DISTINGUISHED DEFORMATIONS OF ISOLATED SINGULARITIES OF PLANE CURVES

Bernd Martin and Gerhard Pfister

Let $f \in \mathbb{C}\{\underline{X}\}$ be an analytic function with isolated critical point at 0, $\underline{X}=(X_1,\dots,X_n)$. An important topological invariant is the Milnor number $\mu(f)=\dim_{\mathbb{C}} Q(f)$, $Q(f)=\mathbb{C}\{\underline{X}\}/(\partial f/\partial X_1,\dots,\partial f/\partial X_n)$.
The idea to study the following technique of "distinguished deformations" goes back to a question of V.I.Arnold (cf.[1]):
Is the μ-constant stratum in the miniversal unfolding of f smooth?
A positive answer is known in the following cases:
n=2 (cf.[6],[7]), if f is semi-quasihomogeneous (cf.[9], resp. for the homogeneous case cf.[2],[8]).
A. Nobile (cf.[4]) introduced the idea of α-deformations, deformations fixing the Hilbert-Samuel-function $H_f(\ell)=\dim_{\mathbb{C}} Q(f)/\underline{m}^{\ell+1}$ of the algebra Q(f). We connect this idea with the methods of Briançon (cf. [2]) to study distinguished deformations of an ideal of finite codimension (applied to the Jacobian ideal) to study the α-deformations in more detail.
We consider here only plane curve singularities (n=2), probably the results are also true in higher dimension.
First we introduce the notion of a vertical standard base and the stair-sequence E(I) of an ideal I (cf.[2]). Then we associate a stair-sequence E_Γ to a Newton polygon Γ such that for almost all functions f with $\Gamma(f)=\Gamma$ and Jacobian ideal $j(f)=(\partial f/\partial X,\partial f/\partial Y)$ $E_\Gamma=E(j(f))$ holds.
In the third chapter we consider distinguished deformations of f in a fixed coordinate system:
A deformation (F,ξ,η) of f and the coordinates X, Y over A is called distinguished if $A\{\xi,\eta\}/(\partial F/\partial\xi,\ \partial F/\partial\eta)+(\xi^a\eta^b,\xi^{a+1})$ is A-flat for all a,b (here A is a local analytic algebra).

This is slightly stronger than normal flatness of $Q(F)=A\{\xi,\eta\}/(\partial F/\partial\xi, \partial F/\partial\eta)$ over A along (ξ,η) which is required for α-deformations.
The canonical morphism from the distinguished deformation functor to unfoldings is an embedding and the functor admits a hull.
For E_Γ-nondegenerate functions (cf.2.2) the reduced d stinguished deformation functor is smooth and any μ-constant deformation over $\mathbb{C}\{T\}$ is distinguished.

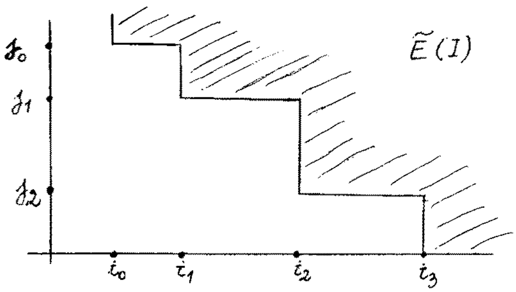
§1. Vertical standard bases and stair sequences

Let A be a local analytic $\mathbb{C}$-algebra. We fix the lexicographic order in $\mathbb{N}^2$ ($\mathbb{N}$ the positive integers).

For any $f=\sum a_{ij}X^iY^j \in A\{X,Y\}$ let $d(f)=\min\{(i,j),\ a_{ij}\neq 0\}$ denote the bidegree and $in(f)=a_{k\ell}$ for $d(f)=(k,\ell)$ the initial form.
Note $d(f)=(i,j)$ iff $f=X^i g(X,Y)$ and $ord_Y g(0,Y)=j$.

Definition 1: The stair-diagram of an ideal $I\subseteq A\{X,Y\}$ is the additive semigroup $\tilde{E}(I)=\{d(f),\ f\in I\}\subseteq\mathbb{N}^2$ and the stair-sequence is the finite minimal base $E(I)=\min\{E, E+\mathbb{N}^2=\tilde{E}(I)\}$ of $\tilde{E}(I)$.
Obviously we have

$$\tilde{E}(I)=\{(i_o,j_o),\dots,(i_s,j_s)\}$$
$$0\leq i_o< i_1<\dots<i_s$$
$$j_o>j_1>\dots>j_s\geq 0$$



Definition 2: A base $\{f_o,\dots,f_s\}$ of I is called (vertical) standard base, if $\{d(f_k)=(i_k,j_k)\}=E(I)$. (1)

Remark: If $A=\mathbb{C}$ than any set $\{f_o,\dots,f_s\}\subseteq I$ with the property (1) is a base of I and there is a unique standard base of I given by the condition

$$\mathrm{supp}(f_k)\cap \tilde{E}(I)=d(f_k).$$

§2. The stair sequence of the Jacobian ideal and the Newton-polyhedron of f

2.1. The stair sequence

Let Γ_+ be the Newton-polygon of a function $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with isolated critical point at 0 ($\Gamma_+\cap\mathbb{N}^2=\mathrm{supp}(f)+\mathbb{N}^2$). Let $E'(\Gamma)=\{(\gamma_o,\delta_o),(\gamma_1,\delta_o-1),\dots,(\gamma_{\delta_o},0)\}\subseteq\mathbb{N}^2$
with

$(\gamma_o,\delta_o)=d(f)$ and $\gamma_\ell=\min\{\gamma,(\gamma,\delta_o-\ell)\in\Gamma_+\}$

and let us denote by $E(\Gamma)\subseteq E'(\Gamma)$ the minimal base of $\Gamma_+\cap\mathbb{N}^2$. Let
$F':=\sum_{\ell=0}^{\delta_o} Z_\ell X^{\gamma_\ell}Y^{\delta_o-\ell}\in\mathbb{Z}[Z_o,\dots,Z_{\delta_o},X,Y]$ and $F_\Gamma=F'_\Gamma|_{E(\Gamma)}$.

For $k=\begin{cases}\delta_o-1 & \text{if } \gamma_o=0\\ \delta_o & \text{if } \gamma_o=1\end{cases}$ we write the derivatives of F'_Γ in the following form:

$$\partial F'_\Gamma/\partial Y=\sum_{\ell=0}^{k} a_\ell Y^{k-\ell} \quad\text{and}\quad \partial F'_\Gamma/\partial X=\sum_{\ell=0}^{k} b_\ell Y^{k-\ell} \tag{2}$$

with $a_\ell,b_\ell\in\mathbb{C}[\underline{Z},X]$.

Let $R_\ell(F'_\Gamma)\in\mathbb{C}[\underline{Z},X]$ be the left side upper $(2\ell,2\ell)$-minor of the $(2k,2k)$ matrix

$$\begin{pmatrix} a_o,\dots\dots,a_k,0\dots\dots0\\ b_o,\dots\dots,b_k,0\dots\dots0\\ 0,a_o,\dots\dots,a_{k-1}a_k,0\dots0\\ \vdots \qquad\qquad\qquad \vdots\\ 0\dots \quad b_0\ b_1,\dots\quad b_k \end{pmatrix} \tag{3}$$

Now for $\ell=0,\dots,k$ let us denote by $r_\ell(\Gamma)$ the order of $R_\ell(F'_\Gamma)$ with respect to X ($r_o=0$), $\alpha_\ell(\Gamma)=r_\ell(\Gamma)-r_{\ell-1}(\Gamma)$ ($\alpha_o=0$) and let E_Γ be the base of

$$\bigcup_{0}^{k} (\alpha_\ell, k-\ell)+\mathbb{N}^2.$$

Lemma: (i) $R_\ell \neq 0$, $\ell=1,..,k$

(ii) $\alpha_{\ell-1} \leq \alpha_\ell$, $\ell=1,...,k$.

2.2. E_Γ-nondegeneracy

Definition 3: f is called E_Γ nondegenerated if $\Gamma(f)=\Gamma$ and $r_\ell(f|_{E'})=$ $=r_\ell(\Gamma)$ for $\ell=1,...,k$.

Remark:

(i) The E_Γ-nondegeneracy of f depends only from the coefficients of the monomials of f on the polygon Γ and "directly" above Γ and is a Zariski-open condition for the "leading forms" $f|_{E'}$.

(ii) The following assertions remain valid if we replace the condition in definition 3 by the weaker one

$$r_\ell(f\ |_{E(\Gamma)})=r_\ell(F_\Gamma) \qquad (r_\ell=\infty, \quad \text{if} \quad R_\ell\equiv 0).$$

More explicit formulas for the $\alpha_\ell(\Gamma)$ are given in [5].

Proposition 1: $E(j(f))=E_\Gamma$, if f is E_Γ-nondegenerated, $j(f)=(\partial f/\partial X, \partial f/\partial Y)\mathbb{C}\{X,Y\}$.

Idea of the prooof:

1. We may choose a base $h_o=\sum_{\ell=0}^{k} a_\ell(X)Y^{k-\ell}$, $h_1=\sum_{\ell=0}^{k} b_\ell(X)Y^{k-\ell}$ of j(f) such that the $a_\ell, b_\ell \in \mathbb{C}\{X\}$ have the same X-order as the corresponding ones in (2) and more over $r_\ell(h_o,h_1)=r_\ell(\Gamma)$.

2. The ℓ-th element of a standard base can be written (using Galligo's algorithm) in the following form

$$ph_o + qh_1 = f_\ell \qquad (h_o=f_o) \tag{4}$$

where $p,q\in\mathbb{C}\{X\}[Y]$ are polynomials in Y of degree ℓ-1 and f_ℓ has degree k-ℓ with respect to Y.

Comparing the coefficients of $Y^{k+\ell-1},...,Y^{k-\ell}$ in the equation (4) we obtain linear equations for the Y-coefficients of p and q:

$$A\cdot w=\begin{pmatrix} a_o & b_o & 0 & 0 & \dots & 0 \\ a_1 & b_1 & a_o & b_o & & \\ \cdot & & a_1 & b_1 & & \\ \cdot & & \cdot & & & \\ \cdot & & \cdot & & & \\ & & \cdot & & & \end{pmatrix}\begin{pmatrix} w_1 \\ \\ \cdot \\ \cdot \\ \cdot \\ w_{2\ell}\end{pmatrix}=\begin{pmatrix} 0 \\ \\ \cdot \\ \cdot \\ \cdot \\ X^{\alpha}\end{pmatrix} \tag{5}$$

Let $d(f_\ell)=(\alpha_\ell k-\ell)$, then α_ℓ is the minimal exponent α, s.t. (5) has a solution in $\mathbb{C}\{X\}$.

3. Let us denote the $(2\ell,i)$-th cofactor of A by A_i. Using Cramer's rule (5) has a solution in $\mathbb{C}\{X\}$ iff $\mathrm{ord}_X|A_i|+\alpha\geq \mathrm{ord}_X|A|=r_\ell(\Gamma)$ for $i=1,\dots,2\ell$. But $\mathrm{ord}_X a_o=0$ implies $\mathrm{ord}_X|A_2|=r_{\ell-1}(\Gamma)$ (if $\gamma_o=0$) resp. $\mathrm{ord}_X b_o=0$ hence $\mathrm{ord}_X|A_1|=r_{\ell-1}(\Gamma)$ (if $\gamma_o=1$). Hence we obtain $\alpha_\ell\geq r_\ell(\Gamma)-r_{\ell-1}(\Gamma)$.

4. But we know that $r_k(\Gamma)\leq\sum\alpha_\ell=\dim_{\mathbb{C}}\ Q(f)=\mu(f)=r_k(\Gamma)$ hence $\alpha_\ell=\alpha_\ell(\Gamma)$, $\ell=1,\dots,k$.

Remark:

Using the proposition the Hilbert-Samuel-function H_f of $Q(f)$ can be expressed in terms of Γ. In general H_f is different from H_g even if f and g are contact equivalent (cf.[4]). But if f is E_Γ-nondegenerated εf (ε an unit) is nondegenerated too, i.e. in this case H_f is an invariant of the contact class of f.

2.3. Examples

1. Let f be homogeneous of degree k+1, then $R_\ell(f)\in\mathbb{C}[X]$ is homogeneous of degree ℓ^2. f is E_Γ-nondegenerated iff $R_\ell(f)\neq 0$, $\ell=1,\dots,k$ i.e. after a suitable linear change of coordinates f becomes E_Γ- nondegenerate.

2. If $E(\Gamma)=\Gamma\cap\mathbb{N}^2$ (Γ=union of compact faces of Γ_+), then any f with $\Gamma(f)=\Gamma$ is E_Γ-nondegenerated and $\alpha_\ell(\Gamma)=2\gamma_\ell-1$ (γ_ℓ as in 2.1).

3. If $\mathrm{mult}(f)=\delta_o+\gamma_o$ (for instance if f is semi-quasihomogeneous, change X and Y if necessary) than $E(\Gamma)=E'(\Gamma)$ and E has only stairs of height 1, i.e. $\alpha_o<\alpha_1<\dots<\alpha_k$.

§3. Distinguished deformations

3.1. Deformations along a coordinate system

Definition 4: Let A be a local analytic algebra with maximal ideal $\underline{m}$. $F \in A\{X,Y\}$ is called a deformation of f, if $\bar{F}=f$ ($\bar{F}=F \bmod \underline{m}$). A triple $(F,\xi,\eta) \in A\{X,Y\}^3$ is called a deformation of f over A along the coordinate system (X,Y) if

$$\bar{F}=f,\ \bar{\xi}=X,\ \bar{\eta}=Y \text{ and } A\{X,Y\}/(\xi,\eta) \simeq A.$$

The last isomorphism lifts to an isomorphism $A\{\xi,\eta\} \simeq A\{X,Y\}$. Two deformations (F,ξ,η) and (F',ξ',η') are called equivalent, if there is an automorphism $\Phi \in \mathrm{Aut}\{X,Y\}$ such that $\Phi(F)=F'$, $\Phi(\xi)=\xi'$, $\Phi(\eta)=\eta'$ and $\bar{\Phi}=\mathrm{id}_{\mathbb{C}\{X,Y\}}$.

3.2. Distinguished deformations

Definition 5: A deformation (F,ξ,η) of f over A along (X,Y) is called distinguished if

$$Q(F)=A\{\xi,\eta\}/(\partial F/\partial\xi,\ \partial F/\partial\eta)$$

is A-flat along the bigrading generated by (ξ,η), i.e. if

$$Q(F)/(\xi^a\eta^b,\xi^{a+1}) \text{ is A-flat for all } a,b \in \mathbb{N}.$$

Remark:

1. The deformation functor on the category of Artin rings

$$\Xi_{f,X,Y}(A)=\{(F,\xi,\eta) \text{ distinguished}\}\ /\text{equivalence}$$

admits a hull.

2. Every distinguished deformation is an α-deformation along $I=(\xi,\eta)$ in the sense of [4].

3. Every α-deformation (F,I) leads to a distinguished deformation for a suitable choice of X,Y and generators ξ,η of I.

4. In the case $A=\mathbb{C}\{T\}$ a distinguished deformation is μ-constant and the Hilbert-Samuel-function H_{f_t} is constant with respect to t. But not every μ-constant deformation is distinguished.

A nontrivial example of a distinguished deformation is a homogeneous deformation of a E_Γ-nondegenerated homogeneous function.

3.3. Characterization of distinguished deformations

Proposition 3: The following conditions are equivalent:

(i) (F,X,Y) is a distinguished deformation

(ii) $E(j(F))=E(j(f))$

(iii) $j(F)=(\partial F/\partial X, \partial F/\partial Y)$ admits a standard base $F_o,\dots,F_s$ and the initial forms (in the sense of 1) $in(F_\ell)$ are units (than $\bar{F}_o,\dots,\bar{F}_s$ is a standard base of $j(f)$).

Proof: (iii)→(i) One has to lift every relation of $\partial f/\partial X$, $\partial f/\partial Y$, X^aY^b, X^{a+1}. Such a relation corresponds to

$$g(X,Y)\varepsilon j(f) \quad \text{and} \quad d(g)=(a,b).$$

Let $G\varepsilon A\{X,Y\}$ be any element over g. By the divisions-theorem (cf.[2]) $G=\sum H_iF_i + R$ and $d(R)=d(G)$ and $R\varepsilon m_A\{X,Y\}$, hence $G-R\varepsilon j(F)$ lifts the given relation.

(i)→(ii) obvious by definition

(ii)→(iii) Again using the division theorem a standard basis $f_o,\dots$ $\dots,f_s$ of $j(f)$ can be lifted to $F_o,\dots F_s\subset j(F)$ in such a way that $d(F_i)=d(f_i)$. But this implies $in(F_i)\varepsilon A^*$.

Corollary: If f is E_Γ-nondegenerated and F a deformation of f by monomials of $\Gamma_+(f)$ than (F,X,Y) is distinguished.

The other direction is true for E_Γ-nondegenerate functions.

Proposition 4: Ler f be E_Γ-nondegenerated and F a deformation of f over $C\{t\}$. Than (F,X,Y) is distinguished iff $\Gamma(f)=\Gamma(F)$.

Idea of the proof: Let (a,b) be the smallest vertice of $\Gamma(F)$ not belonging to $\Gamma(f)$. Than there is a minimal ℓ such that $r_\ell(F)<r_\ell(f)$. Similarly to 2.2.(4) one can construct an element $h\varepsilon j(F)$ with $d(h)=(r_\ell(F)-r_{\ell-1},k-\ell)\notin\tilde{E}(j(f))$.

Corollary: If f is E_Γ-nondegenerated than the reduced hull H_{red} of $\mathcal{E}_{f,X,Y}$ is smooth of dimension m(f).

Here m(f)=modality of f=number of monomials of an admissible monomial base of Q(f) (cf. [3]) which are in $\Gamma_+(f)$.

Remark:

1. Even for E_Γ-nondegenerate functions in general $\varepsilon_{f,X,Y}$ is not a smooth functor. This shows the following example:
Let $f=Y^3+aX^2Y^2+bX^3Y+cX^5$. If $b\neq 0$ than $r_1=2$, $r_2=7$, f is E_Γ-nondegenerated and equivalent to the smooth function simple function E_7. Let $F=f+\varepsilon X^4$. Than $R_1(F)=R_1(f)$ and $R_2(F)=81\varepsilon^2X^6+\ldots$, hence for $\varepsilon^2=0$,F is distinguished and dim $\varepsilon_{f,X,Y}(\mathbb{C}[\varepsilon])>0$.
For more details cf [5].

2. Every deformation of f is equivalent by versality to a deformation in terms of an admissible base. Similarly to proposition 4,holds:
For E_Γ-nondgenerate functions f every admissible μ-constant deformation of f over $\mathbb{C}\{T\}$ is distinguished.

References

1 V.I.Arnold, On some problems in singularity theory, Proc.Indian Acad.Sc.90(1), 1-9(1981)

2 J.Briançon, Contribution à l'Etude des Déformations de Germes de Sous-Espaces Analytiques de $\mathbb{C}^n$, Thése Nice 1976

3 A.G.Kouchnirenko, Polyedres de Newton et nombres de Milnor, Invent.Math.32, 1-31 (1976)

4 A.Nobile, On certain numbers associated to isolated critical points, Math.Z.177, 503-517 (1981)

5 B.Martin, On distinguished deformations of plane curve singularities, Preprint Berlin 1983

6 M.Oka, On stability of the Newton boundary, Preprint Arcata 1982

7 J.Wahl, Equisingular deformations of plane algebroid curves, Trans. AMS 193, 143-170 (1974)

8 A.M.Gabrielov, A.G.Koushnirenko, Description of the deformations with constant Milnor number for homogeneous functions (In Russian), Funkt.Anal.Pril.9/4, 67-68 (1975).

9 A.N.Varchenko- Evaluation of the codimension of the μ-constant stratum by the mixed Hodge Structure (In Russian) Vesnik MGU 6, 28-31 (1982).

Bernd Martin and Gerhard Pfister
Humboldt-Universität zu Berlin
Sektion Mathematik
108 Berlin, Unter den Linden 6
DDR