Moduli of irreducible plane curve singularities with the semi-group <a,b>

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The moduli space of irreducible plane curve singularities with the semi-group <a,b> and minimal Tjurina-number is a quasi-smooth algebraic variety. Its dimension is the modality of the corresponding monomial curve with respect to the μ-constant stratum under the action of the contact group.

1. Introduction

The aim of this paper is to describe the moduli space of irreducible plane curve singularities with the semi-group <a,b>. There was already an approach by Washburn (cf. [9]) but it turned out to be wrong in general.

We use the following approach: Consider the germ (X_o, o) in $(\mathbb{C}^2, o)$ defined by $0 = x^a + y^b$, a and b being relatively prime, and let $\underline{X} \dashrightarrow \underline{H}$ be a good representative of the versal μ-constant deformation, μ the Milnor-number of the singularity (X_o, o). Because of the $\mathbb{C}^*$-action $\underline{H}$ can be chosen as $\mathbb{C}^n$ for a suitable n and $\underline{X}$ as a hypersurface in $\underline{H} \times \mathbb{C}^2$. $\underline{H}$ already "contains" all singularities we are hunting for. Along the integral-manifolds of the Lie-algebra V, the kernel of the Kodaira-Spencer map, the family $\underline{X} \dashrightarrow \underline{H}$ is analytically trivial. To obtain a moduli space we have to look for the quotient of $\underline{H}$ by the group G=exp V. Obviously a good quotient can only exist on the strata $\underline{S}_\tau$ of fixed orbit-dimension. $\{\underline{S}_\tau\}$ turns out to be the stratification defined by fixing the Tjurina-number τ, i.e. the dimension of the base of the versal deformation of the singularity in the corresponding fibre of the family.

Notice that the τ-constant stratum in the whole versal deformation of x^a+y^b is in general not contained in the μ-constant stratum $\underline{H}$.

The quotient $\underline{S}_\tau/G$ always exists in the analytic category (cf. [4]) and it is the moduli space of all plane curve singularities with semi-group $\langle a,b \rangle$ and Tjurina-number τ.

We prove that on the open stratum $\underline{S}_{\tau_{min}}$ the quotient $\underline{S}_{\tau_{min}}/G$ in the sense of Mumford (cf. [5]) does exist and is a quasi-smooth algebraic variety, i.e. locally an open subset of a weighted projective space. The dimension of this variety has already been computed by by Delorme [2].

It turns out that $\tau >= \tau'$ implies $\dim \underline{S}_\tau/G <= \dim \underline{S}_{\tau'}/G$, i.e. especially $\dim \underline{S}_{\tau_{min}}/G = m + \tau_{min} - \mu$ is the modality of x^a+y^b with respect to the μ-constant stratum, under the action of the contact group (m the modality with respect to right equivalence, cf. [1]). We give an example that this is not true for surface singularities in $\mathbb{C}^3$. In the example $\underline{S}_\tau$ is empty for some τ, $\tau_{min} <= \tau <= \mu$. This is not possible for curves.

2. The Kodaira-Spencer map of the versal μ-constant family

In this chapter we will study the Kodaira-Spencer map of the versal μ-constant family $\underline{X} \dashrightarrow \underline{H}$ of the germ of the singularity (X_o, o) defined by the polynomial $f = x^a + y^b$, $(a,b) = 1$, in $(\mathbb{C}^2, o)$. f is quasihomogeneous with weights b,a and degree d = ab.

Let $B = \{m_1, \dots, m_\mu\}$ be a monomial base of $R_o = \mathbb{C}[x,y]/\Delta f$, $\Delta f = (\partial f/\partial x, \partial f, \partial y) = (x^{a-1}, y^{b-1})$, ordered by degree: $\deg(x^e y^{e'}) = eb + e'a$.

We may choose $m_1 = 1$ and the hessian $m = x^{a-2}y^{b-2}$.

If it does not lead to any confusion we will not distinguish between the monomials and their exponents.

Denote $(\, , \,)$ the bilinear form defined by the coefficient of the hessian m_μ of the product in R_0:

Let $f, g \in R_0$ and $fg = a_1 m_1 + \ldots + a_\mu m_\mu$, then $(f,g) := a_\mu$.

Denote by $B^0 = \{n_1, \ldots, n_\mu\}$ the dual basis with respect to this bilinear form, i.e. $n_i = m_\mu / m_i$.

Denote $B_u = \{n_1, \ldots, n_r, \deg(n_i) > d\}$

(notice that $r = (a-3)(b-3)/2 + b/a - 1$)

$$B_1 = \{m_1, \ldots, m_r\} = B_u^0$$

$$F = f + \sum_{i=1}^{r} t_i n_i$$

$H = \mathbb{C}[t_1, \ldots, t_r]$

$\underline{H} = \operatorname{Spec} H$ and $\underline{X} = \operatorname{Spec} H[x,y]/F$.

Because of the $\mathbb{C}^*$-action $\underline{X} \dashrightarrow \underline{H}$ is a good representative of the versal μ-constant deformation of $(X_0, 0)$.

Obviously there is a $\mathbb{C}^*$-action on $\underline{H}$ and the Lie-algebra of derivations $\operatorname{Der}(H)$ defined by $\deg(t_i) = d - \deg(n_i)$ and $\deg(\partial/\partial t_i) = -\deg(t_i)$.

The Kodaira-Spencer map is given by the map

$$g: \operatorname{Der}(H) \dashrightarrow H[x,y]/(F, \Delta F),$$

$\Delta F = (\partial F/\partial x, \partial F/\partial y)$, $g(d) = \operatorname{cls}(dF)$, (cf. [4]).

We will study the kernel $V = \operatorname{Ker}(g)$ of the Kodaira-Spencer map.

It is not difficult to see that V is a graded Lie-algebra. Denote by V_+ the Lie-algebra of all vectorfields of V of degree >0, by V_0 the Lie-algebra generated (as vectorspace) by V_+ and the Eulervectorfield of $\operatorname{Der} H$.

Proposition 2.1.: V_0 is of finite dimension and V_0 generates V as an H-module. V_+ is nilpotent and V_0 is solvable.

Proof: $H[x,y]/F =: R$ is a free H-module generated by the elements of B. The multiplication by F is an H-linear map. Denote by $K = (k_{ij})$ the matrix of this map with respect to the bases B and B^0. The image of F is contained in the submodule generated by the elements of B_u. Denote by d_i the vectorfield $\sum_{j=1}^{r} k_{ji} \partial/\partial t_j$. By definition of V the vectorfields d_i, $i = 1, \ldots, r$ generate V, notice that $d_i = 0$ if $i > r$. By definition d_i is homogeneous of degree $\deg(m_i)$, i.e. an element of V_0.

For homogeneous vectorfields d and d' we have $\deg(d, d') = \deg(d) + \deg(d')$, but the degree of a vectorfield of V_0 is bounded by $\deg(m_r)$, notice that the elements of H have negative degree, i.e. V_+ is nilpotent. Finally $V_+ = [V_0, d_1]$, $d * d_1$ is the Eulervectorfield on H and $[d_1, d] = (\deg(d/d)\, d$.

Remark 2.2: It is not difficult to prove that

$[d_i, d_j] = (\deg(d_j) - \deg(d_i))/d * d_k + d$,

if $m_i m_j = m_k$, d a homogeneous vectorfield of the submodule generated by $d_{k+1}, \ldots, d_r$.

The way to study the kernel of the Kodaira-Spencer map and the action on H is to study the Lie-algebra V_0 and the matrix corresponding to its generators d_i. Because of $m_i F = 0$ in R if $i > r$ and the image of the multiplication by F being contained in the submodule generated by $n_1, \ldots, n_r$ we may consider F to be a map on these submodules.

Denote by M_1 the submodule of R generated by B_1 and by M_u the submodule of R generated by B_u. For technical reasons we will consider the map $E: M_1 \dashrightarrow M_u$ corresponding to the multiplication by $-dF$.

Let us denote by $C(\underline{t})$ the matrix corresponding to E with respect to the bases B_1 and B_u and by $CL(\underline{t})$ the matrix of the linear terms with respect to $\underline{t}$ of $C(\underline{t})$.

The following lemma will give some simple properties of these matrices:

Lemma 2.3.: (1) $CL(\underline{t})$ is symmetric;

(2) $C_{ij}(\underline{t}) = 0$ iff $CL_{ij}(\underline{t}) = 0$;

(3) let $j_i = \max\{j, C_{ji} <> 0\}$, then $j_i >= j_{i+1}$;

(4) for $j <= j_i$ there are integers $k(i,j)$ with the following properties: $k(i,j) < k(i,j+1)$, $k(i,j) < k(i+1,j)$;

(5) if $j <= j_i$, then $CL_{ij} = (\deg(n_{k(i,j)}) - d)\, t_{k(i,j)}$

Proof: (1) is clear because of the choice of the bases.

Now $x^e y^{e'} \in B_1$ iff $eb + e'a < ab - 2b - 2a$, i.e. $m_i m_j \in B_1$ implies $m_i m_k \in B_1$ for $k <= j$. Let j_i be maximal such that $m_i m_{j_i} \in B_1$. For $j <= j_i$ define $k(i,j)$ by $m_i m_j = m_{k(i,j)}$. Then we have also $m_i n_{k(i,j)} = n_j$.

By definition of E

$$Em_1 = \sum_{j=1}^{r} (\deg(n_j) - d)\, t_j n_j , \text{ i.e.}$$

$$Em_i = \sum_{j=1}^{r} (\deg(n_j) - d)\, t_j n_j m_i$$

$$= \sum_{j=1}^{r} C_{ji} n_j$$

If $j <= j_1$ then $n_j = m_i n_{k(i,j)}$, i.e.

$CL_{ji} = (\deg(n_{k(i,j)}) - d)\, t_{k(i,j)}$. Suppose now $C_{ji} <> 0$ for $j > j_1$ and choose j maximal with this property, then $0 = Em_i m_j = C_{ji} n_1$ which is a contradiction.

Remark: In the higher dimensional case B_1 cannot be characterized simply by degree. This is the reason why (2) of the lemma fails.

Proposition 2.4. (T. Yano): There is a basis B_1' of M_1 with the following properties:

(1) $m_i' = m_i + h_i$, h_i homogeneous of degree m_i in the submodule generated by $h_{i+1}, \ldots, h_r$

(2) the matrix of E with respect to B_1' is symmetric with linear part $CL(\underline{t})$.

Proof: Consider, on $M_1 \times M_u$, the pairing (,) defined by the coefficient of the hessian of the product of two elemente of M_1 resp. M_u: Let $h, g \in R$ and $hg = c_1 n_1 + \ldots + c_r n_r$, then $(h,g) = c_1$. (,) has the following properties:

(i) $(m_i, n_i) = 1$,

(ii) $(m_i, n_j) = 0$ if $i + j$,

(iii) if $(m_i, n_j) <> 0$, then it is homogeneous of degree $\deg(m_i) + \deg(n_j) - \deg(n_1)$.

Denote by K the matrix of this pairing with respect to the bases B_1 and B_u. Obviously the map E induces a symmetric bilinear form on M_1 defined by (,E) with a symmetric matrix G with respect to the basis B_1.

Notice that $(m_i, Em_j) = 0$ if $m_i m_j \in B_1$! Now

(iv) $G = K{*}C(\underline{t})$.

This implies that $C(\underline{t}){*}K^{-1}$ is symmetric. The basis change induced by K^{-1} on M_1 has the required properties (because of (i) to (iv)).

Remark: A basis change in M_1 corresponds to a choice of other generators of the kernel of the Kodaira-Spencer map.

Corollary 2.5.: There is a basis B_u' of M_u and an automorphism w of H with the following properties:

(1) w is homogeneous,

(2) $n_i' = n_i + h_i$, h_i homogeneous of degree n_i in the submodule generated by $n_1, \ldots, n_{i-1}$,

(3) the matrix of E with respect to B_1 and B_u' is $CL(w(\underline{t}))$.

Proof: With the notations of the proposition the basis change on M_u induced by K has the required property (2) and the matrix of E with respect to B_1 and B'_u is G. Now

$(m_i, Em_j) = 0$ if $m_i m_j$ is not in B_1. On the other hand

if $Em_i = \sum_{j=1}^{r} C_{ji} n_j$ and $m_i m_j = m_k$, $k=k(i,j)$, then

$(m_i, Em_j) = c_{k1}$. But $c_{k1} = (\deg(n_k)-d)\,t_k$ + higher order terms and $(\deg(n_k)-d)\,t_k$ is the ij-th element in $CL(\underline{t})$.

Let us consider the example $f = x^5 + y^{11}$:

$B_1 = \{1, y, y^2, x, y^3, xy, y^4, xy^2, x^2\}$

$B_u = \{x^3y^9, x^3y^8, x^3y^7, x^2y^9, x^3y^6, x^2y^8, x^3y^5, x^2y^7, xy^9\}$

$H = \mathbb{C}[t_1, \ldots, t_9]$, $r = 9$, $\mu = 40$

If we choose a basis B'_1 given by the proposition we get the following symmetric matrix $C' = (C'_{ji})$ and generators of the kernel of the Kodaira-Spencer map $\sum C'_{ji}\, \partial/\partial t_j$

$23t_1$	$18t_2$	$13t_3$	$12t_4$	$8t_5$	$7t_6$	$3t_7$	$2t_8$	t_9
$18t_2$	$13t_3+E$	$8t_5+D$	$7t_6+C$	$3t_7+B$	$2t_8+A$	0	0	0
$13t_3$	$8t_5+D$	$3t_7+F$	$2t_8$	0	0	0	0	0
$12t_4$	$7t_6+C$	$2t_8$	t_9	0	0	0	0	0
$8t_5$	$3t_7+B$	0	0	0	0	0	0	0
$7t_6$	$2t_8+A$	0	0	0	0	0	0	0
$3t_7$	0	0	0	0	0	0	0	0
$2t_8$	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	0	0

$A = -9t_9^2/11 \quad B = -7t_9t_8/11 \quad C = 3t_9t_7^3/11$

$H = -8t_9t_6/11 + 2t_9^2t_7^2/11$

$E = -117t_9t_4/11 + 16t_8t_7t_5/11 - 7t_9t_8t_7t_6/11 + t_9^2t_8t_7^2/121$

$F = t_9t_8$

We are now intrested in the flattening stratification of $H[x,y]/(F, \Delta F)$ as an H -module. This stratification coincides obviously with the stratification given by the rank of $C(\underline{t})$: For $\underline{t} \in \underline{H}$ denote by $\mu(F(\underline{t}))$ resp. $\tau(F(\underline{t}))$ the Milnor-number resp. the Tjurina-number of the singularity in the fibre $\underline{X}_{\underline{t}}$ of the family.

Remark 2.6.: $\mu = \mu(F(\underline{t})) = \tau(F(\underline{t})) + rk(C(\underline{t}))$, $\underline{t} \in \underline{H}$.

Let us denote by $\{\underline{S}_\tau\}$ the flattening stratification of $\underline{H}$, i.e. the underlying set is the set of all $\underline{t} \in \underline{H}$ with fixed Tjurina-number τ.

Remark: In general $\underline{S}_\tau$ is not the τ-constant stratum of the whole versal deformation of f:

The family $x^5 + y^{11} + tx^2y^7 + 2t'x^4y^2 + t'^2x^3y^4$ is for small $t \neq 0$ and arbitrary t' τ-constant ($\tau = 34$) but not μ-constant ($\mu(t'=0) = 40$, $\mu(t' \neq 0) = 39$).

In our example we get the following stratification:

$\underline{S}_{34}$: $4t_8^2 - 3t_9t_7 - t_9^2t_8 =: g \neq 0$

$\underline{S}_{35}$: $g = 0$ and $2t_8+A \neq 0$ or $3t_7+B \neq 0$ or
$4t_8(7t_6+C)(8t_5+D) - t_9(8t_5+D)^2 - (3t_7-t_9t_2)(7t_6+C)^2 =: h \neq 0$

$\underline{S}_{36}$: $2t_8+A = 3t_7+B = h = 0$ and
$(7t_6+C)^2 - t_9(13t_3+E) \neq 0$ or
$11(7t_6+C)(8t_5+D) - 9t_9^2(13t_3+E) \neq 0$ or
$(8t_5+D)(9t_9(7t_6+C) - 11(8t_5+D)) \neq 0$ or
$(7t_6+C)(9t_9(7t_6+C) - 11(8t_5+D)) \neq 0$

$\underline{S}_{37}$: the polynomials defining $\underline{S}_{36}$ equal to zero and
$t_9 \neq 0$ or $7t_6+C \neq 0$ or $8t_5+D \neq 0$ or

$13t_3+E <> 0$

$\underline{S}_{38}$: $t_9=\ldots=t_5=t_3=0$ and $t_4<>0$ or $t_2<>0$

$\underline{S}_{39}$: $t_9=\ldots=t_2=0$ and $t_1<>0$

$\underline{S}_{40}$: $t_9=\ldots=t_1=0$

Notice that $\underline{S}_{34}$ and $\underline{S}_{35}$ are smooth but $\underline{S}_{36}$ is not reduced.

Especially $\underline{S}_{\tau}$ is not empty for $\tau_{min} <= \tau <= \mu$. We will prove that this is always true in our situation.

Remark: $\underline{S}_{\tau}$ may be empty in the surface case:

$f=x^3+y^{10}+z^{19}$ $\mu = 324$, $\tau_{min}=246$, $\underline{S}_{247} = \emptyset$,

$\underline{S}_{248}$ is an open set in a hypersurface in $\underline{H}$.

Theorem: 2.7.: $\tau(F(\underline{t}))$, $\underline{t} \in \underline{H}$ takes every possible value between τ_{min} and $\mu=\mu(f)$.

Proof: Because of Corollary 2.5. it is enough to show that for the linear matrix every rank between 0 and the maximal rank is possible. Any minor of the linear matrix $CL(\underline{t})$ has the form

$M(\underline{t}) = (d_{a(i,j)} t_{a(i,j)})$ with $a(i,j) \in \{0,\ldots,r\}$,

$i,j=1,\ldots,m$, $d_{a(i,j)} = \deg(n_{a(i,j)})-d$ if $a(i,j)<>0$ and $d_0=0$. The $a(i,j)$ satisfy the following properties (cf. 2.3):

$a(i,j)<>0$ implies $0<>a(i,j-1)<a(i,j)$ and $0<>a(i-1,j)<a(i,j)$.

Lemma: 2.8.: With the notations above, $\det(M(\underline{t})) <> 0$ iff $a(m-i+1,i) <> 0$ for all $i=1,\ldots,m$.

Proof: If $a(m-i+1,i) = 0$ for some i, then, because of the properties of the $a(i,j)$, $\det(M(\underline{t}))$ vanishes. Assume that $a(m-i+1,i)$ is not zero for all i. Use induction on the number of non vanishing $a(i,j)$. Let $s = \max\{a(i,j) , a(i,j)<>0\}$ and assume that $\det(M(\underline{t},t_s=0)) = 0$. By induction hypothesis $M(\underline{t},t_s=0)$ has an antidiagonal element $a'(m-i+1,i)=0$. This implies that $\det(M(\underline{t}))=d_s\, t_s \det(A)\det(B)$:

$$\left|\begin{array}{cccc} & * & A & \\ & * & & \\ & * & & \\ *\ldots\ldots* & d_s t_s & 0 \ldots & 0 \\ & 0 & 0 \ldots & 0 \\ B & \vdots & \vdots & \vdots \\ & 0 & 0 & 0 \end{array}\right| \quad m-i+1$$

$$i$$

By induction hypothesis $\det(A)<>0$ and $\det(B)<>0$.
As a corollary of this lemma we get the following result.

Remark 2.9.: Denote any string of elements of the matrix $CL(t)$ parallel to the antidiagonal also by a antidiagonal. Then the rank of $CL(\underline{t})$ is the length of the maximal antidiagonal containing no zeros, i.e. with the notations of 2.3.

$$rk(CL(\underline{t}) = \min j_i + i - 1 .$$

Now we prove 2.7. using induction on τ Consider the maximal antidiagonal of the linear matrix containing no zeros. Among the $k(i,j)$ for which $t_{k(i,j)}$ occur on the corresponding antidiagonal of $CL(\underline{t})$, let $k(i_l, j_l)=:k$ be maximal, $l=1,\ldots,p$.

On the subset defined by $t_k=\ldots=t_r=0$ the rank has decreased by 1. Now we apply the induction hypothesis to prove 2.7..

In the next chapter we are looking for moduli-spaces of germs of plane curve singularities having the same topological type as (X_0,o), i.e. we look for the quotient of $\underline{H}$ under the action of the kernel of the Kodaira-Spencer map V. Such a quotient can only exist on the strata of the flattening stratification $\{\underline{S}_{\tau}\}$ of $\underline{H}$. The qoutient exists always in the analytic category ([4]). Let us denote the dimension of this moduli-space $\underline{S}_{\tau}/V$ by $m(f,\tau)$. Then $m(f,\tau)=\dim \underline{S}_{\tau} - rk(C(\underline{t}|\underline{S}_{\tau}))$.

Denote by $m(f) = \max m(f,\tau)$ the modality of f with respect to the μ-constant stratum, under the action of the contact group.

Theorem 2.10.: (1) $m(f) = \dim \underline{H} + \tau_{min} - \mu$.

(2) $\tau <= \tau'$ implies $m(f,\tau) >= m(f,\tau')$.

Proof: We have to prove (2). **Analysing the proof of** 2.7. **it is** enough to prove the following lemma for any linear matrix similar to $CL(\underline{t})$:

Lemma 2.11.: For $i,j = 1,\ldots, m$ let $a(i,j) \in \{0,\ldots,r\}$ satisfying the property: $a(m-i+1,i) <> 0$ for $i = 1,\ldots,m$ and $a(i,j) <> 0$ implies $0 <> a(i-1,j) < a(i,j)$ and $0 <> a(i,j-1) < a(i,j)$.

Let $d_{a(i,j)} \in \mathbb{C}$ such that $d_0=0$ and $d_{a(i,j)} <> 0$ if $a(i,j) <> 0$. Let I_p be the ideal generated by the $(p+1)$-minors of the matrix $M(\underline{t}) := (d_{a(i,j)}\, t_{a(i,j)})$. Then

$$\dim \mathbb{C}[t_1,\ldots,t_r]/I_p \;<=\; r-m+p .$$

Proof: Use induction on m and on the number of different $t_{a(i,j)}$'s contained in the matrix. If $p+1=m$, the assumption is true because $\det(M(\underline{t})) <> 0$ (lemma 2.8.).

Let $s = \max\{a(i,j),\ a(i,j) <> 0\}$ and let U be a component of the zero set of I_p.

Case 1: U is contained in the hypersurface defined by $t_s=0$.

Consider the matrix $M'(\underline{t})$ obtained from $M(\underline{t})$ by deleting the last row and the last column. By definition of s and the properties of the $a(i,j)$ the matrix $M'(\underline{t}, t_s=0)$ satisfies the properties required in the lemma and we may apply the induction hypothesis.

Case 2: U is not contained in the hypersurface defined by t_s. Suppose t_s occurs l times in the matrix $M(\underline{t})$. Then, on the open set defined by $t_s <> 0$ the rank of $M(\underline{t})$ is at least l. In particular this implies $l <= p$. Consider the matrix $M'(\underline{t})$ obtained by deleting the rows and columns in which t_s occurs. For $t_s <> 0$ it is easy to see that $rk(M(\underline{t})) = rk(M'(\underline{t}))+l$, therefore $\{\underline{t},\ rk(M(\underline{t})) <= p\} = \{\underline{t},\ rk(M'(\underline{t})) <= p-l\}$. Now $M'(\underline{t})$ satisfies the properties required in the lemma and does not contain t_s. We can apply the induction hypotheses, and the lemma is proved.

Remark: Theorem 2.10. **fails also in the case** of surface singularities. Consider the same example as before:

$f = x^3+y^{10}+z^{19}$ $m(f,246) = 88$ and $m(f,248) = 89$.

We will describe the open set S_{min} **more precisely in terms of the matrix C. A decreacing filtration** $F^* = F^pM_u$ **on** M_u **is introduced, induced by a filtration of the monomial basis B compatible with the dual filtration on** M_1.

(The dual filtration F^* **on** M_1 **is defined by:**

$$m_i \in F^pM_1 \quad \text{iff} \quad n_i \in F^{l-p}M_u\ )$$

Let us denote the matrix corresponding to $gr_pE : gr_pM_1 \dashrightarrow gr_pM_u$ **by** C^p **and the radical of the ideal generated by the maximal minors of** C^p **by** I_p.

Lemma 2.12.: **There is a filtration** F^* **of** M_u **satisfying the following properties:**

(1) $F^0M_u = M_u \supseteq F^iM_u \supseteq F^{i+1}M_u \supseteq F^lM_u = 0$

(2) $n_i \in F^pM_u$ implies $n_j \in F^pM_u$ if $j<i$

(3) $rk(gr_pM_1) <= rg(gr_pM_u)$ if $2p<=l+1$ (rk as H -modul)

(4) **The elements of** C^p **are invariant with respect to the action of** V_+

(5) $I^p = I^{l-p}$ for all p

(6) $I^p \subseteq I^{p-1}$ or $I^p \subseteq I^{p-2}$ if $2p <= l+1$

(7) $m_i \in F^p$ implies $m_im_j \in F^{p+1}$ if $m_im_j \in B_l$

(8) $m_i \in F^p$ implies $m_i \cdot n_j \in F^p$

(9) S_{min} is the open set defined by $I_{l'}$ or $I_{l'} \cap I_{l'-1}$, $l' := [(l+1)/2]$

Consider the example x^5+y^{11}. In this case F^* is the (x,y)-adic filtration:

gr_0M_u is generated by $xy^9,\ x^2y^7,\ x^3y^5$

gr_1M_u is generated by $x^2y^8,\ x^3y^6$

gr_2M_u is generated by $x^2y^9,\ x^3y^7$

$gr_3 M_u$ is generated by x^3y^8

$gr_4 M_u$ is generated by x^3y^9

For the graded pieces C^p of the matrix we get:

$C^0 = (t_9,\ 2t_8,\ 3t_7)$

$C^1 = (A', B'),\ A' = 2t_8 + A,\ B' := 3t_7 + B$

$$C^2 = \begin{pmatrix} A' & B' \\ t_9 & 2t_8 \end{pmatrix}$$

$$C^3 = \begin{pmatrix} B' - 9/11 t_9 A' \\ A' \end{pmatrix}$$

$$C^4 = \begin{pmatrix} B' - 9/11 t_9 A' \\ A' \\ t_9 \end{pmatrix}$$

$I_0 = I_4 = (t_7, t_8, t_9)$

$I_1 = I_3 = (A', B')\quad I_2 = (t_8A' - t_9B')$

$\tau_{min} = 34$ and S_{34} defined by $t_8A' - t_9B' \neq 0$

Construction of a suitable filtration of the monomial basis:

Lemma 2.13.: There is a map dh: B -----> **N** having the following properties:

(1) dh is injectiv

(2) If $e_1, e_2 \in M_1$ (resp. M_u) and $\deg(e_1) < \deg(e_2)$, then $dh(e_1) < dh(e_2)$

(3) For $e_1 \in B_1$ and $e_2 \in B_u$ we have:
if for a suitable $e_3 \in B_u$ with
$dh(e_3) = dh(e_1) + dh(e_2)$, then we have
$e_3 = e_1 + e_2$.

(4) $dh(e) + dh(e^0) = dh(n_1) =: d_*$

(5) Let $k_* := dh((0,1))$, then
$0 < \#B_1 \cap dh^{-1}(\{k+1, \dots, k+k_*\}) < a-1$
if $k + k_* <= d_*$

(6) (5) also holds for B_u.

Proof:

Define
$$dh(e) := \begin{cases} \deg(e) & \text{if } \deg(e) < ab \\ \deg(e) - ab & \text{if } \deg(e) >= ab \end{cases}$$

(1) and (2) are clear by definition.

(3) Suppose $e_1 \in B_1$; $e_2, e_3 \in B_u$ and $dh(e_3) = dh(e_1) + dh(e_2)$, then we have $dh(e_3) = dh(e_1 + e_2)$ and $\deg e_3 = \deg(e_1 + e_2)$ and this implies $e_3 = e_1 + e_2$. (4) is a consequence of (3).

(5) For a fixed number l we regard the following sequence of monomials $(1,0), (1,1), \dots, (1,t)$ from B, then the values of dh on the sequence increase by $k_* = dh(0,1) = a$ as long as $\deg(1,t) < d$. Suppose $\deg((1,t+1)) > d$, then $dh(1,t+1)$ is smaller than k_*. Therefore exactly one monomial $(0,t_0) \in B_1$, exactly one monomial $(a-2, t_{a-2}) \in B_u$ and at most one monomial $(1, t_1)$, $0<1<a-2$, belong to I_k,

$$I_k := dh^{-1}(\{k+1, \dots, k+k_*\}).$$

Lemma 2.14.: If $k < (d_* - k_*)/2$, then

$$\#B_1 \cap I_k <= \#B_u \cap I_k.$$

Proof: Suppose $B_1 \cap I_k$ contains l monomials, which necessarily are of the form $(0,t_0)$, $(1,t_1), \dots, (l-1, t_{l-1})$. Then $B_u \cap I_k$ contains either a-l-1 monomials $(1,t_1), \dots (a-2, t_{a-2})$ or a-l-2 monomials $(l+1, t_{l+1}), \dots, (a-2, t_{a-2})$. The second case occurs iff $k < dh(l, b-1) <= k + k_*$. Hence the statement of the lemma is equivalent to the inequality:

$$dh([a/2], b-1) > (d_* + k_*)/2$$

which is easy to check.

Let $c(e) := \#\{h \in B_u,\ dh(h) <= dh(e)\} - \#\{h \in B_1,\ dh(h) <= dh(e)\}$,

then $c(e) = c(e^0) + 1$ if $e \in B_u$. Let $c = \max c(e)$, then a maximal anti-diagonal of $C(\underline{t})$ containing no zeros has length $r-c$, hence rank $C(\underline{t}) = r - c$.

Choose $\underline{e}_* \in B_u$ with $dh(\underline{e}_*) < d_*/2$, such that $c = c(\underline{e}_*)$ and $dh(\underline{e}_*)$ maximal. Let $\underline{e}_0 \in B_1$, such that $dh(\underline{e}_0) > dh(\underline{e}_*)$ and $dh(\underline{e}_0)$ minimal.

Lemma 2.15.: $dh(\underline{e}_0^0) - dh(\underline{e}_0) < k_*$

Proof: Let $I := dh^{-1}(\{dh(\underline{e}_0),\dots,dh(\underline{e}_0)+r_*-1\})$ and suppose $dh(\underline{e}_0) <= (d_* - k_*)/2$, then, by 2.14. we have

$$\# B_1 \cap I <= \# B_u \cap I$$

Let $\underline{h} \in B_u \cap I$ maximal, then $c(\underline{h}) = c$, hence $dh(\underline{h}) > d_*/2$, i.e. $dh(\underline{h}^0) < d_*/2$ and $c(\underline{h}^0) = c-1$ for the dual $\underline{h}^0$.

Let $\underline{h}_0 \in B_1$ maximal with $dh(\underline{h}_0) < dh(\underline{h})$ then $c(\underline{h}_0^0) = c$, hence $dh(\underline{h}_0^0) <= dh(\underline{e}_*)$ and $dh(\underline{h}^0) <= dh(\underline{e}_0)$. We get

$dh(\underline{e}_0)+k_* > dh(\underline{h}) >= dh(\underline{e}_0^0)$ and

$dh(\underline{e}_0^0) - dh\ \underline{e}_0) < k_*$ q.e.d.

We are now ready to construct the filtration. We start with a filtration on B. Let

$S^i := dh^{-1}(\{dh(\underline{e}_0)-ik_*,\dots,dh(\underline{e}_0^0)-ik_*\})$ and

$R^i = dh^{-1}(\{dh(\underline{e}_0^0)-(i+1)k_*-1,\dots$

$\dots,dh(\underline{e}_0)-ik_*-1\})$.

Because of $(S^0)^o = S^0$ and $(R^0)^o = R^1$ we get

$$(S^i)^o = S^{-i} \quad \text{and} \quad (R^i)^o = R^{1-i}$$

Notice that it is possible that

$\underline{e}_*^0 = \underline{e}_0$, i.e. $S^i = \emptyset$ for all i,

or $\underline{e}_*^0 = \underline{e}_0 + (0,1)$, i.e. $R^i = \emptyset$ for all i.

In these cases we will get the (x,y)-adic filtration on M_u.
Denote by F^*B the filtration defined by S^i and R^i as follows: $F^0B := B$, suppose F^iB is defined and the minimal element $\underline{h} \in F^iB$ belongs to S^j (resp. to R^j), $j < 0$, then

$$F^{i+1} := \begin{cases} F^i - S^j, & \text{if } R^{j+1} \cap B_u \neq \emptyset \text{ and } R^{j+1} \cap B_1 \neq \emptyset \\ F^i - S^j - R^{j+1}, & \text{else} \end{cases}$$

resp.

$$F^{i+1} := \begin{cases} F^i - R^j, & \text{if } S^j \cap B_u \neq \emptyset \text{ and } S^j \cap B_1 \neq \emptyset \\ F^i - R^j - S^j, & \text{else} \end{cases}$$

To obtain symmetry we define for $j = 0$

$F^{i+1} := F^i - S^0$ (resp. $F^i - R^0$)

and if $j > 0$

$$F^{i+1} := \begin{cases} F^i - S^j, & \text{if } S^j \cap B_u \neq \emptyset \text{ and } S^j \cap B_1 = \emptyset \\ F^i - S^j - R^{j+1}, & \text{else} \end{cases}$$

for $\underline{h} \in R^j$ it is defined in a similar way.

Let l be minimal such that $F^l = \emptyset$, then, because of the duality $B_u^0 = B_1$, we have $(F^iB)^o = B - F^{l-i}$.
F^* induces filtrations on B_u and B_1 and by duality we have $(F^iB_u)^o = B_1 - F^{l-1}B_1$.

Let us denote the induced filtration on M_1 and M_u also by F^*. This filtration has the properties required in 2.12.

Proof of Lemma 2.12.:

(1) and (2) are obvious.

(3) $gr_p M_1$ is generated by
$\{x^e y^{e'}, dh(e,e') \in gr_p B_1\}$ (similarly for $gr_p M_u$)
But $gr_p B_u$ is $S^j \cap B_u$ or $R^j \cap B_u$ or $(S^j \cup R^j) \cap B_u$ or $(S^j \cup R^{j+1}) \cap B_u$ for some j, similarly $gr_p B_1$. Now, by definition $\#S^0 \cap B_u = \#S^0 \cap B_1$ and

$\#R^0 \cap B_u >= \#R^0 \cap B_1$.

Using 2.13. we have $\#S^j \cap B_u >= S^j \cap B_1$,
$\#R^j \cap B_u >= \#R^j \cap B_1$ if $j < 0$.

(4) holds because of the fact that

$$dh(\underline{e}_1) - dh(\underline{e}_2) < k_*$$

for any two elements $\underline{e}_1$ and $\underline{e}_2$ of gr_pB and that t_q is invariant under the action of V_+ if $q>j_2$ (lemma 2.4): if an element of C^P depends on t_q with $dh(n_q)>k_*$ (we may assume the linear part to depend on t_q) and this element is in the i-th column and the j-th row of $C(\underline{t})$, then

$$m_j n_q = n_i \quad \text{and } m_j, n_i \in gr_pB$$

but than $\quad dh(n_q) = dh(n_i) - dh(m_j) < k_*$!

(5) is a consequence of the duality and the fact that the change of the matrix $C(\underline{t})$ to the symmetric matrix does not change I_{1-p}.

(6) is a consequence of 2.13.:

Suppose $gr_pB = S^j$, $j<=0$, and $gr_{p-2}B = S^{j-1}$, then

(i) $gr_{p-2}B_u = \{\underline{e} - (0,1),\ \underline{e} \in gr_pB_u\} \cup L$,
L is empty or contains just one element.

(ii) $gr_{p-2}B_1 = \{\underline{e} - (0,1),\ \underline{e} \in gr_pB_1\} \setminus T$,
T is empty or contains just one element.

Furthermore $\quad \# gr_pB_u >= \# gr_pB_1 =: d_p$.

I_{p-2} is the radical of the ideal generated by the d_{p-2}-minors of the matrix C^{p-2}. Let $m_{i(1)},\dots,m_{i(d-2)}$ generate $gr_{p-2}M_1$. Suppose I_{p-2} vanishes at a point $\underline{t}$, then the leading forms of $m_{i(1)}E,\dots,m_{i(d-2)}E$ with respect to the graduation, i.e. in $gr_{p-2}M_u$, are dependent.

Now, because of (ii) the leading forms of

$$ym_{i(1)}E,\dots,ym_{i(d-2)}$$

define rows of the matrix C^P. By (i) they depend on $\underline{t}$, too. This implies that the corresponding d_{p-2}-minors of C^P vanish at $\underline{t}$. But $d_p >= d_{p-2}$ implies that I_p vanishes at $\underline{t}$. All the other cases are similar.

(7) and (8) are obvious by definition of F^*.

(9) By the choice of $\underline{e}_*$, $c(\underline{e}_*) = c$, the matrix $C(\underline{t})$ has maximal rank r-c at a general point. The rank decreases if the rank of a graded piece decreases. Because of (5) and (6),

$S_{\underline{t}min}$ is defined by $I_{1'} \wedge I_{1'-1}$ if $S^0 \neq \emptyset$ and $R^0 \wedge B_1 \neq \emptyset$ or by $I_{1'}$, if $S^0 = \emptyset$ or $R^0 \wedge B_1 = \emptyset$ $(1' := [(1+1)/2])$.

From the last results one can obtain formulas for the maximal rank of the matrix $C(\underline{t})$. We represent

$$b/a = [r_1,\dots,r_k] = r_1 + \cfrac{1}{r_2 + \cfrac{1}{\cdots}}$$

as a continued fraction then define l_i and t_i inductively:

$l_k = 0$, $t_k = 1$

$l_{i-1} = l_i + t_i r_i$ and $t_{i-1}=0$ if $t_i=1$ and l_i even,

$t_i=1$ else.

Corollary: (cf. [2])

$$\text{rank } C(\underline{t}) = (a-2)(b-2)/4 - (l_0-2)/4 + t_1(r_1+t_2-2)/2$$

If for instance $b = r*a\pm1$, we get

$$\text{rank } C(\underline{t}) = \begin{cases} (a-2)(b-3)/4 & a \text{ even} \\ (a-1)(b-r-3)/4 & a \text{ odd} \end{cases}$$

3. Moduli of irreducible plane curve singularities with semi-group $\langle a,b\rangle$

We will construct the moduli space, of irreducible plane curve singularities with the semi-group $\Gamma = \langle a,b\rangle$ and minimal Tjurina-number

There was already an approach by Washburn (cf. [9]) but his construction is wrong in general, also his dimension formula. He uses as filtration $F^*$the (x,y)-adic filtration on M_u and M_1. But the corresponding graduation was not compatible with the multiplication $E: M_1 \longrightarrow M_u$. This is true only in very special cases (cf.2.13.), not true for $a=5$, $b=12$.

Theorem 3.1.: Let $\Gamma = \langle a,b \rangle$, $(a,b)=1$, be a semi-group. The fine moduli space $\pi: \mathfrak{X}^{\Gamma}_{\tau_{min}} \dashrightarrow T^{\Gamma}_{\tau_{min}}$ of all plane curve singularities with the semi-group Γ and minimal Tjurina-number τ_{min} exsits.

(1) $T^{\Gamma}_{\tau_{min}}$ is a quasismooth scheme, i.e. locally an open subset in a wighted projective space of dimension

$(a-4)(b-4)/4 + l_0/4 + (2-t_1)(r_1-2)/2 - t_1 t_2/2$

(2) $\mathfrak{X}^{\Gamma}_{\tau_{min}}$ is an algebraic space and there is an affine covering $\{U_i\}$ of $T^{\Gamma}_{\tau_{min}}$ such that $\pi^{-1}(U_i)$ are affine schemes.

Proof: Suppose $a < b$. For short, let $\tau := \tau_{min}$.
Let (X_0,o) be a germ of a plane curve with singularity at o, having as semigroup. Consider $\underline{X} \dashrightarrow \underline{H}$ to be the versal μ- constant deformation of the singularity defined by $x^a + y^b$.

There is a $t \in \underline{H}$ such that $(\underline{X}_0,o) \cong (\underline{X}_t,o)$ (cf. [1]).

Lemma 3.2.:

If for $t^1, t^2 \in \underline{H}$ $(\underline{X}_{t^1},o) \cong (\underline{X}_{t^2},o)$, then t^1 and t^2 are in an analytically trivial subfamily of $\underline{X} \dashrightarrow \underline{H}$.

Proof: The $\mathbb{C}^*$-action induces a canonical filtration on the automorphism group $\mathbb{E}$ of $\mathbb{C}[[x,y]]$:

$\mathbb{E}_l = \{ \varphi \in \mathbb{E}, \deg(\varphi(x)-x) >= l+a, \deg(\varphi(y)-y) >= l+b \}$

$\deg(\varphi) := l$ iff $\varphi \in \mathbb{E}_l - \mathbb{E}_{l+1}$.

$(\underline{X}_{t^1},o) \cong (\underline{X}_{t^2},o)$, i.e. there is an $\varphi \in \mathbb{E}$ and a unit $u \in \mathbb{C}[[x,y]]$ such that

$$F(x,y,\underline{t}^1) = u(x,y)F(\varphi(x),\varphi(y),\underline{t}^2).$$

We will prove that $\deg \varphi > 0$.

If this is true $\underline{t}^1$, $\underline{t}^2$ are in an analytically trivial subfamily of $\underline{X} \dashrightarrow \underline{H}$ induced by the $\mathbb{C}^*$-action.

Consider the map induced by φ and the corresponding map $\bar{\varphi}$ of the normalizations.

$$\begin{array}{ccc} \varphi: \mathbb{C}[[x,y]]/F(x,y,\underline{t}^1) & \dashrightarrow & \mathbb{C}[[x,y]]/F(x,y,\underline{t}^2) \\ \mathbb{C}[[t^a + \text{higher order}, t^{b}+\ldots]] & \dashrightarrow & \mathbb{C}[[t^{a}+\ldots, t^{b}+\ldots]] \\ \bar{\varphi}: \mathbb{C}[[t]] & \dashrightarrow & \mathbb{C}[[t]] \end{array}$$

$\bar{\varphi}(t) = t*h(t)$, $h(t)$ a unit in $\mathbb{C}[[t]]$.

Then it is clear that $\deg \varphi > 0$.

Using the lemma we get $T^{\Gamma}_{\tau} = S_{\tau}/V$, S_{τ} the open stratum in the flattening stratification of $\underline{H}$ with respect to the kernel of the Kodaira-Spencer map V. By general results

$$\mathfrak{X}^{\Gamma}_{\tau} \dashrightarrow T^{\Gamma}_{\tau} = S_{\tau}/V$$

exists in the category of algebraic spaces (cf. [4], [6]).

We will prove that S_{τ}/V is locally an open subset in a weighted projective space.

V is a graded Lie-algebra generated as an H-module by the elements d_i, $\deg d_i = \deg m_i$ (cf. proposition 2.1.).

It is enough to study $S_{\tau}/V_0 = (S_{\tau}/V_+)/\mathbb{C}^*$ ($\exp V_+$ is a normal subgroup in $\exp V_0$ and $\exp V_0 / \exp V_+ = \mathbb{C}^*$).

Lemma 3.3.: Let R be a commutative algebra over a field k. $d_1,\ldots,d_q \in \mathrm{Der}_k A$ with the following properties:

(i) $[d_i,d_j] = 0$ for all i,j

(ii) d_i nilpotent, i.e. for all $a \in R$ there is an $n(a)$ such that $d_i^{n(a)}(a) = 0$

(iii) There are $z_1,\ldots,z_q \in R$ such that

- $d_i z_j$ is invariant with respect to the action of the Lie-algebra $\sum d_i k =: L$
- $\det(d_i z_j)$ is invertible in R

Then $R^L[z_1,\ldots,z_q] = R$.

The proof of this lemma is not difficult and we will omit it.

Now we study the action of V_+ on S_{τ}.

Using (4), (6) and (9) of 2.12. we can cover S_{τ} by invariant affine open sets defined by the product of suitable minors of $\mathbb{C}^p$, $p <= l' = [(l+1)/2]$.

Let $U = \mathrm{Spec}\, \mathbb{C}[\underline{t}]_h$, $h = h_1 \dots h_l$, be one of these open sets, h_i minors of C^i.

Let $1 = i_1, \dots, i_{t(1)}, \dots, i_{t(1)+1}, \dots, i_{t(2)}, \dots, i_{t(l)}$ define the columns and $j_1 \dots j_{s(1)} \dots j_{s(l)}$ the rows of $C(\underline{t})$ corresponding to these minors, $t(l) = s(l) = \mu - \tau - 1 = \mathrm{rk}\, C(\underline{t}) - 1$. Because of 2.12. (7) and (3) and 2.2 $[d_{j_{s(i)}}, d_k]$ is in the $\mathbb{C}[\underline{t}]_h$-module generated by $d_{j_{s(i+1)}}, \dots, d_{j_{s(l)}}$. Starting with $d_{j_{s(l-1)+1}}, \dots, d_{j_{s(l)}}$ we apply 3.3. l times and get $\mathbb{C}[\underline{t}]_h^{V_+}[t_{i_1}, \dots, t_{i_{t(l)}}] = \mathbb{C}[\underline{t}]_h$.

We may choose homogeneous invariant functions $g_{i_{t(l)+1}}, \dots, g_{i_q} \in \mathbb{C}[\underline{t}]$ generating $\mathbb{C}[\underline{t}]_h^{V_+}$ determined by $g_{i_k}/h = t_{i_k} \bmod (t_{i_1}, \dots, t_{i_{t(l)}})$

$U/V_+ = \mathrm{Spec}\, \mathbb{C}[g_{i_{t(l)+1}}, \dots g_{i_q}]_h$,

$x^a + y^b + \sum_{j > t(l)} g_{i_j}/h * n_{i_j}$ is the corresponding family.

U/V_0 is the open set defined by h in the corresponding weighted projective space.

Now it is clear that the quotients of the invariant affine open sets covering S_τ by V_0 glue to a quasismooth scheme T_τ^Γ. The corresponding families glue in the etal topology.

Let us consider our example x^5+y^{11}:

For shortness let $A' = 2t_8 + A$ and $B' = 3t_7 + B$.

$S_\tau = \mathrm{Spec}\, \mathbb{C}[\underline{t}]_{t_8 A' - t_9 B'}$. Let us consider U defined by $h := t_9 A'(t_8 A' - t_9 B')$. Then $i_v = v$, $v = 1, \dots, 4$, $i_5 = 6$.

$\mathbb{C}[\underline{t}]_h^{V_+} = \mathbb{C}[t_9, t_8, t_7, A' t_5 - B' t_6]_h$ with the corresponding family x^5+y^{11} $+ t_9 xy^9 + t_8 x^2y^7 + t_7 x^3y^5 + (t_5 - t_6 B'/A') x^3y^6$

Similarly we get the invariants on the other open sets. S_τ/V is the open set $D_+(t_8 A' - t_9 B')$ in $\mathbb{P}^3_{(1:2:3:10)} =:$ $\mathrm{Proj}\, \mathbb{C}[t_9, t_8, t_7, w]$. $T_\tau^\Gamma = S_\tau/V_0$ is covered by the open sets $U_1 = D_+(A') \cap T_\tau^\Gamma$ and $U_2 = D_+(B') \cap T_\tau^\Gamma$. On U_1 resp. U_2 we have the universal families

$x^5+y^{11}+t_9 xy^9+t_8 x^2y^7+t_7 x^3y^5+w/A' x^3y^6$

resp.

$x^5+y^{11}+t_9 xy^9+t_8 x^2y^7+t_7 x^3y^5+w/B' x^2y^8$

4. Moduli of reducible plane curve singularities of quasi-homogeneous type

In the case of reducible plane curves one gets similare results:

We can construct the moduli space of all plane curve singularities having the topological type of a quasi-homogeneous plane curve singularity, more precisely connected by a topological trivial family with a singularity defined by a non-degenerated quasi-homogeneous polynomial of degree d with respect to the weights w_1, w_2. The following three cases will occur:

(i) k branches with semigroup $\Gamma = \langle a_0, b_0 \rangle$; $(a_0, b_0) = 1$

$w_1 = d/ka_0$, $w_2 = d/kb_0$; $f = x^a + y^b$

(ii) k branches with semigroup Γ and one smooth branch:

$w_1 = d/ka_0$, $w_2 = (d-w_1)/kb_0$; $f = x^a + xy^b$

(iii) k branches with semigroup Γ and two smooth branches:

$w_1 = (d-w_2)/ka_0$, $w_2 = (d-w_1)/kb_0$;

$f = x^a y + xy^b$

Theorem

Fix the quasi-homogeneous type $(\underline{w}; d)$ and the Tjurina-number τ, then a coarse moduli space $T_\tau(\underline{w}; d)$, of all plane curve singularities with that qh topological type and Tjurina-number τ exists in the category of algebraic spaces. For $\tau = \tau_{min}$ the moduli space T_τ is a scheme (except may be in the homogeneous case, i.e. $w_1 = w_2 = d$, if d is even).

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BUCHSBAUM CURVES IN $\mathbf{P}^3$

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This paper is a somewhat expanded version of a talk that the author gave at the International Conference "Algebraic Geometry" at Humboldt-Universität zu Berlin. The author is very grateful to Humboldt-Universität and in particular to Professors W. Kleinert, H. Kurke, G. Pfister and M. Roczen for their generous hospitality.

In the first section of this paper we begin by briefly reviewing the basic results of liaison of curves in $\mathbf{P}^3$. We then give a definition of a Buchsbaum curve in terms of liaison which, while not the usual definition, is shown in [GMV] and [BSV] to be equivalent to it. From this point of view we mention some things that are known about special cases and then we recall a technique introduced in [M1] to attack the general case.

The second section contains two applications of this technique. First, we give a geometric consequence of the Buchsbaum property by proving that every Buchsbaum curve is connected except for the curve consisting of two skew lines. Second, we give an example of two curves, each a configuration of lines on a quadric plus two additional lines, which are extremely similar in an important way (their Hartshorne- Rao modules have the same dimension in each component) but we show that one is Buchsbaum and the other is not.

§1 Liaison and the Buchsbaum Property

Throughout this paper k shall denote an algebraically closed field, $S = k[X_0, \ldots, X_3]$ and $\mathbf{P}^3 = \mathbf{P}^3_k$. By a <u>curve</u> we mean a closed one-dimensional subscheme of $\mathbf{P}^3$ which is locally Cohen-Macaulay and equidimensional. Finally, if M is a graded S-module then $M^\vee = \mathrm{Hom}_k(M,k)$ is the dual module. We have in particular $M^\vee_n = M_{-n}$.

Two curves C and C' are said to be <u>directly linked</u> by a complete intersection X, denoted $C \sim_X C'$, if $I(X) : I(C) = I(C')$. This "usually" means (if C and C' have no common component) that $C \cup C' = X$. This notion generates an equivalence relation among curves in $\mathbf{P}^3$ called <u>liaison</u>, and we write $C \sim C'$ when C and C' are in the same <u>liaison</u> (equivalence) <u>class</u>. If C and C' are linked in an even number of steps then we say that they are <u>evenly linked</u>, and similarly for C and C' to be <u>oddly linked</u>.