THE MODULI OF IRREDUCIBLE CURVE SINGULARITIES WITH THE SEMIGROUP $\Gamma = \langle 5,11 \rangle$

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We compute the moduli space of all irreducible curve singularities with the semigroup $\Gamma = \langle 5, 11 \rangle$ and describe the behaviour of several invariants on the components of this moduli space.

1. INTRODUCTION

The moduli space $M_{a,b}$ of all germs of irreducible curve singularities with the same semigroup $\Gamma = \langle a, b \rangle$ is a disjoint union of analytic varieties $M_{a,b,t}$, $t = 0, \ldots, g$. The generic component $M_{a,b,g}$ of $M_{a,b}$ is an algebraic variety, locally an open subset in a weighted projective space (cf. [2]). It turns out that the other components are algebraic spaces but need not be algebraic varieties as $M_{5,11,5}$ in the case of $\Gamma = \langle 5,11 \rangle$.

$M_{a,b}$ is constructed in the following way.

Consider the family $p : X \to \mathbf{C}^r$, X the subset of $\mathbf{C}^r \times \mathbf{C}^2$ defined by the equation

$$F := x^a + y^b + \Sigma_B t_{ij} x^i y^j = 0,$$

$$B = \{(i, j),\ ib + ja > ab,\ i < a - 1,\ j < b - 1\}$$

p the projection.

This family has the following property (cf. [2]).

It is the versal deformation of the monomial curve $(X_0, 0)$ defined by $x^a + y^b = 0$ with constant semigroup Γ. Moreover, if $(Y, 0)$ is any germ of an irreducible plane curve singularity with the semigroup $\Gamma = \langle a, b \rangle$, then there is an t in $\mathbf{C}^r$ such that $(p^{-1}(t), 0) = (Y, 0)$. Using the relative Kodaira-Spencer map of this family one can compute the analytically trivial subfamilies.

Consider the relative Kodaira-Spencer map

$$\theta : \mathrm{Der}_{\mathbf{c}}(\mathbf{C}[t]) \to \mathbf{C}[t, x, y]/(F, \partial F/\partial x, \partial F/\partial y) =: T^1$$

$\theta(\delta) :=$ class (δF), $t = (t_{ij})$. The kernel K of the Kodaira-Spencer map is a sub Lie-algebra of the Lie-algebra of derivations of $\mathbf{C}^r$. Along the integral manifolds of K the family $X \to \mathbf{C}^r$ is trivial.

It turns out that $M_{a,b} = \mathbf{C}^r/K$ and $M_{a,b,i} = S_i/K$, S_i the set of all points in $\mathbf{C}^r$ such that the integral manifolds of K through t have dimension i. S_g is a dense open subset in $\mathbf{C}^r$ defined by the nonvanishing of one or two polynomials.

The stratification $\{S_i\}$ of $\mathbf{C}^r$ is the reduced flattening stratification of the relative T^1 of the family. The singularities corresponding to points of S_i have Tjurina number $\mu - i$, $\mu = (a-1)(b-1)$ the Milnor number. There is an algorithm to compute generators of K (cf. [2], [3]).

Let $\{a_i\}_{i=1,\ldots,\mu}$ be a free base of $\mathbf{C}[[t,x,y]]/(\partial F/\partial x, \partial F/\partial y)$ as a $\mathbf{C}[t]$ — module and let $a_i F \equiv \Sigma_B h_{i,kl}\, x^k y^l \mod (\partial F/\partial x, \partial F/\partial y)$ then $\delta_i = \Sigma_B h_{i,kl}\, \partial/\partial t_{kl}$ generate K as $\mathbf{C}[t]$-module. Because of the $\mathbf{C}^*$ — action on the monomial curve $(X_0, 0)$, K is a graded Lie-algebra.

$$\deg \partial/\partial t_{kl} = -\deg t_{kl} = kb + la - ab > 0.$$

If we choose the a_i to be homogeneous then $h_{i,kl}$ is a (weighted) homogeneous polynomial of degree $\deg a_i + \deg t_{kl}$ and δ_i is homogeneous of degree $\deg a_i \geqslant 0$.

Let K^+ (resp. K^0) be the Lie-algebra generated (as $\mathbf{C}$— vector space) by all homogeneous vector fields of K of degree $\geqslant 0$ (resp. > 0) then K^+ (resp. K^0) is a finite dimensional solvable (resp. nilpotent) Lie-algebra. K^+ has the same integral manifolds as K, because of $\delta_i \in K^+$ and $M_{a,b,i} = S_i/K^+ = S_i/\exp(K^+)$. $G := \exp(K^+)$ resp. $G^0 := \exp(K^0)$ is a solvable (resp. nilpotent) algebraic group acting regularly on the quasiaffine algebraic varieties $S_i \cdot S_i/G$ is an algebraic variety iff S_i can be covered by open G-invariant sets such that the G-invariant functions separate the orbits (cf. [1]). For the semigroup $\Gamma = \langle 5,11 \rangle$ we construct explicitely $M_{5,11,i}$ and the corresponding universal family, compute the singularities of $M_{5,11,i}$ and study the behaviour of several invariants as the irregularity, Steenbrink's invariants α, β, γ and the polar curves.

The singularities of the moduli space are related to the automorphism group of the germ of the singularity in the corresponding fibre.

2. THE REDUCED FLATTENING STRATIFICATION OF THE KODAIRA-SPENCER MAP

The versal deformation with constant semi-group of the monomial curve $(X_0, 0)$ defined by $f = x^5 + y^{11}$ is given by the following polynomial.

$$F := f + t_1\, xy^9 + t_2\, x^2y^7 + t_3\, x^3y^5 + t_4\, x^2y^8 +$$

$$+ t_5\, x^3y^6 + t_6\, x^2y^9 + t_7\, x^3y^7 + t_8\, x^3y^8 + t_9\, x^3y^9.$$

We choose a suitable free base $\{a_1, \ldots, a_{40}\}$ of the $\mathbf{C}[t]$-module $\mathbf{C}[x,y,t]/(\partial F/\partial x, \partial F/\partial y)$ with the properties:

— a_i is quasihomogeneous in x, y and t;

— for $t = 0$ we get the monomial base of $\mathbf{C}[x,y]/(x^4, y^{10})$ ordered by degree: $a_1 = 1$, $a_2(0) = y$, $a_3(0) = y^2$, $a_4(0) = x$, $a_5(0) = y^3$, $a_6(0) = xy$, $a_7(0) = y^4$, $a_8(0) = xy^2$, $a_9(0) = x^2$.

The matrix of coefficients $(h_{i,kl})$, $i \leqslant 9$, with respect to that base has the following form

$$\begin{pmatrix} t_1 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 12t_6 & 13t_7 & 18t_8 & 23t_9 \\ 0 & 0 & 0 & A & B & C & D & E & 18t_8 \\ 0 & 0 & 0 & 0 & 0 & 2t_2 & 3t_3 + t_1t_2 & D & 13t_7 \\ 0 & 0 & 0 & 0 & 0 & t_1 & 2t_2 & C & 12t_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & 8t_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 7t_4 \\ 0 & \ldots & & & & & & \ldots\ldots 0 & 3t_3 \\ 0 & \ldots & & & & & & \ldots\ldots 0 & 2t_2 \\ 0 & \ldots & & & & & & \ldots\ldots 0 & t_1 \end{pmatrix}$$

$A = 2t_2 - 9/11\, t_1^2$, $\qquad B = 3t_3 - 7/11\, t_1t_2$,

$C = 7t_4 + 3/11\, t_1t_3^2$, $\qquad D = 8t_5 - 8/11\, t_1t_4 + 2/11\, t_1^2t_3^2$,

$E = 13t_7 - 117/11\, t_1t_6 + 3/11\, t_1^2t_3t_5 + 55/11^2\, t_1t_2t_3t_4 + 7/(5 \cdot 11^3) t_1^3t_2^2t_3^2$

(notice that $a_iF = 0 \mod (\partial F/\partial x, \partial F/\partial y)$ if $i > 9$).

The reduced stratification $\{S_i\}$ or $\mathbf{C}^9$ given by the rank of that matrix is defined by the following equations and inequalities

$$S_6 : 4t_2^2 - 3t_1t_3 - t_1^2t_2 \neq 0$$

$$S_5 : 4t_2^2 - 3t_1t_3 - t_1^2t_2 = 0 \text{ and } A \neq 0 \text{ or } B \neq 0 \text{ or } I \neq 0,$$

$$I := (A + 1/11t_1^2)\, t_1((t_1D - 2t_2C)^2 - 2(A^2 + 1/11\, t_1^2A)\,(t_1t_7 - 2t_2t_6)).$$

$$S_4 : A = B = t_1(9t_1C - 11D) = 0 \text{ and } C^2 - t_1E \neq 0 \text{ or } D^2 - (9/11)^2\, t_1^3E \neq 0$$

$$S_3 : A = B = D - 9/11\, t_1C = C^2 - t_1E = 0 \text{ and } t_1 \neq 0 \text{ or } E \neq 0$$

$$S_2 : t_1 = t_2 = \ldots = t_5 = t_7 = 0 \text{ and } t_6 \neq 0 \text{ or } t_8 \neq 0$$

$$S_1 : t_1 = \ldots = t_8 = 0 \text{ and } t_9 \neq 0$$

$$S_0 : t_1 = \ldots = t_9 = 0.$$

3. CONSTRUCTION OF THE MODULI SPACES

(1) $M_{5,11,6}$:

On the affine G_0-invariant open subset S_6 of $\mathbf{C}^9$ the ring of G_0-invariants is obviously generated by t_1, t_2, t_3 and $I_0 := t_4B - t_5A$. The orbits are separated by the invariant functions.

On the invariant affine open subset defined by $A \neq 0$

$$I_0' := -(1/A)I_0 = t_5 - (B/A)t_4$$

is invariant and G_0-orbits look like

$$G_0(t_1, \dots, t_9) = G_0(t_1, \dots, t_4, I_0', t_6, \dots, t_9) =$$
$$= (t_1, t_2, t_3, t_4 + g_1, I_0', t_6 + g_2, \dots, t_9 + g_5)$$

(with respect to the coordinates $(t_1, \dots, t_4, I_0', t_6, \dots, t_9)$ on the open set $A \neq 0$; $g_1, \dots, g_5$ – arbitrary constants).

Hence the geometric quotient exists here and is given by Spec $\mathbf{C}[t_1, t_2, t_3, I_0']_{ZA}$, $Z := 4t_2^2 - 3t_1t_3 - t_1^2t_2 = 2t_2A - t_1B$. Dividing out by the $\mathbf{C}^*$-action we get an open subset $D(ZA) \subset \mathbf{P}^3(1,2,3,8)$ in the weighted projective space.

The universal family is given by

$$f + t_1xy^9 + t_2x^2y^7 + t_3x^3y^5 + I_0'x^3y^6.$$

Similarly on the invariant affine open subset defined by $B \neq 0$

$$I_0'' := (1/B)I_0 = t_4 - (A/B)\,t_5$$

is invariant and G_0-orbits are given here by

$$G_0(t_1, \dots, t_9) = G_0(t_1, t_2, t_3, I_0'', t_5 \dots, t_9) =$$
$$= (t_1, t_2, t_3, I_0'', t_5 + g_1, \dots, t_9 + g_5).$$

Hence the geometric quotient is given by Spec $\mathbf{C}[t_1, t_2, t_3, I_0'']_{ZB}$. Dividing out by the $\mathbf{C}^*$-action we get an open subset $D(ZB) \subset \mathbf{P}^3(1,2,3,7)$ and the universal family

$$f + t_1xy^9 + t_2x^2y^7 + t_3x^3y^5 + I_0''\,x^2y^8.$$

Both parts glue together to the moduli space $M_{5,11,6} := D(Z)$ in $\mathbf{P}^3(1,2,3,10)$ with coordinates $(t_1 : t_2 : t_3 : I_0)$. The singular locus is defined by $t_1 = t_3 = 0$, i.e. the points with coordinates $(0:1:0:i)$. The universal families glue together in etale topology.

(2) $M_{5,11,5}$:

The geometric quotient of S_5 by G_0 does not exist as algebraic variety. S_5 is defined by $Z = 2t_2A - t_1B = 0$ and covered by the three invariant affine open sets

$$U_1 : A \neq 0 \text{ resp. } U_2 : B \neq 0 \text{ resp. } U_3 : I \neq 0,$$

It is not difficult to see that the G_0-invariants on S_5 are generated by

$$t_1,\ t_2,\ t_3,\ I_0 \text{ and } I_1' \text{ (resp. } I_0''\text{) on } U_1 \text{ (resp. on } U_2).$$

$$I_1' := (1/A^2)\,[A^2t_7 - ABt_6 + (BC - AD)t_4 + ((1/2)B - (4/11)t_1A)t_4^2]$$

$$I_1'' := (1/B^3)\,[B^3t_6 - AB^2t_7 + (ABD - B^2C)t_5 + ((1/2)AB - (4/11)\,t_1A^2)t_5^2]$$

The orbits are separated by the invariant functions because the G_0-orbits look like

$$G_0(t_1, \dots, t_9) = G_0(t_1, \dots, t_4, I_0', t_6, I_1', t_8, t_9) =$$
$$= (t_1, t_2, t_3, t_4 + g_1, I_0', t_6 + g_2, I_1', t_8 + g_3, t_9 + g_4)$$

resp.

$$G_0(t_1, \dots, t_9) = G_0(t_1, t_2, t_3, I_0'', t_5, I_1'', t_7, t_8, t_9) =$$
$$= (t_1, t_2, t_3, I_0'', t_5 + g_1, I_1'', t_7 + g_2, \dots, t_9 + g_4).$$

Hence the geometric quotient exists over U_1(resp. U_2) and is given by $V(Z) \cap D(A) \subset \mathbf{P}^4(1,2,3,8,13)$ resp. $V(Z) \cap D(B) \subset \mathbf{P}^4(1,2,3,7,12)$.

Universal families are given by

$$f + t_1xy^9 + t_2x^2y^7 + t_3x^3y^5 + I_0'x^3y^6 + I_1'x^3y^7$$

resp.

$$f + t_1xy^9 + t_2x^2y^7 + t_3x^3y^5 + I_0''\,x^2y^8 + I_1''x^2y^7.$$

Both families glue together in etale topology over the base variety $M_{5,11,5} \cap D(A, B) \subset \mathbf{P}^4(1,2,3,10,17)$ with coordinates $(t_1 : t_2 : t_3 : I_0 : I_1)$ and singular point $t_1 = t_2 = I_0 = I_1 = 0$ i.e. $(0:0:1:0:0)$, where

$$I_1 := A^2I_1' = ABI_1'' + (15/2\,B + 4/11\,t_1A)\,I_0'' +$$
$$+ (3/11\,B - 2/11\,t_1A)t_1t_3^2\,I_0''.$$

But on U_3 a geometric quotient does not exist as algebraic variety.

The G_0 – invariants on U_3 are generated by t_1, I_0 and I_2

$$I_2 = t_1AI_1',$$

(t_3 is omitted because of the relation $Z = 0$ and $t_1 \neq 0$ on U_3). This is obvious for $A \neq 0$. But if H is invariant we have

$$A^rH = P(t_1, t_2, I_0, I_2) = P_1(t_1, A, I_0, I_2) = AP_2 + P_3(t_1, I_0, I_2).$$

Putting $A = 0$ we get $P_3 = 0$! But the quotient map

$$\pi : \mathbf{C}^8 \to \mathbf{C}^4,\ \pi(t_1, t_2, t_4, \dots, t_9) = (t_1, t_2, I_0, I_2)$$

restricted to $D(I)$ is not a geometric quotient because the fibres do not separate the orbits. Look at points with $A = 0$ (then $B = 0$ too). G_0-orbits are of the form

$$G_0(t_1, 9/22\,t_1^2, t_4, \dots, t_9) = (t_1,\ 9/22\,t_1^2,\ t_4, t_5, t_6 + g_1, \dots, t_9 + g_4).$$

The fibre $\pi^{-1}(\tau_1, \tau_2, i_0, i_2)$, $\tau_2 = 9/22\,\tau_1^2$, consists of two orbits $G_0(\tau_1, \tau_2, \tau_4, \tau_5, 0, 0, 0, 0)$ where τ_4 and τ_5 are two different solutions of

$$i_0 = \tau_1t_5 - 9/11\,\tau_1^2t_4$$

$$i_2 = 143/22\,\tau_1^2t_4^2 - 8\,\tau_1t_4t_5 + (5^13^27^2/2^211^6)\,\tau_1^9t_4.$$

In the category of algebraic spaces the following etale covering gives the geometric quotient. Let

$$H := AW^2 + 2(t_1D - 2t_2C)W + 2(A + 1/11t_1^2)\,(t_1t_7 - 2t_2t_6);$$

then $H'^2 = 4(AH + I) / (A + 1/11\, t_1^2))$ and

$$\Phi : \mathbf{C}[t_1, \ldots, t_2]_x \to \mathbf{C}[t_1, \ldots, t_9]_x [W]/H \text{ is etale.}$$

Φ is a G_0-morphism by $g(W) := W - g_1(A + 1/11t_1^2)$. The G_0-invariants are t_1, t_2, I_0 and $J := AW + t_1D - 2t_2C$.

$$\pi : V(H) \cap D(t_1H') \subset \mathbf{C}^9 \to D(t_1H'),\ \pi(t_1, \ldots, t_9, W) = (t_1, t_2, I_0, J)$$

is a geometric quotient with respect to the G_0-action. The fibre $\pi^{-1}(\tau_1, 9/22\ \tau_1^2, i_0, j)$ is just the G_0-orbit of $(\tau_1, 9/22\ \tau_1^2, \tau_4, \tau_5, 0, \ldots, 0)$, where τ_4 and τ_5 are the unique solutions of

$$i_0 = \tau_1 (t_5 - 9/11\ \tau_1 t_4)$$

$$j = \tau_1 D - 9/11\ \tau_1^2 C = 8i_0 - (105/2^1\, 11^4)\ \tau_1^9 + 1/11\ \tau_1^2\, t_4.$$

The universal family is given by

$$f + t_1xy^9 + t_2x^2y^7 + (2t_2A/3t_1 + 7/33t_1t_2)\, x^3y^5 +$$
$$+ (t_4 + AW/(A + 1/11t_1^2))\, x^2y^8 + (t_5 + (4/11t_1AW/(A + 1/11t_1^2)\, x^3y^6.$$

(3) $M_{5,11,4}$

On S_4 we have $A = B = t_1(D - 9/11\, t_1C) = 0$ and an invariant open affine covering is given by

$$U_4 : C^2 - t_1E \neq 0 \text{ and } U_5 : D^2 - 81/121\, t_1^3E \neq 0.$$

The invariants are generated by t_1, t_4, t_5 and I_3' (resp. I_3'') with

$$I_3' := t_7 - (CD - 9/11\, t_1^2E)/(C^2 - t_1E)\, t_6$$

$$I_3'' := t_6(CD - 9/11\, t_1^2E)/(D^2 - 81/121\, t_1^3\, E)t_7.$$

On $U_4 \cap U_5$ we have an invariant function

$$I_3 := (CD - 9/11\, t_1^2E)\, I_3' = (D^2 - 81/121\ \tau_1^3E)\, I_3''.$$

The invariant functions separate the G_0-orbits on U_4 and U_5 which are given by

$$G_0(t_1, \ldots, t_9) = G_0(t_1, 9/22t_1^2, 21/242t_1^3, t_4, t_5, t_6, I_3', t_8, t_9) =$$
$$= (t_1, \ldots, t_5, t_6 + g_1, I_3', t_8 + g_2, t_9 + g_3)$$

resp.

$$G_0(t_1, \ldots, t_9) = G_0(t_1, 9/22\ t_1^2, 21/242\ t_1^3, t_4, t_5, I_3'', t_7, t_8, t_9) =$$
$$= (t_1, \ldots, t_5, T_3'', t_7 + g_1, t_8 + g_2, t_9 + g_3)$$

hence the geometric quotient exists and is given by

$$V(t_1(9t_1C - 11D)) \cap D(C^2 - t_1E) \subset \mathbf{P}^3(1,7,8,13)$$

resp.

$$V(t_1(9t_1C - 11D)) \cap D(D^2 - 81/121\, t_1^3E) \subset \mathbf{P}^3(1,7,8,12)$$

Both pieces glue together to

$M_{5,11,4} = V(t_1(9t_1C - 11D)) \cap D(C^2 - t_1E, D^2 - 81/121t_1^3E)$ in $\mathbf{P}^3(1,7,8,28)$ with coordinates $(t_1 : t_4 : t_5 : I_3)$.

The singular points are $(0 : 1 : 0 : i)$ and $(0 : 0 : 1 : i)$.

Notice that in the set of singular points $(0 : 1 : 0 : i)$ only $(0 : 1 : 0 : 0)$ is a singular point of the weighted projective space.

The universal families

$$f + t_1xy^9 + 9/22t_1^2x^2y^7 + 21/242\ t_1^3x^3y^5 + t_4x^2y^8 + t_5x^3y^6 + I_3'x^3y^7$$

resp.

$$f + t_1xy^9 + 9/22\ t_1^2x^2y^7 + 21/242\ t_1^3x^3y^5 + t_4x^2y^8 + t_5x^3y^6 + I_3''x^2y^9$$

glue together in etale topology.

(4) $M_{5,11,3}$

On S_3 we have $A = B = D - 9/11t_1C = C^2 - t_1E = 0$ with open invariant affine covering

$$U_6 : t_1 \neq 0 \text{ and } U_7 : E \neq 0$$

i.e. S_3 is a smooth variety : an open subset of

$$\mathrm{Spec}\ \mathbf{C}[t_1, t_4, t_6, t_7, t_8, t_9]/(C^2 - t_1E).$$

The invariant functions on U_6 (resp. U_7) are generated by

t_1, t_4, I_4', I_5'	(resp. t_1, t_4, I_4'', I_5'')
$I_4' := t_7 - 9/11\, t_1t_6$	$I_4'' := t_7 - (9/11\, t_1t_8C)/E$
$I_5' := t_8 - (C/t_1)t_6$	$I_5'' := t_6 - (C - E)\, t_8$

On $U_6 \cap U_7$ we have

$$I_4 := I_4' = I_4'' - 9/11\, t_1I_5'',\ I_5 := CI_5' = -EI_5''.$$

The invariant functions separate orbits on U_6 and U_7 because the G_0-orbits are given by

$$G_0(t_1, t_4, t_6, \ldots, t_9) = G_0(t_1, t_4, t_6, I_4', I_5', t_9) =$$
$$= (t_1, t_4, t_6 + g_1, I_4', I_5', t_9 + g_2)$$

resp.

$$G_0(t_1, t_4, t_6, \ldots, t_9) = G_0(t_1, t_4, I_5'', I_4'', t_8, t_9) =$$
$$= (t_1, t_4, I_5'', I_4'', t_8 + g_1, t_9 + g_2)$$

hence the geometric quotients exists and are given by

$$V(C^2 - t_1E) \cap D(t_1) \subset \mathbf{P}^3(1,7,13,18)$$

resp.

$$V(C^2 - t_1E) \cap D(E) \subset \mathbf{P}^3(1,7,13,12).$$

Both pieces glue together to

$$M_{5,11,3} = V(C^2 - t_1E) \cap D(t_1, E) \subset \mathbf{P}^3(1,7,13,25)$$

with coordinates $(t_1 : t_4 : I_4 : I_5)$ and one singular point $(0:0:1:0)$. The universal families given by

$$f + t_1xy^9 + 9/22\, t_1^2x^2y^7 + 21/242\, t_1^3x^3y^5 + t_4x^2y^8 + (71/88t_1t_4 + 5\cdot 21^2/32\cdot 11^4t_1)\, x^3y^6 + I_4'x^3y^7 + I_5'x^3y^8$$

resp.

$$f + t_1xy^9 + 9/22t_1^2x^2y^7 + 21/242\, t_1^3x^3y^5 + t_4x^2y^6 + (71/88\, t_1\, t_4 + 5\cdot 21^2/32\cdot 11^4t_1^5)x^3y^6 + I_4''x^3y^7 + I_5''x^2y^9$$

glue together in etale topology.

(5) As a trivial exercise one gets

$$M_{5,11,2} = \mathbf{P}^1(12,\ 18)$$

and universal family (in the category of algebraic varieties!)

$$f + t_6x^2y^9 + t_8x^3y^8$$

$M_{5,11,1}$ resp. $M_{5,11,0}$ is just one point corresponding to

$$f + x^3y^9 \text{ resp. } f.$$

Thus we got the following results

$M_{5,11}$ *has exactly seven components with dimensions* 3,3,2,2,1,0,0 *and*

$M_{5,11,6}$ *is a quasismooth open subset of the weighted projective space* $\mathbf{P}^3$ (1,2,3,10);

$M_{5,11,5}$ *is a 3-dimensional algebraic space*;

$M_{5,11,4}$ *and* $M_{5,11,3}$ *are open sets of (hyper-) surfaces in* $\mathbf{P}^3$(1,7,8,28) resp. $\mathbf{P}^3$ (1,7,13,25);

$M_{5,11,2}$ *is the weighted projective line* $\mathbf{P}^1$ (12,18).

Universal families exist for $M_{5,11,6}, \ldots, M_{5,11,3}$ *in the category of algebraic spaces and for* $M_{5,11,2}$ *as algebraic variety.*

4. THE AUTOMORPHISM GROUP

We will describe the automorphism group of a germ $(X, 0)$ of an irreducible curve with the semigroup $\langle 5, 11\rangle$. Let X be defined by $f \in \mathbf{C}\{x, y\}$ and denote $R := \mathbf{C}\{x, y\}/f$.

Let $\mathrm{Der}_{\mathbf{C}}^0(R)$ be the sub Lie-algebra generated (as R-module) by $\partial f/\partial y\ \partial/\partial x + \partial f/\partial x\ \partial/\partial y$ of the Lie-algebra $\mathrm{Der}_{\mathbf{C}}(R)$. If $(X, 0)$ is not quasihomogeneous then $\mathrm{Der}_{\mathbf{C}}(R)/\mathrm{Der}_{\mathbf{C}}^0(R)$ is a finite dimensional nilpotent Lie-algebra. If $(X, 0)$ is quasihomogeneous then it is solvable. Let us denote by

$$\mathrm{Aut}_{\mathbf{C}}^0(R) \text{ resp. } \mathrm{Aut}_{\mathbf{C}}^c(R)$$

the corresponding normal subgroups of $\mathrm{Aut}_{\mathbf{C}}(R)$, i.e.

$$\mathrm{Aut}_{\mathbf{C}}^0(R) = \exp\ \mathrm{Der}_{\mathbf{C}}^0(R),\quad \mathrm{Aut}_{\mathbf{C}}^c(R) = \exp\ \mathrm{Der}_{\mathbf{C}}(R).$$

Then $\mathrm{Aut}_{\mathbf{C}}(R)/\mathrm{Aut}_{\mathbf{C}}^0(R)$ is an algebraic group (solvable resp. nilpotent) and $\mathrm{Aut}_{\mathbf{C}}^c(R)/\mathrm{Aut}_{\mathbf{C}}^0(R)$ is its connected component of the identity.

If $(X, 0)$ is generic then $\mathrm{Aut}_{\mathbf{C}}(R)/\mathrm{Aut}_{\mathbf{C}}^0(R)$ is connected. $(X, 0)$ corresponds to a suitable point $t \in \cup M_{5,11,i}$. The $M_{5,11,i}$ are locally quasiprojective with respect to a weighted projective space $\mathbf{P}_w$.

It turns out that the isotropy group G_t of $t \in \mathbf{P}_w$ is isomorphic to

$$\mathrm{Aut}_{\mathbf{C}}(R)/\mathrm{Aut}_{\mathbf{C}}^c(R).$$

This is a special property of our moduli spaces and is not true in general for families of singularities defined on subsets of a weighted projective space.

It is also not clear whether this fact will be true for other semi groups $\langle a, b\rangle$. It is true for the moduli spaces corresponding to the minimal Tjurina number.

If the weights of the corresponding projective space $\mathbf{P}_w$ are reduced, points with a nontrivial isotropy group are the singular points of $\mathbf{P}_w$. In this case these singular points of the moduli space correspond to the singular curves with a non connected automorphism group (Idea of A. Dimca).

We give a list of all singularities with the semigroup $\langle 5, 11\rangle$ and automorphism group not connected.

i	points in $M_{5,11,i}$	$\mathrm{Aut}_{\mathbf{C}}(R)/\mathrm{Aut}_{\mathbf{C}}^c(R)$
6	$(0:1:0:i) \in \mathbf{P}^3(1, 2, 3, 8)$	$\mathbf{Z}/2\mathbf{Z}$
5	$(0:0:1:0:0) \in \mathbf{P}^4(1, 2, 3, 7, 12)$	$\mathbf{Z}/3\mathbf{Z}$
4	$(0:1:0:0) \in \mathbf{P}^3(1, 7, 8, 13)$	$\mathbf{Z}/7\mathbf{Z}$
	$(0:0:1:i) \in \mathbf{P}^3(1, 7, 8, 12)$	$\mathbf{Z}/4\mathbf{Z}$
	$(0:0:1:0) \in \mathbf{P}^3(1, 7, 8, 12)$	$\mathbf{Z}/8\mathbf{Z}$
3	$(0:0:1:0) \in \mathbf{P}^3(1, 7, 13, 12)$	$\mathbf{Z}/13\mathbf{Z}$
2	$(1:0) \in \mathbf{P}^1(12, 18)$	$\mathbf{Z}/12\mathbf{Z}$
	$(0:1) \in \mathbf{P}^1(12, 18)$	$\mathbf{Z}/18\mathbf{Z}$
	else	$\mathbf{Z}/6\mathbf{Z}$
1	point	$\mathbf{Z}/23\mathbf{Z}$

For $i = 3,4,5$ and 6 these points in $M_{5,11,i}$ are singular points.

5. STEENBRINK'S INVARIANTS

Let us consider the singularities with the semigroup $\langle 5, 11\rangle$ as surface singularities by adding z^2 to the corresponding equation. We compute the geometric genus p_g, the irregularity q, the genus g, and α, β, γ (cf. [4]):

Let $Y \to X$ be a good resolution of the normal surface singularity $(X, 0)$ and E the reduced exceptional divisor, $\{E_i\}_{i=1\ldots k}$ the components of E and F their disjoint union. Then

$$g = \Sigma g(E_i),\ p_g = \dim H^1(O_Y),\ q = \dim H^0(\Omega^1_{Y\setminus E})/H^0(\Omega^1_Y)$$

$$\alpha = \dim H^0(\Omega^2_Y)/dH^0(\Omega^1_Y(\log E)(-E)),\ \beta = \dim H^0(\Omega'_F)/\mathrm{Im}\, H^0(\Omega^1_Y)$$
$$= \mathrm{rk}(H^1_E(\Omega^1_Y) \to H^1(\Omega^1_Y)) - k.$$

For the singularities considered here we get

$$p_g = 4,\ g = 0,\ \beta = 0,\ \alpha = 36 + q - \tau,\ \gamma = \tau - 2q - 32$$

τ	34	35	36	37	38	39	40
g	0	1 if $A \neq 0$ or $B \neq 0$ 0 else	1	2 if $t_1 \neq 0$ 2 else	2	3	4

6. THE POLAR CURVES

For $t \in S_i$ denote by P_t the polar curve of the corresponding singularity defined by $F_t = 0$.

There are three possible types of polar curves

I P_t is irreducible with Milnor number 24 defined by an equation $x^4 + y^9 +$ terms of degree >1 with respect to the weights 1/4, 1/9 iff $t_1 \neq 0$.

II P_t is reducible with Milnor number 25 defined by an equation $x^4 + xy^7 +$ terms of degree >1 with respect to the weights 1/4, 3/28 iff $t_1 = 0$ $t_2 \neq 0$.

III P_t is reducible with Milnor number 27 defined by an equation $x^4 + y^{10} +$ term of degree >1 with respect to the weights 1/4, 1/10 iff $t_1 = t_2 = 0$

τ	34	35	36	37	38	39	40
type	I, II	I, III	I, III	I, III	III	III	III

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