Singularities of Moduli Spaces

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Let (X,o) be the germ of an irreducible curve singularity in the complex plane C^2 defined by the equation $f=0$, f being holomorphic in a neighbourhood of o. Let $B_d=\{(x,y),\ |(x,y)|<d\}$ be a small ball and $K_d=\{(x,y),\ |(x,y)|<d, f(x,y)=0\}$. For small d the homeomorphy class of (B_d,K_d) does not depend on d and is called the topological type of the singularity. Denote by $T_{a,b}$ the topological type of the singularity defined by $f=x^a+y^b$, a and b relatively prime.

The moduli space $M_{a,b}$ of all germs of plane curve singularities with the same topological type $T_{a,b}$ is a disjoined union of analytic varieties $M_{a,b,t}$, $t=1,\ldots,g$. The generic component $M_{a,b,g}$ of $M_{a,b}$ is an algebraic variety, locally an open subset in a weighted projective space P_w. There are examples that the other components are not algebraic varieties (cf.<1>). $M_{a,b}$ is constructed in the following way: Consider the family $p:X \longrightarrow C^r$, X the subset of $C^r \times C^2$ defined by the equation

$$F = x^a+y^b+ \sum_B t_{ij}x^iy^j,$$

$B=\{(i,j), ib+ja>ab, i<a-1, j<b-1\}$, p the projection.

This family has the following properties: If (Y,o) is any germ of an irreducible plane curve singularity with the topological type $T_{a,b}$ then there is an t in C^r such that $(p^{-1}(t),o)=(Y,o)$. Using the relative Kodaira-Spencer map of this family one can compute the analytically trivial subfamilies: The kernel K of the Kodaira-Spencer map is a sub Lie-algebra of the derivation of C^r. Along the integral manifolds of K the family $X \longrightarrow C^r$ is trivial. It turns out that $M_{a,b}=C^r/K$ and $M_{a,b,i}=S_i/K$, S_i the set of all points t in C^r such that the integral manifold through t has dimension i, (cf.<3>). Let $\{a_i\}_{i=1,\ldots,n}$ be a free base of $C[[t,x,y]]/(\partial F/\partial x, \partial F/\partial y)$ as a $C[[t]]$-module and let $a_iF = \sum_B h_{i,kl}x^ky^l \bmod (\partial F/\partial x, \partial F/\partial y)$ then $\delta_i = \sum_B h_{i,kl}\partial/\partial t_{kl}$, generate K as $C[[t]]$-module. Moreover S_i is determined by $\mathrm{rank}(h_{i,kl}(t))=i$. There is an algorithm to compute generators of K, (cf.<2>).

The generic component $M_{a,b,g}$ od $M_{a,b}$ may have singularities, according to singularities of the corresponding weighted projective space P_w. The singular points of $M_{a,b,g}$ are of special intrest because they correspond to singular curves with a non connected automorphism gruop G, the quotient of G by its connected component turns out to be the isotropy group of the corresponding point in P_w, (cf.<5>). These singularities are charaterized by the weights and can be computed in a combinatorial way.
The weights of P_w depend only on a and b and can also be computed in a combinatorical way, (cf.<2>,<3>). $M_{a,b,g}$ is defined in P_w by the non vanishing of one or two polynomials with integer coefficients. These polynomials are certain minors of the matrix $(h_{i,ki})$ and can also be described in a combinatorial way in terms of a and b, (cf.<2>,<3>).
The components of the singular locus of P_w are suitable intersections of coordinate hyperplanes such that the weights of the corresponding coordinates have a non trivial common factor. Now using Buchbergers algorithm (cf.<4>) we get the singular locus of $M_{a,b,g}$.

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