-1,s-1,\Delta}$ is free of rank $\binom{m-s+1}{p+1}$ at $\mathfrak{p}$ and df_s belongs to a free basis. So we get that

$$\bar{\psi}_1 - d\bar{\varphi}_1 = 0 \quad \text{in} \quad \tilde{\Omega}^p_{s,s,\Delta} .$$

So again there are φ_2 and ψ_2 of degree $q-2m_s$, such that

$$\psi_1 - d\varphi_1 = 2df_s \wedge \varphi_2 + f_s\psi_2$$

and we get

$$d\,(\omega - f_s\varphi_1) = 2f_s df_s \wedge \varphi_2 + f_s^2\psi_2 = d(f_s^2\varphi_2) + f_s^2\,(\psi_2 + d\varphi)$$

hence $0 = f_s\,(2df_s \wedge (\psi_2 - d\varphi_2) + f_s d\psi_2)$ and again

$$\bar{\psi}_2 - d\bar{\varphi}_2 = 0 \quad \text{in} \quad \tilde{\Omega}^p_{s,s,\Delta} .$$

If we continue like this, we get for some l with

$$lm_s < q \leqq (l+1)\,m_s$$

$$\psi_l - d\varphi_l = 0 \quad \text{in} \quad \tilde{\Omega}^p_{s-1,s-1,\Delta} \quad \text{and therefor}$$

$$d\,(\omega - f_s\varphi_1 - \dots - f_s^l\varphi_l) = 0 \in \tilde{\Omega}^p_{s-1,s-1,\Delta} .$$

By induction hypothesis there exist $\eta \in \tilde{\Omega}^{p-2}_{s-1,s-1,\Delta}$, such that $\omega = f_s\varphi_1 + \dots + f_s^l\varphi_l + d\eta$ hence

$$\bar{\omega} = d\bar{\eta} \in \tilde{\Omega}^{p-1}_{s,s,\Delta}$$

(respectively for $p=1$: $\omega - f_s\varphi_1 - \dots - f_s^l\varphi_l \in K$ i.e. $\bar{\omega} \in K$.)

Step 2: $s > t$, localized at some $\mathfrak{p} \notin V(f_t, \mathfrak{J}(f_s))$. Suppose the proposition is true for $(s-1, t)$. Choose $\bar{\omega}$ as in the first step. Let $\omega \in \tilde{\Omega}^{p-1}_{s-1,t,\Delta}$ be a homogeneous representative of $\bar{\omega}$, then there exist some $\varphi_1 \in \tilde{\Omega}^{p-1}_{s-1,t,\Delta}$ of degree $q-m_s$, such that

$$d\omega = df_s \wedge \varphi_1$$

hence

$$0 = df_s \wedge d\varphi_1 \in \tilde{\Omega}^{p+1}_{s-1,t,\Delta}\,.$$

But df_s belongs to a free basis of $\tilde{\Omega}^{p+1}_{s-1,t,\Delta}$ at $\mathfrak{p}$, hence

$$d\varphi_1 = df_s \wedge \varphi_2, \text{ -degree } \quad \varphi_2 = q - 2m_s\,.$$

Similarly to the first step we get for some $1: d\varphi_l = 0$. For $p \geqq 2$ this means $\varphi_l = d\psi_l \in \tilde{\Omega}^{p-1}_{s-1,t,\Delta}$. Going back we see

$$d\varphi_{l-1} = df_s \wedge d\psi_{l-1} = -d\,(df_s \wedge \psi_{l-1})$$

hence

$$\varphi_{l-1} + df_s \wedge \psi_{l-1} = d\psi_{l-2}$$

and so on ... hence $d\omega = df_s \wedge d\psi_1$ which implies

$$\omega + df_s \wedge \psi_1 = d\psi_0 \quad \text{and} \quad \bar{\omega} = d\bar{\psi}_0 \in \tilde{\Omega}^{p}_{s,t,\Delta}\,.$$

For $p=1: \varphi_l \in \mathfrak{K}_{s-1,t,\Delta}$. Going back we see

$$d\varphi_{l-1} = \varphi_l \cdot df_s = d(f_s\varphi_l), \quad \text{hence} \quad \varphi_{l-1} - f_s\varphi_l \in \mathfrak{K}_{s-1,t,\Delta}$$

and so on ... Finally $\omega - f_s\varphi_1 \in \mathfrak{K}_{s-1,t,\Delta}$, i.e. $\bar{\omega} \in \mathfrak{K}_{s,t,\Delta}$.

Step 3: The sequence $\tilde{\Omega}^0_{s,t,\Delta} \to \ldots \to \tilde{\Omega}^{d_\Delta - s}_{s,t,\Delta}$ is exact outside the closed set $V(f_t, \mathfrak{J}(f_s))$ of dimension $\max\{0, s-t-1\}$. But $\operatorname{cod} h\, \tilde{\Omega}_{s,t,\Delta} \geqq d_\Delta - p - t$ and using (2.1.) it is not difficult to see that the sequence is exact everywhere.

2.6. Remark. *$0 \to \mathfrak{K}_{s,t} \to \bigoplus_{\Delta \in \Psi_m} \mathfrak{K}_{s,t,\Delta} \to \ldots \to \bigoplus_{\Delta \in \Psi_s} \mathfrak{K}_{s,t,\Delta}$ is exact.*

The proof is the same as in 1.2. (iii) knowing that $f_s^{(\Delta)}$ is a regular sequence in A_Δ, $d_\Delta \geqq s$.

2.7. Corollary. *For $t \leqq s$ $0 \to \mathfrak{K}_{s,t} \to \tilde{\Omega}^0_{s,t} \xrightarrow{d} \ldots \xrightarrow{d} \tilde{\Omega}^{m-s}_{s,t}$ is exact.*

Proof. The first row is exact because in the following diagramme all collumns are exact (2.4.), (2.6.) and all other rows are exact (2.5.).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{K}_{s,t} & \to & \tilde{\Omega}^0_{s,t} & \to \ldots \to & \tilde{\Omega}^{m-s-1}_{s,t} & \to & \tilde{\Omega}^{m-s}_{s,t} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \bigoplus_{\Delta \in \Psi_m} \mathfrak{K}_{s,t,\Delta} & \to & \bigoplus \tilde{\Omega}^0_{s,t,\Delta} & \to \ldots \to & \bigoplus \tilde{\Omega}^{m-s-1}_{s,t,\Delta} & \to & \bigoplus \tilde{\Omega}^{m-s}_{s,t,\Delta} \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{\Delta \in \Psi_{s+1}} \mathfrak{K}_{s,t,\Delta} & \to & \bigoplus \tilde{\Omega}^0_{s,t,\Delta} & \to & \bigoplus \tilde{\Omega}^1_{s,t,\Delta} & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \bigoplus_{\Delta \in \Psi_s} \mathfrak{K}_{s,t,\Delta} & \to & \bigoplus \tilde{\Omega}^0_{s,t,\Delta} & & & &
\end{array}$$

Now we will try to get similar results about $\bar{\Omega}_{s,t}$. Let $I \subseteqq \{1, \ldots, m\}$ be a subset and

$$\dot{\Omega}_I^p := \left\{ \omega \in \dot{\Omega}^p \;\middle|\; \begin{array}{l} \omega = \sum\limits_{i_1 < \cdots < i_p} f_{i_1 \ldots i_p} \dfrac{dX_{i_1} \wedge \ldots \wedge dX_{i_p}}{X_{i_1} \ldots X_{i_p}} \\ \text{and} \quad X_{i_k} / f_{i_1 \ldots i_p} \quad \text{if} \quad i_k \notin I \end{array} \right\}$$

i.e. $\dot{\Omega}_I^p$ is the subset of all differential forms which have at most X_i, $i \in I$ in the dominator. With this notation we have:

$$\tilde{\Omega}_I^p := \operatorname{gr} \dot{\Omega}_I^p \quad \text{and} \quad \tilde{\Omega}_\emptyset^p = \bar{\Omega}_p .$$

2.8. Proposition. *For $t \leqq s$, $s + p \leqq m$*

$$\left(\sum_{i=1}^{s} \tilde{\Omega}^p \wedge df_i + \sum_{j=1}^{t} f_j \tilde{\Omega}^{p+1} \right) \cap \tilde{\Omega}_I^{p+1} = \sum_{i=1}^{s} \tilde{\Omega}_I^p \wedge df_i + \sum_{j=1}^{t} f_j \tilde{\Omega}_I^{p+1}$$

2.9. Corollary.

(i) $0 \to \Omega_{s,I}^p \xrightarrow{\wedge df_s} \tilde{\Omega}_{s-1,I}^{p+1} \to \tilde{\Omega}_{s,I}^{p+1} \to 0$

is exact for all I;

(ii) $\bar{\Omega}_{s,t}^p$ *is a graded submodule of* $\tilde{\Omega}_{s,t}^p$;

(iii) *The graded sequence (A) is exact:*

$$0 \to \bar{\Omega}_{s-1}^0 \xrightarrow{\wedge df_s} \ldots \to \bar{\Omega}_{s-1}^{p+1} \to \bar{\Omega}_s^{p+1} \to 0 \;;$$

(iv) f_s *is a regular sequence of* $\bar{\Omega}_s^p$.

Proof of 2.8. We will prove the proposition by using induction on t, s, m and $|I| = \operatorname{card} I$.

Step 1: $t = 0$ and $s = 1$ $m = 1$ implies $p = 0$ and all is trivial. Suppose now the proposition is true for $(m-1)$ variables and for m and all $J \subseteqq \{1, \ldots, m\}$ with $|J| > |I| = 1$. We start with $\tilde{\omega} \in \tilde{\Omega}^p$ such that $\tilde{\omega} \wedge df_1 \in \tilde{\Omega}_I^{p+1}$. We choose $J \supset I$ minimal, such that $\tilde{\omega} \in \tilde{\Omega}_J^p$. Without restriction of generality we can suppose that

$$J = \{1, \ldots, l+1\} \quad \text{and} \quad I = \{2, \ldots, l+1\} .$$

Now we can write

$$\tilde{\omega} = \omega_0 \wedge \frac{dX_1}{X_1} + \omega_1, \quad \omega_1 \in \tilde{\Omega}_I^p \quad \text{and} \quad \omega_0 \in \tilde{\Omega}_I^{p-1}$$

does not depend on X_1. We are intrested to find $\omega_0' \in \tilde{\Omega}_I^p$, such that $\omega_0 \wedge \frac{dX_1}{X_1} \wedge df_1 = \omega_0' \wedge df_1$. Now let $f_1 = \sum_{i=0}^{q} f_{1,i}, X_1^i$. We get $\omega_0 \wedge df_{1,0} = 0$, since

$$\omega_0 \wedge \frac{dX_1}{X_1} \wedge df_1 \in \tilde{\Omega}_I^{p+1} .$$

Now we apply induction hypothesis for $(m-1)$ variables together with 2.9. (i) to ω_0. This is possible because f is non-degenerate. So we get $\omega_2 \in \tilde{\Omega}_I^{p-2}$ (not depending on X_1), such that $\omega_0 = \omega_2 \wedge df_{1,0}$. If we now put $\varphi = -\sum_{i=1}^{q} X_1^i df_{1,i} \wedge \frac{dX_1}{X_1}$ and $\omega_0' = \omega_2 \wedge \varphi$, then $\omega_0 \wedge \frac{dX_1}{X_1} \wedge df_1 = \omega_0' \wedge df_1$ can easily be computed.

Step 2: $t=0$, suppose the proposition is true for $(s-1, m, I)$, $(s, m-1, I)$ and (s, m, J) with $|J| > |I|$. We start with $\Phi = \sum_{i=1}^{s} \tilde{\omega}_i \wedge df_i \in \tilde{\Omega}_I^{p+1}$, $\tilde{\omega}_s \in \tilde{\Omega}_J^p$. We look for $\omega \in \Omega_I^p$ and $\tilde{\omega}_0' \in \Omega_J^p$, such that

$$\Phi = \sum_{i=1}^{s-1} \tilde{\omega}_i' \wedge df_i + \omega \wedge df_s .$$

With the same notations as in the first step, we have

$$\tilde{\omega}_i = \omega_{i,0} \wedge \frac{dX_1}{X_1} + \omega_{i,1} \quad \text{and} \quad \left(\text{putting } f_i = \sum_{j=0}^{q} f_{i,j} X_1^j\right)$$

$$\sum_{i=1}^{s} \omega_{i,0} \wedge df_{i,0} = 0 \in \tilde{\Omega}_I^p \quad \text{without} \quad X_1 .$$

Again by induction hypothesis and 2.9. (i) for $(s, m-1, I)$ we obtain $\xi_i \in \tilde{\Omega}_I^{p-2}$ not depending on X_1, such that

$$\omega_{s,0} = \sum_{i=1}^{s} \xi_i \wedge df_{i,0} .$$

Now putting $\varphi_i = -\sum_{j=1}^{q} X_1^j df_{i,j} \wedge \frac{dX_1}{X_1}$ and using

$$df_{s,0} \wedge \frac{dX_1}{X_1} \wedge df_s = \varphi_s \wedge df_s \quad \text{and for} \quad i < s$$

$$df_{i,0} \wedge \frac{dX_1}{X_1} \wedge df_s + df_{s,0} \wedge \frac{dX_1}{X_1} \wedge df_i = \varphi_i \wedge df_s + \varphi_s \wedge df_i$$

we get

$$\begin{aligned}
\Phi &= \sum_{i=1}^{s} \tilde{\omega}_i \wedge df_i \\
&= \sum_{i=1}^{s-1} \left(\tilde{\omega}_i - \xi_i \wedge df_{s,0} \wedge \frac{dX_1}{X_1}\right) \wedge df_i + \sum_{i=1}^{s-1} \xi_i \wedge df_{s,0} \wedge \frac{dX_1}{X_1} \wedge df_i \\
&\quad + \left(\omega_{s,1} + \sum_{i=1}^{s} \xi_i \wedge df_{i,0} \wedge \frac{dX_1}{X_1}\right) \wedge df_s \\
&= \sum_{i=1}^{s-1} \left(\tilde{\omega}_i - \xi_i \wedge df_{s,0} \wedge \frac{dX_1}{X_1} + \xi_i \wedge \varphi_s\right) \wedge df_i \\
&\quad + \left(\omega_{s,1} + \sum_{i=1}^{s} \xi_i \wedge \varphi_i\right) \wedge df_s .
\end{aligned}$$

Now we have finished by putting

$$\tilde{\omega}_i' = \tilde{\omega}_i - \xi_i \wedge df_{s,0} \wedge \frac{dX_1}{X_1} + \xi_i \wedge \varphi_s \in \tilde{\Omega}_J^p$$

and

$$\omega = \omega_{s,1} + \sum_{i=1}^{s} \xi_i \wedge \varphi_i \in \tilde{\Omega}_I^p .$$

Step 3: Suppose the proposition is true for $(t-1, s, m, I)$, $(t, s, m-1, I)$ and $(t, s\ m, J)$ with $|J|>|I|$. Let

$$\Phi = \sum_{i=1}^{s} \tilde{\omega}_i \wedge df_i + \sum_{j=1}^{t} f_j \varphi_j \in \tilde{\Omega}_I^{p+1}, \quad \tilde{\omega}_i \in \tilde{\Omega}_J^p, \quad \varphi_j \in \tilde{\Omega}_J^{p+1}.$$

Using the same notations as before we get

$$\sum_{i=1}^{s} \omega_{i,0} \wedge df_{i,0} + \sum_{j=1}^{t} f_{j,0} \varphi_{j,0} = 0 - \text{in} \quad \tilde{\Omega}_I^p \text{ (without } X_1) .$$

according to 2.8. (iv) it follows that

$$\varphi_{t,0} = \sum_{i=1}^{s} \xi_i \wedge df_{i,0} + \sum_{j=1}^{t-1} f_{j,0} \eta_j,$$

$$\xi_i \in \tilde{\Omega}_I^{p-1}, \quad \eta_j \in \tilde{\Omega}_I^p \quad \text{(not depending on } X_1) .$$

Hence we get

$$\Phi = \sum_{i=1}^{s} \left(\tilde{\omega}_i - f_t \xi_i \wedge \frac{dX_1}{X_1}\right) \wedge df_i + \sum_{j=1}^{t-1} f_j \left(\tilde{\varphi}_j + f_t \eta_j \wedge \frac{dX_1}{X_1}\right)$$
$$+ f_t \left(\varphi_{t,1} + \left[\varphi_{t,0} - \sum_{i=1}^{s} \xi_i \wedge df_i - \sum_{j=1}^{t-1} f_j \eta_j\right] \wedge \frac{dX_1}{X_1}\right) .$$

But by construction X_1 divices the expression in the square brackets, hence the last summand belongs to Ω_I^p. The proof is finished.

2.10. Corollary. *For* $t \leqq s$

$0 \to \mathfrak{K}_{s,t} \to \bar{\Omega}_{s,t}^0 \xrightarrow{d} \dots \xrightarrow{d} \bar{\Omega}_{s,t}^{m-s}$ *is exact.* (For $t=0$ this is the graded sequence (B)).

Proof. We have to show that

$$\left(d\tilde{\Omega}^p + \sum_{i=1}^{s} \tilde{\Omega}^p \wedge df_i + \sum_{j=1}^{t} f_j \tilde{\Omega}^{p+1}\right) \cap \Omega_I^{p+1} = d\tilde{\Omega}_I^p + \sum_{i=1}^{s} \tilde{\Omega}_I^p \wedge df_i + \sum_{j=1}^{t} f_j \tilde{\Omega}_I^{p+1} .$$

This is done by induction in the same way as 2.8. For $s=t=0$ it suffices to show for $\tilde{\omega} \in \tilde{\Omega}_{\{1,\dots,l\}}^p$, such that $d\tilde{\omega} \in \tilde{\Omega}_{\{2,\dots,l\}}^{p+1}$, that there is some $\omega \in \tilde{\Omega}_I^p$ with $d\tilde{\omega} = d\omega$. We write $\tilde{\omega} = \omega_1 + \omega_0 \wedge \frac{dX_1}{X_1}$. Then of course $d\left(\omega_0 \wedge \frac{dX_1}{X_1}\right) = d\omega_0 \wedge \frac{dX_1}{X_1} \in \Omega_I^{p+1}$, which implies $d\omega_0 = 0$, hence $d\tilde{\omega} = d\omega_1$. For the induction step we calculate like this: Assume

$$\Phi = d\tilde{\omega} + \sum_{i=1}^{s} \tilde{\varphi}_i \wedge df_i + \sum_{j=1}^{t} f_j \tilde{\psi}_j \quad \text{and} \quad \Phi \wedge \frac{dX_1}{X_1} \in \tilde{\Omega}_I^{p+1} ,$$

$\tilde{\omega}$, $\tilde{\varphi}_i$, $\tilde{\psi}_j \in \tilde{\Omega}^{\cdot}_I$ not depending on X_1. Then $d\tilde{\omega}+\sum_{i=1}^{s} \tilde{\varphi}_i \wedge df_{i,0}+\sum_{j=1}^{t} f_{j,0}\tilde{\psi}_j=0\in\tilde{\Omega}^p_I$ (without X_1), hence we have all without X_1 in $\tilde{\Omega}^{p-1}_I$:

$$\tilde{\omega}=d\eta+\sum_{i=1}^{s} \varphi_i\wedge df_{i,0}+\sum_{j=1}^{t} f_{j,0}\psi_j$$

($f_{i,j}$ as in 2.8.) and

$$\begin{aligned}\Phi &= d\tilde{\omega}+\sum d\varphi_i\wedge df_i-\sum d(f_j\psi_j)-\sum \psi_j\wedge df_j\\ &\quad+\sum f_j d\psi_j-\sum d\varphi_i\wedge df_i+\sum \tilde{\varphi}_i\wedge df_i+\sum f_j\tilde{\psi}_j\\ &= d\,\underbrace{(\tilde{\omega}-d\eta-\sum \varphi_i\wedge df_i-\sum f_j\psi_j)}_{\omega^*}+\sum(\tilde{\varphi}_i+d\varphi_i)\wedge df_i\\ &\quad-\sum \psi_j\wedge df_j+\sum f_j\,(d\psi_j+\tilde{\psi}_j)\,.\end{aligned}$$

But $\omega^*\wedge \frac{dX_1}{X_1}\in\tilde{\Omega}^p_I$, so we reduced it to 2.8.

All we have to do now is to look for the graded sequence (C).

2.11. Corollary. *f_r is a regular sequence of $\bar{V}^r := \bar{\Omega}^{m-r}_{r,r}/d\bar{\Omega}^{m-r-1}_{r,r}$.*

Proof. For $0\leqq t<r$ we have to show that the last row in the diagramme is exact. But all the other rows are exact by (2.9. (iv)) and all columns are exact by (2.10.), hence the last row is exact too.

$$\begin{array}{ccccccc} 0\to & \bar{\Omega}^0_{r,t} & \xrightarrow{f_{t+1}} & \bar{\Omega}^0_{r,t} & \to & \bar{\Omega}^0_{r,t+1} & \to 0\\ & \downarrow d & & \downarrow d & & \downarrow d & \\ & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \bar{\Omega}^{m-r}_{r,t} & \xrightarrow{f_{t+1}} & \bar{\Omega}^{m-r}_{r,t} & \to & \bar{\Omega}^{m-r}_{r,t+1} & \to 0\\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \bar{V}^t & \xrightarrow{f_{t+1}} & \bar{V}^t & \to & \bar{V}^{t+1} & \to 0\\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

3. Poincaré series

With the same assumtion of f as in chapter 2 we will compute here the Poincaré series of the $\bar{\Omega}^p_s$ using the exact sequences of (2.9.), (2.10.) and (2.11.). Let us start with some general facts about Poincaré series. Let $G=\oplus G_q$ be a graded K-algebra and $M=\oplus M_q$ a graded G-module, then

$$P(M)=\sum_{q\in\mathbf{Z}} (\dim_K M_q)\, T^q\in\mathbf{Z}((T))$$

is called the Poincaré serie of M. Now it is not difficult to get the following properties of Poincaré series:

3.1. Lemma.

(i) *Let* $G=K[Y_1,\ldots,Y_r]$ *be a free* K*-algebra and* $deg(Y_i)=d_i$, *then* $P(G)=\prod_{i=1}^{r}(1-T^{d_i})^{-1}$.

(ii) *Let* $0\to\bar{M}_r\overset{\varphi_r}{\to}\ldots\overset{\varphi_1}{\to}\bar{M}_0\to 0$ *be an exact sequence of graded* G*-modules and deg* $\varphi_i=d_i$, *then* $\sum_{i=0}^{r}(-1)^i T^{d_1+\cdots+d_i}P(\bar{M}_i)=0$.

(iii) *Let* $h_1\in G_{q_1},\ldots,h_s\in G_{q_s}$ *be a regular* G*-sequence, then* $P(G/(h_1,\ldots,h_s))=(1-T^{q_1})\cdot\ldots\cdot(1-T^{q_s})\,P(G)$.

Now this lemma will be applied to our graded algebra $\bar{A}$. Let σ be any subdivision of simplexes of Γ_f such that the vertices of these simplexes are exactly the points of $\Gamma_f\cap\mathfrak{M}_f$. For any face Δ let $\sigma_{\Delta,d}$ be the set of all $\Delta'\in\sigma$ such that $\Delta'\subset\Delta$ is not on the boundary of Δ and dim $\Delta'=d-1$. If $\Delta'\in\sigma$ is such a simplex, we put $P_{\Delta'}$ to be the set of the integral points of the half open parallelepiped defined by Δ' and $v_{\Delta'}(T):=\sum_{\underline{\mu}\in P'_\Delta}T^{Mh(\underline{\mu})}$. Then $\frac{v_{\Delta'}(1)}{d_{\Delta'}!}$ is just the volume of the $d_{\Delta'}$-dimensional pyramid defined by Δ' with vertex 0. Here $d_{\Delta'}=\dim\Delta'+1$.

3.2. Proposition. $P(\bar{A})=\sum_{\Delta\in\sigma}(-1)^{m-d_\Delta}\frac{v_\Delta(T)}{(1-T^M)^{d_\Delta}}$.

$\sigma_0\subseteqq\sigma$ is the set of all simplexes which are not contained in a coordinate hyperplane.

Proof. We apply lemma 3.1. (ii) to the following exact sequences:

$$0\to\bar{A}\to\bigoplus_{\Delta\in\Psi_m}A_\Delta\to\ldots\to\bigoplus_{\Delta\in\Psi_1}A_\Delta\to 0$$

$$0\to A_\Delta\to\bigoplus_{\Delta'\in\sigma_{\Delta,d}}A_{\Delta'}\to\ldots\to\bigoplus_{\Delta'\in\sigma_{\Delta,1}}A_\Delta\to 0^{2)}$$

We have to show that $P(A_{\Delta'})=\frac{v_{\Delta'}(T)}{(1-T^M)^{d_{\Delta'}}}$. Let $\mu_1,\ldots,\mu_d$ be the vertices of $\Delta'\in\sigma_0$, then $h(\mu_i)=1$ and $A_0=K[X^{\underline{\mu}_1},\ldots,X^{\underline{\mu}_d}]$ is a free K-algebra with deg $X^{\underline{\mu}_i}=M$. By lemma 3.1. (i) we get $P(A_0)=\frac{1}{(1-T^M)^d}$. Now we know that $A_{\Delta'}=\bigoplus_{\underline{\mu}\in P'_\Delta}X^{\underline{\mu}}A_0$ and we get

$$P(A_{\Delta'})=\sum_{\underline{\mu}\in P_{\Delta'}}P(X^{\underline{\mu}}A_0)=\sum_{\underline{\mu}\in P_{\Delta'}}T^{Mh(\underline{\mu})}P(A_0)=\frac{v_{\Delta'}(T)}{(1-T^M)^d}$$

Hence the proposition is proved.

Now we are ready to compute the Poincaré series of $\bar{\Omega}^p$. Let $\gamma\subseteqq\{1,\ldots,m\}$ be a subset. By $A^{(\gamma)}$ we will denote the ring $K\langle X_{i_1},\ldots,X_{i_r}\rangle$, $\{i_1,\ldots,i_r\}=\{1,\ldots,m\}\setminus\gamma$.

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2) This sequence is exact as sequence of K-vector spaces

Furthermore, let $A^{[l]} = \bigoplus_{|\gamma|=m-l} A^{(\gamma)}$ $(\bar{A}^{[m]} = \bar{A})$ and $\bar{A}^{[l]}$ be the corresponding graded algebra. Using proposition 3.2. we obtain

$$P(\bar{A}^{[l]}) = \sum_{\Delta \in \sigma^{(l)}} (-1)^{l-d_\Delta} \frac{v_\Delta(T)}{(1-T^M)^{d_\Delta}}$$

and $\sigma^{(l)} \subseteqq \sigma$ is the set of all simplexes contained in an l-dimensional coordinate hyperplane and not in a lower dimensional coordinate hyperplane.

– Now let us have a look at the following exact sequences which one gets similarly to proposition 1.2.

$$0 \to (\bigcup_{j\in\gamma} \bar{A}^{(j)}) \to \bigoplus_{j\in\gamma} \bar{A}^{(j)} \to \bigoplus_{j_1,j_2\in\gamma} \bar{A}^{(j_1,j_2)} \to \ldots \to \bar{A}^{(\gamma)} \to 0 .$$

As a consequence we get

$$P(\bigcup_{j\in\gamma} \bar{A}^{(j)}) = \sum_{\substack{l=1 \\ |\gamma'|=l \\ \gamma'\subseteqq\gamma}}^{|\gamma|} (-1)^{l+1} P(\bar{A}^{(\gamma')}) .$$

If we sum up about all γ with $|\gamma| = p$, we get

$$\sum_{|\gamma|=p} P(\bigcup_{j\in\gamma} \bar{A}^{(j)}) = \sum_{l=1}^{p} (-1)^{l+1} P(\bar{A}^{[m-l]}) \binom{m-l}{p-l}$$

since every γ' with $|\gamma'| = l$ occurs exactly $\binom{p}{l}\binom{m}{p}\binom{m}{l}^{-1} = \binom{m-l}{p-l}$ times in this sum. Now it is not difficult to prove the following proposition:

3.3. Proposition. $P(\bar{\Omega}^p) = \sum_{l=0}^{p} (-1)^l \binom{m-l}{p-l} p(\bar{A}^{[m-l]})$.

Proof. We know that $\Omega^p = \bigoplus_{i_1<\cdots<i_p} AdX_{i_1} \wedge \ldots \wedge dX_{i_p}$ and $\frac{\partial}{\partial X_{i_1} \ldots \partial X_{i_p}}$ gives us a surjective graded map of A to $AdX_{i_1} \wedge \ldots \wedge dX_{i_p}$ with cernel $\bigcup_{j\in\{i_1,\ldots,i_p\}} A^{(j)}$. So we get

$$P(\bar{\Omega}^p) = \binom{m}{p} P(\bar{A}) - \sum_{l=1}^{p} (-1)^{l+1} \binom{m-l}{p-l} P(\bar{A}^{[m-l]}) .$$

This result we can use to get the POINCARÉ series of $\bar{\Omega}^p_{f_s}$. Using the exact sequence of 2.5. (iii) we obtain

$$P(\bar{\Omega}^p_{f_s}) = \sum_{i=0}^{p} (-1)^i T^{im_s} P(\bar{\Omega}^{p-i}_{f_{s-1}})$$

if we notice that $\wedge df_i$ is a graded map of the degree $\deg(\wedge df_i) = \deg f_i = m_i$). Repeating this process we get

$$P(\bar{\Omega}_{f_s}) = \sum_{0\leqq|i|\leqq p} (-1)^{|i|} T^{\langle i, M\cdot M\rangle} P(\bar{\Omega}^{p-|i|})$$

(with $\boldsymbol{i}=(i_1, \ldots, i_s)$, $|\boldsymbol{i}|=i_1+\ldots+i_s$ and $\langle \boldsymbol{i}, \boldsymbol{M}\cdot\boldsymbol{M}\rangle = i_1 m_1+\ldots+i_s m_s$). Now we can apply 3.3. to obtain:

3.4. Proposition. $P(\bar{\Omega}^p_{f_s}) = \sum_{0\leq l+|i|\leq p} (-1)^{l+|i|} \binom{m-l}{p-l-|\boldsymbol{i}|} \cdot T^{\langle i, M\cdot M\rangle} P(\bar{A}^{m-l})$.

Finally we are interested in the Poincaré series of

$$V^r = \bar{\Omega}^{m-r}_{f_r}/d\bar{\Omega}^{m-r-1}_{f_r} + \sum_{j=1}^{r} f_j \bar{\Omega}^{m-r}_{f_r} .$$

We know that $f_1, \ldots, f_r$ is a regular sequence of $\bar{\Omega}^{m-r}_{f_r}/d\bar{\Omega}^{m-r-1}_{f_r}$ (Corollary 2.11.) and the sequence of Corollary 2.10. is exact.

Using lemma 3.1. we get

$$P(\bar{V}^r) = \prod_{i=1}^{r} (1-T^{m_i}) \left(P(\bar{\Omega}^{m-r}_{fr}) + \ldots + (-1)^{m-r} P(\Omega^0_{f_r}) + (-1)^{m-r+1} P(K[\boldsymbol{f}])\right) .$$

Now $K[\boldsymbol{f}]$ is a free K-algebra ($\boldsymbol{f}=(f_1, \ldots, f_r)$) so we get

$$P(K[\boldsymbol{f}]) = \prod_{i=1}^{r} (1-T^{m_i})^{-1} .$$

By using this we obtain

$$P(\bar{V}^r) = (-1)^{m-r+1} + \sum_{k=0}^{m-r} (-1)^k P(\bar{\Omega}^{m-r-k}_{f_r}) \prod_{i=1}^{r} (1-T^{m_i}) .$$

We use 3.4. to obtain

$$\begin{aligned} P(V^r) &= (-1)^{m-r+1} + \prod_{i=1}^{r} (1-T^{m_i}) \sum_{0\leq l+|i|+k\leq m-r} (-1)^{l+|i|+k} \\ &\quad \times \binom{m-l}{r+k+|\boldsymbol{i}|} T^{\langle i, M\cdot M\rangle} P(\bar{A}^{(m-l)}) \\ &= (-1)^{m-r+1} + \prod_{i=1}^{r} (1-T^{m_i}) \sum_{\substack{0\leq |i|+k\leq m-r-l_\Delta \\ \Delta\in\sigma^{(m-l_\Delta)}}} (-1)^{|i|+k-d_\Delta+m} \\ &\quad \times \binom{m-l_\Delta}{r+k+|\boldsymbol{i}|} T^{\langle i, M\cdot M\rangle} \frac{v_\Delta(T)}{(1-T^M)^{d_\Delta}} \end{aligned}$$

(using 3.2.).

4. Proof of the theorem

Let us first suppose that $\boldsymbol{f}$ is non-degenerate. Then we can apply the results of chapter 3. Now we know that the Milnor number of $\boldsymbol{f}$ is $\mu(\mathfrak{O}_x) = \dim_K V^r$ (cf. [1] resp. chapter 1). On the other hand we have $\dim_K V^r = P(V^r)|_{T=1}$. It remains to show that $\nu(\Gamma_f, M) = P(V^r)|_{T=1}$. Using the last result of chapter 3 we can write

$$P(V^r) = (-1)^{m-r+1} + \prod_{i=1}^{r} (1-T^{m_i}) \sum_{\substack{0 \leqq |i|+k \leqq m-r-l_\Delta \\ \Delta \in \sigma^{(m-l_\Delta)}}} (-1)^{|i|+k-d_\Delta+m} \times \binom{m-l_\Delta}{r+k+|i|} T^{\langle i, M\cdot M\rangle} \frac{v_\Delta(T)}{(1-T^M)^{d_\Delta}}.$$

If we write shortly

$$C_\Delta(T) = \sum_{0 \leqq |i|+k \leqq m-l_\Delta-r} (-1)^{|i|+k} \binom{m-l_\Delta}{r+k+|i|} T^{\langle i, M\cdot M\rangle}$$

$$= \sum_{0 \leqq |i| \leqq m-l_\Delta-r} \left(\sum_{0 \leqq k \leqq m-l_\Delta-r-|i|} (-1)^k \binom{m-l_\Delta}{r+k+|i|} \right) (-1)^{|i|} T^{\langle i, M\cdot M\rangle}$$

$$= \sum_{0 \leqq |i| \leqq m-l_\Delta-r} (1)^{|i|} \binom{m-l_\Delta-1}{r-1+|i|} T^{\langle i, M\cdot M\rangle}$$

$$= \sum_{j_1+\cdots+j_r = m-l_\Delta-r} (1-T^{m_1})^{j_1} \cdot \ldots \cdot (1-T^{m_r})^{j_r}$$

we get

$$P(V^r) = (-1)^{m-r+1} + \sum_{\Delta \in \sigma^{(m-l)}} (-1)^{m-d_\Delta} \cdot \frac{\prod_{i=1}^{r} (1-T^{m_i})}{(1-T^M)^{d_\Delta}} \cdot C_\Delta(T) \cdot V_\Delta(T).$$

In $T=1$ $C_\Delta(T)$ has a zero of order $m-l_\Delta-r \geqq d_\Delta - r$ (because $\Delta \subseteqq (m-l_\Delta)$-dimensional hyperplane and $d_\Delta = \dim \Delta + 1$). So we obtain

$$P(\bar{V}^r)|_{T=1} = (-1)^{m-r+1} + \sum_{\Delta \in \sigma^{(m-l)}} (-1)^{m-d_\Delta} \times \frac{\prod_{i=1}^{r} (1-T^{m_i})}{(1-T^M)^{d_\Delta}} \cdot C_\Delta(T) \cdot V_\Delta(T)|_{T=1},$$

i.e. we just have to sum up about all simplexes of maximal dimension of the corresponding coordinate hyperplane, respectively. Now

$$C_\Delta(T) = \sum_{j_1+\cdots+j_r = m-l_\Delta-r} (1-T^{m_1})^{j_1} \cdot \ldots \cdot (1-T^{m_r})^{j_r}.$$

Using $m - l_\Delta = d_\Delta$ we get

$$\frac{(1-T^{m_1}) \cdot \ldots \cdot (1-T^{m_r})}{(1-T^M)^{d_\Delta}} C_\Delta(T)$$

$$= \frac{1}{(1-T^M)^{d_\Delta}} \sum_{j_1+\cdots+j_r = d_\Delta - r} (1-T^{m_1})^{j_1+1} \cdot \ldots \cdot (1-T^{m_1})^{j_r+1}$$

$$= (1+T+\ldots+T^{M-1})^{-d_\Delta} \sum_{j_1+\cdots+j_r = d_\Delta - r} \prod_{k=1}^{r} (1+T+\ldots+T^{m_k-1})^{j_k+1}.$$

This yields

$$P(\bar{V}')\,|_{T=1}=(-1)^{m-r+1}+\sum_{\substack{\Delta\in\sigma^{(m-l)}\\ m-l_\Delta=d_\Delta}}(-1)^{m-d_\Delta}M^{-d_\Delta}$$

$$\times\sum_{j_1+\cdots+j_r=d_\Delta-r}m_1^{j_1+1}\cdot\ldots\cdot m_r^{j_r+1}V_\Delta(1)\,.$$

Let us recall the notations of chapter 1:

$$M_i=\frac{m_i}{M},\quad \lambda_d=d!\prod_{i=1}^{r}M_i\sum_{j_1+\cdots+j_r=d-r}M_1^{j_1}\cdot\ldots\cdot M_r^{j_r}\,.$$

This means

$$P(\bar{V}')\,|_{T=1}=(-1)^{m-r+1}\sum_{\substack{\Delta\in\sigma^{(m-l_\Delta)}\\ m-l_\Delta=d_\Delta}}(-1)^{m-d_\Delta}\frac{\lambda_{d_\Delta}}{d_\Delta!}V_\Delta(1)\,.$$

Now we know that $\frac{V_\Delta(1)}{d_\Delta!}$ is just the volume of the d-dimensional pyramid defined by Δ with vertex 0. All d-dimensional volumes of the d-dimensional simplexes together give the d-dimensional volumes of $\Gamma_-(f)$ and we get

$$P(\bar{V}')\,|_{T=1}=(-1)^{m-r+1}\sum_{d=r}^{m}(-1)^{m-d}\lambda_d V_d=\nu(\Gamma_f,\boldsymbol{M})$$

and the theorem is proved in case f is non-degenerate.

Let us now consider the general case.

Let V_i be the K-vector space generated by the
$X_1^{\mu_1}\cdot\ldots\cdot X_m^{\mu_m}$ with $h(\mu)=M_i$, $\mu=(\mu_1,\ldots,\mu_m)$
(h – the function defined by the NEWTON polygon Γ_f). Furthermore, let $V=V_1\oplus\oplus\ldots\oplus V_r$, then the inertial form $f^{(\Gamma)}$ of f is an element of V.

4.1. Proposition. *$U=\{g\in V,\ g$ non-degenerate$\}\subseteq V$ contains a ZARISKI open dense subset of V.*

Using the proposition it is not difficult to prove the theorem in the general case. Suppose f is degenerate and consider the family $F_\lambda=f+\lambda\cdot g$ and $g\in U$ non-degenerate. Then F_λ is non-degenerate for almost all λ. Now μ is upper semi-continuous, then $\mu(F_0)=\mu(\mathfrak{O}_X)\geqq\nu(\Gamma_f,\boldsymbol{M})$, because $\mu(F)=\nu(\Gamma_f,\boldsymbol{M})$. The theorem is proved.

Proof of proposition 4.1.: Let $U_{\Delta,s}=\{g\in V$, such that $g_s^{(\Delta)}$ is non-degenerate$\}$, $d_\Delta\geqq s$. It is enough to show that $U_{\Delta,s}$ contains an open dense set. We can use the following lemma of KOUCHNIRENKO (cf. [4]):

4.2. Lemma. *Let $\Delta\in\psi$ be a face of Γ. The homogenous $g_1,\ldots,g_k\in\boldsymbol{A}_\Delta$ generate an $\mathfrak{M}_\Delta$-primary ideal iff for any face $\Delta'\subseteq\Delta$ the polynomials $g_1^{(\Delta')},\ldots,g_k^{(\Delta')}$ have no common zero in $(K\setminus\{0\})^m$.*

Remark. *If Δ is in a coordinate plane defined by $X_{j_1}=\ldots=X_{j_k}=0$ then this lemma holds if $(K\backslash\{0\})^m$ is replaced by $(K\backslash\{0\})^{(\gamma)}$, $\gamma=\{j_1,\ldots,j_k\}$.*

Let $\boldsymbol{g}\in V$, then $\boldsymbol{g}\in U_{\Delta,s}$ iff for all $\Delta'\subseteq\Delta$ the ideal $(\boldsymbol{g}_s^{(\Delta')},\mathfrak{J}(\boldsymbol{g}_s^{(\Delta')}))$ has no zero in $(K\backslash\{0\})^m$. So it is enough to show that the $W_{\Delta',s}=\{\boldsymbol{g}\in V$, s.t. $(\boldsymbol{g}_s^{(\Delta')},\mathfrak{J}(\boldsymbol{g}_s^{(\Delta')}))$ has no zero in $(K\backslash\{0\})^m$ contains an open dense subset for all faces Δ' of Γ and all s, $1\leqq s\leqq r$. Now we fix a face Δ and choose an injective homomorphism $\varphi:\mathbf{Z}^m\to\mathbf{Z}^m$ such that for a positive integer t: $\varphi(\Delta)\subseteq\{(n_1,\ldots,n_m),\ n_i\geqq 0,\ n_m=t\}$ and points $(0,\ldots,0,t)$, $(0,\ldots,p_{m-1,1},t),\ldots,(0,\ldots,0,p_{m-d_\Delta+1,d_\Delta-1},\ldots,t)$ are in $\varphi(\Delta)$, $p_{ij}>0$.

φ give us a finite injective homomorphism $\varphi_*:K[X]\to K[X]$. Let us suppose that $(\boldsymbol{f}_s^{(\Delta)},\mathfrak{J}(\boldsymbol{f}_s^{(\Delta)}))$ has a zero in $(K\backslash\{0\})^m$, then the ideal generated by $\boldsymbol{f}_s^{(\Delta)}$ and the s-minors of $\left(\dfrac{\partial f_i^{(\Delta)}}{\partial x_j}\right)$ has also a zero $\xi=(\xi_1,\ldots,\xi_m)$, $\xi_i\neq 0$. Then $\varphi(\xi)$ is a zero of $\left(\varphi_*\boldsymbol{f}_s^{(\Delta)},\ s\text{-minors of}\left(\dfrac{\partial\varphi_*\boldsymbol{f}_i^{(\Delta)}}{\partial x_j}\right)\right)$. For this reason we can replace Δ by $\varphi(\Delta)$. If $d_\Delta\leqq s$ because of the special type of Δ, we can construct a family of $\mathfrak{m}_\Delta$-regular sequences using polynomials with general coefficients $f_1^{(\Delta)}$ depending on X_m only, $f_2^{(\Delta)}$ on $X_m, X_{m-1},\ldots$ up to $f_{d_\Delta}^{(\Delta)}$ depending on $X_m,\ldots,X_{m-d_\Delta+1}$. If $d_\Delta>s$, we get $f_i^{(\Delta)}=X^{\alpha_i\cdot t}(g_i(X_1,\ldots,X_{m-1})-a_i)$, $a_i\in K$, $i=1,\ldots,s$. By the same reason the g_s form a regular sequence of $K[X]$ for $\boldsymbol{f}_s^{(\Delta)}$ from an open set.

We consider the map $\sigma_g:\mathbf{A}^{m-1}\to\mathbf{A}^s$ of the affin K-spaces given by $\sigma_g(\xi_1,\ldots,\xi_{m-1})=(g_1(\xi),\ldots,g_s(\xi))$. Since g is a regular sequence, the set of critical values of σ_g, i.e. the set of all $(a_1,\ldots,a_s)\in\mathbf{A}^s$ such that $g_i(\xi)=a_i$ and rank $\left(\dfrac{\sigma g_i}{\sigma X_j}(\xi)\right)<s$, is contained in a proper closed subset of $\mathbf{A}^s$. So, for an open set U_g of $\mathbf{A}^s$, we have that the ideal generated by $\left(g_1-a_1,\ldots,g_s-a_s,\ s\text{-minors of}\left(\dfrac{\sigma g_i}{\sigma X_j}\right)\right)$ has no zero. Hence we found an open subset $\{\boldsymbol{f}_s^{(\Delta)}, \boldsymbol{g}$ is a regular sequence and $\boldsymbol{a}\in U_g\}$ contained in $W_{\Delta,s}$. Proposition 4.1. is proved.

A more usefull criterion for non-degeneration we find in the following situation:

4.3. Proposition. Let $\boldsymbol{M}=(1,\ldots,1)$ *be the weight of $\boldsymbol{f}$ and let any face of Γ be a simplex. If $f_i=\sum a_{i\alpha}X^{\underline{\mu}_\alpha}$, $\underline{\mu}_\alpha$ runs over all vertices of Γ, the $\boldsymbol{f}$ is non-degenerate iff for all $s\leqq r$ and all faces Δ of Γ with $d_\Delta=s$, vertices $\underline{\mu}_{j_1},\ldots,\underline{\mu}_{j_s}$: $\det(a_{i,j_k})\neq 0$.*

4.4. Corollary. *If any face of Γ is a simplex, any equation $\boldsymbol{f}$ with $\Gamma_f\cap N^m=\{$vertices of the simplexes of $\Gamma\}$ is non-degenerate.*

Proof of 4.3. Again using 4.2. we have to show that for any Δ and any s the ideal $(\boldsymbol{f}_s^{(\Delta)},\mathfrak{J}(\boldsymbol{f}_s^{(\Delta)}))$ has no zero in $(K\backslash\{0\})^m$. For $d_\Delta\leqq s$ it is trivially seen that $(f_1^{(\Delta)},\ldots,f_{d_\Delta}^{(\Delta)})=(X^{\underline{\mu}_{j1}},\ldots,X^{\underline{\mu}_{jd}})$ because the matrix of coefficients of $\boldsymbol{f}_{d_\Delta}^{(\Delta)}$ has full rank. Hence our ideal contains monomials with all possible X_i, hence no zero is from $(K\setminus\{0\})^m$. For $d_\Delta > s$ the ideal $\mathfrak{J}(f_s^{(\Delta)})$ is generated by $\binom{m}{s}$ equations $\det\left(X_{k_i}\frac{\sigma f_j^{(\Delta)}}{\sigma X_i}\right) = \sum_{0 \leq j_1 < \cdots < j_s \leq d} \det(\mu_{k_i j_l}) \det(a_{i j_l}) X^{\mu_{j_1} + \cdots + \mu_{j_s}}$. Since the $\mu_1, \ldots, \mu_d$ are linear independent, the matrix $(\det(\mu_{k_i j_l}))$ has full rank $\binom{d}{s}$ and $\mathfrak{J}(f_s^{(\Delta)})$ is generated by the $\det(a_{i j_l}) X^{\mu_{j_1} + \cdots + \mu_{j_s}}$, $1 \leq j_1 < \ldots < j_s \leq d$. So again our ideal has no zero in $(K\setminus\{0\})^m$. The proposition is proved.

References

[1] Greuel, G. M., Der singuläre Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, Math. Ann. **214**, 235–266 (1975).
[2] Greuel, G. M., Hamm, H. A., Invarianten quasihomogener vollständiger Durchschnitte, Inv. math. **49**, 67–86 (1978).
[3] Hochster, M., Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. **96**, 318–337 (1972).
[4] Kouchnirenko, A. G., Polyèdres de Newton et nombres de Milnor, Inv. math. **32**, 1–31 (1976).

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