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The tangent cone algorithm and some applications to local algebraic geometry

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1. About two conjectures

Our interest to use computers to investigate problems of algebraic geometry started because we couldn't solve two conjectures we have been interested for a long time.

The first conjecture goes back to a result of K. Saito cf. [S] who proved the following theorem.

Theorem. *Let* $f \in \mathbb{C}[[x_1,\dots,x_n]]$ *defining an isolated singularity* $(X,0)\subseteq(\mathbb{C}^n,0)$ (*i.e.* $\dim_{\mathbb{C}} \mathbb{C}[[x_1,\dots,x_n]]/(f,\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}) =: \tau(f) < \infty$).

The following conditions are equivalent:

(1) $f \in (\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})$, *i.e.* $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[[x_1,\dots,x_n]]/(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}) =: \mu(f)$

(2) There is an automorphism $\varphi : \mathbb{C}[[x_1,\dots,x_n]] \to \mathbb{C}[[x_1,\dots,x_n]]$ *such that* $\varphi(f) = p$ *is a weighted homogeneous polynomial (i.e. there are* $w_1,\dots,w_n,d$ *positive integers such that* $p(\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n) = \lambda^d p$); f *is called quasi-homogeneous.*

(3) The Poincaré-complex

$$O \to \mathbb{C} \to \mathcal{O}_{X,0} \to \Omega^1_{X,0} \to \dots \to \Omega^n_{X,O} \to 0$$

is exact.

Now everything has still a meaning if instead of hypersurface-singularities we consider complete intersection singularities resp. curve singularities in general (cf.[GMP]), because μ, the Milnor number, is a topological invariant of the singularity and τ, the Tjurina number, is the dimension of the miniversal deformation. One can prove that for complete intersections, resp. Gorenstein curve singularities, (1) and (2) are equivalent. We didn't succeed with (3) being equivalent to (2). With the help of the tangent cone algorithm and an Atari we found a counterexample

Let us consider $f, g \in \mathbf{C}[[x,y,z]]$ defining an isolated curve singularity $(X,0)$. In this case

$$\tau(f,g) = \dim \mathbf{C}[[x,y,z]]/(f,g,M_1,M_2,M_3),$$

M_i denoting the 2-minors of the Jacobian matrix $\frac{\partial(f,g)}{\partial(x,y,z)}$.

In the general case,one defines

$$\tau = \dim_K (K[[x_1,\ldots,x_n]]/(f_1,\ldots,f_m))^m/J$$

$$J = \left(\begin{pmatrix} \partial f_1/\partial x_1 \\ \vdots \\ \partial f_m/\partial x_1 \end{pmatrix}, \ldots, \begin{pmatrix} \partial f_1/\partial x_n \\ \vdots \\ \partial f_m/\partial x_n \end{pmatrix} \right)$$

Again in the space curve case, one has

$$\mu(f,g) = \dim_{\mathbf{C}} \mathbf{C}[[x,y,z]]/(f,M_1,M_2,M_3) - \dim_{\mathbf{C}} \mathbf{C}[[x,y,z]]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$$

assuming that both dimensions are finite; in the general case,

$$\mu = \sum_{k=1}^{m} (-1)^{m-k} \dim K[[x_1\ldots x_n]]/(f_1,\ldots,f_{k-1}, \frac{\partial(f_1\ldots f_k)}{\partial(x_{v_1},\ldots,x_{v_k})}),$$

$1 \le v_1 < \ldots < v_k \le n$.

The Poincaré-complex is given by

$$0 \to \mathbf{C} \to \mathcal{O}_{X,0} \to \Omega^1_{X,0} \to \Omega^2_{X,0} \to \Omega^3_{X,0} \to 0;$$

here

$$\Omega^i_{X,0} = \bigwedge^i \Omega^1_{X,0}$$

$$\Omega^1_{X,0} = \Omega^1_{\mathbf{C}^3,0}/f\Omega^1_{\mathbf{C}^3,0} + g\Omega^1_{\mathbf{C}^3,0} + df\mathcal{O}_{\mathbf{C}^3,0} + dg\mathcal{O}_{\mathbf{C}^3,0}$$

$$\Omega^1_{\mathbf{C}^3,0} = \mathcal{O}_{\mathbf{C}^3,0}dx + \mathcal{O}_{\mathbf{C}^3,0}dy + \mathcal{O}_{\mathbf{C}^3,0}dz.$$

With the identification $\Omega^1_{\mathbf{C}^3,0} = \mathbf{C}[[x,y,z]]^3$ we get

$$\Omega^3_{X,0} = \mathbf{C}[[x,y,z]]/(\frac{\partial f}{\partial x},\ldots,\frac{\partial g}{\partial z})$$

$$\Omega^2_{X,0} = \mathbf{C}[[x,y,z]]^3/\mathcal{U}$$

$\mathcal{U}$ generated by $\begin{pmatrix} f\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\f\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\f \end{pmatrix}, \begin{pmatrix} g\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\g\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\g \end{pmatrix}, \begin{pmatrix} f_y\\f_z\\0 \end{pmatrix},$
$\begin{pmatrix} f_x\\0\\ \end{pmatrix}, \begin{pmatrix} 0\\f_x\\ \end{pmatrix}, \begin{pmatrix} g_y\\g_z\\ \end{pmatrix}, \begin{pmatrix} g_x\\0\\ \end{pmatrix}, \begin{pmatrix} 0\\g_x\\ \end{pmatrix}.$

For a large class of examples, one can prove that the Poincaré-complex is exact iff $\mu = \dim \Omega^2_{X,0} - \dim \Omega^3_{X,0}$.

On the other hand $(X,0)$ is quasihomogeneous iff $\mu = \tau$.

The following (unimodular) singularities have exact Poincaré complex but they are not quasihomogeneous:

$$f = xy + z^{l-1},\ g = xz + y^{k-1} + yz^2 \quad 4 \le l \le k,\ 5 \le k$$

$$f = xy, \quad g = xz + yz^2 + z^3 + z^{3+i} \quad i > 1.$$

The second conjecture was a conjecture of Mumford concerning the structure of moduli spaces of curve singularities.

If you start with any (irreducible) plane curve singularity defined by $f(x,y) = 0$, $f \in \mathbf{C}[[x,y]]$, then one is interested in the (local) moduli space of it, i.e. a space which classifies all plane curve singularities with the same topological type. The approach to construct it is the following (cf.[LaP], [Me]): we construct a family $F(x,y,\mathbf{t})$, $\mathbf{t} \in \mathbf{C}^N$ which contains all singular curves we are interested in (the miniversal μ-constant deformation). This is usually not so difficult. For instance, if $f = x^p + y^q$ and $gcd(p,q) = 1$ then

$$F = f + \sum_{\substack{qi+pj>pq \\ i \le p-2 \\ j \le q-2}} t_{ij} x^i y^j$$

is such a minimal family.

But this family still contains trivial subfamilies, i.e. the underlying parameter space is not a moduli space in the sense that different points correspond to different (non-isomorphic) singularities.

Let $\mathbf{C}^N$ be the parameter space, the moduli space will be $\mathbf{C}^N/\sim =: \mathfrak{A}$. How to study $\sim$?

The trivial subfamilies of the family above are given as integral manifolds of a Lie algebra L, the kernel of the Kodaira-Spencer map,

$$\mathrm{Der}_{\mathbf{C}}\mathbf{C}[\mathbf{t}] \to \mathbf{C}[\mathbf{t}][[x,y]]/(F, \partial F/\partial x, \partial F/\partial y)$$
$$\delta \rightsquigarrow [\delta F]$$

and $\mathfrak{A} = \mathbf{C}^N/L$.

There is no chance in general for $\mathfrak{A}$ having a nice structure, because the orbits under the action of L will have different dimensions. This implies that $\mathfrak{A}$ is not Hausdorff.

We stratify $\mathbf{C}^N$ by the orbit dimension: $\mathbf{C}^N = \coprod S_i$. The conjecture was now that S_i/L exists as a geometric quotient i.e. especially it has the structure of an algebraic varity. This we could only prove for the stratum

Again with the help of the tangent cone algorithms and the Atari we could compute the kernel of the Kodaira-Spencer map and give an example that S_i/L is not an algebraic variety (cf. [LP]).

How to compute the kernel of the Kodaira-spencer map for a family $F = f + \sum_t \alpha X^\alpha$ of hypersurface singularities with constant Milnor number μ?

Generators can be computed as follows:

Choose a monomial base $\{X^\alpha\}$ of $\mathbf{C}[[X]]/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ and assume it is also a base of the free $\mathbf{C}[\mathbf{t}]$-module $\mathbf{C}[\mathbf{t}][[x_1, \ldots, x_n]]/\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)$ which is often the case. Consider $X^\alpha F = \sum h_{\alpha\beta}(\mathbf{t}) X^\beta \mod (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$. Then $\{\delta_\alpha\}$, $\delta_\alpha = \sum h_{\alpha\beta} \frac{\partial}{\partial T_\alpha}$ generate the kernel of the Kodaira-Spencer map.

Notice that in this situation often the standard base (cf. §2) is constant and we know in advance that computations can be stopped at a certain degree with respect to $\mathbf{X}$. This is important because the set of parameters $\mathbf{t}$ is usually large.

2. The tangent cone algorithm

What is the tangent cone of an algebric variety X at some point $P \in X$? It is the set of all tangents (i.e. lines intersecting X at P with a higher multiplicity then the multiplicity of X at P).

In terms of the local ring $\mathcal{O}_{X,P}$ resp. $\hat{\mathcal{O}}_{X,P}$ it is just given by the graded ring $gr_{\mathfrak{m}}\mathcal{O}_{X,P}$.

Let $\mathcal{O}_{X,P} = K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/\mathfrak{a}$, resp. $\hat{\mathcal{O}}_{X,P} = K[[x_1, \ldots, x_n]]/\mathfrak{a}$, $\mathfrak{a} \subset K[x_1, \ldots, x_n]$ the ideal defining X. Then $gr_{\mathfrak{m}}\mathcal{O}_{X,P} = gr_{\mathfrak{m}}R/gr_{\mathfrak{m}}\mathfrak{a}R \simeq K[x_1, \ldots, x_n]/I(\mathfrak{a})$, $R = K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ resp. $K[[x_1, \ldots, x_n]]$. $I(\mathfrak{a})$ is the ideal generated by $\{I(f), f \in \mathfrak{a} \setminus \{0\}\}$. If we consider $f \in R$ as powerseries, $f = \sum_{m \geq 0} f_m$, f_m homogeneous of degree m, then $I(f) = f_k$, $k = ord(f) = \min\{m, f_m \neq 0\}$.

One is interested now to compute generators of $I(\mathfrak{a})$ because the tangent cone contains already a lot of information about X and is much easier to handle. One example is the Hilbert polynomial:

Let $F(l) := \dim_K \mathcal{O}_{X,P}/\mathfrak{m}^l$. It is well known that there is a polynomial P such that $F(l) = P(l)$ if $l \gg 0$, $P(l) = \frac{m(\mathcal{O}_{X,P})}{d!} l^d + \ldots$, $d = \dim \mathcal{O}_{X,P}$ and $m(\mathcal{O}_{X,P})$ the multiplicity. Now $F(l) = \sum_{i=0}^{l-1} \dim_k (K[x_1, \ldots x_n]/I(\mathfrak{a}))_i$ and the Hilbert-polynomial of $K[x_1, \ldots, x_n]/I(\mathfrak{a})$ can be computed.

Especially if $\dim_K \mathcal{O}_{X,P} < \infty$ than

$$\dim_K \mathcal{O}_{X,P} = \dim_K K[x_1, \ldots, x_n]/I(\mathfrak{a})$$

and the computation of the dimension is reduced to a combinatorial problem.

To do this with a computer we need some ordering between the monomials in R.

Let $T = \{x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \alpha_i \geq 0\}$, T is a semigroup.

We consider a total ordering $<$ of T compatible with the semigroup structure. It is known (Robbiano, cf. [R]) that such an ordering can be realized by a suitable matrix $\mathcal{U} \in M_{n,s}(\mathbf{R})$, i.e. if u_j is the j-th row of $\mathcal{U}$ then $X^\alpha < X^\beta$ iff there is an i such that $u_j \cdot \alpha = u_j \cdot \beta$ for $j < i$ and $u_i \cdot \alpha < u_i \cdot \beta$ (here $\cdot$ means just the ordinary scalar product).

If such an ordering is fixed we may speak about $L(f)$, $C(f)$ the leading monomial, resp. the leading coefficient for $f \in K[x_1, \ldots, x_n]$.

Let $\mathfrak{a} \subseteq K[x_1 \ldots x_n]$ be an ideal and $L(\mathfrak{a})$ the ideal generated by $\{L(f), f \in \mathfrak{a} \setminus \{0\}\}$ than we have the notion of a standard base: $f_1, \ldots, f_m$ is a standard base of $\mathfrak{a}$ if $L(\mathfrak{a})$ is generated by $L(f_1), \ldots, L(f_m)$.

If $<$ is a well ordering than a standard base is also called a Gröbner base.

Now let us consider the ordering given by the matrix

$$\begin{pmatrix} -1 & -1 & \ldots & -1 & -1 \\ 0 & -1 & \ldots & -1 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & -1 \end{pmatrix}$$

the "dual" of the graded lexicographic ordering.

Let us assume we have $g_1, \ldots, g_t$ a standard base of $\mathfrak{a}$ with respect to that ordering; then $I(g_1), \ldots, I(g_t)$ generate $I(\mathfrak{a})$. So we got generators of $I(\mathfrak{a})$ and computed the tangent cone. The problem of this ordering is that it is not a well ordering. We cannot apply Buchberger's algorithm (cf. [B]) wich lives on well-orderings.

There is one way to avoid this problem:

We choose generators of $\mathfrak{a}$: $f_1, \ldots, f_k$ and make them homogeneous by adding one variable z:

$$f_i^h = z^{d_i} f_i(x_1/z, \ldots, x_n/z), \qquad d_i = \deg f_i.$$

We considers the ordering

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 1 & 1 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Using Buchberger's algorithm we get a Gröbner base of $(f_1^h, \ldots, f_k^h)$ $G_1, \ldots, G_t$ and $g_1 = G_1(z=1), \ldots, g_t = G_t(z=1)$ is a standard base of $\mathfrak{a}$.

The problem is that this way is very time and space consuming. Let us consider an example:

$\mathfrak{a}$ has a standard base of 5 elements with respect to the ordering defined by $\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$ and it took in our implementation 0.1 seconds to compute it.

The corresponding homogenization has a Gröbner base of 19 elements and it took 2.4 seconds to compute it.

T. Mora (cf. [M]) developped the tangent cone algorithm which is a modification of Buchberger's algorithm and works for a large class of orderings wich are not well-orderings.

Let us recall the idea of Buchberger's algorithm (cf. [B]; [MPT]): we need the notion of an S-polynomial and of a normal form.

(1) Let $f_1, f_2 \in K[x_1, \ldots, x_n]$, $G = \mathrm{lcm}(L(f_1), L(f_2))$; then $Spoly(f_1, f_2) = (C(f_2)G/L(f_1))f_1 - (C(f_1)G/L(f_2))f_2$.
(2) Let G be a finite set of polynomials. We define $NF(G, h) := h'$ computed by the algorithm

```
h' := h
while exists f ∈ G such that L(f)/L(h') do
      choose any f ∈ G such that L(f)/L(h')
      h' := h' - (C(h')/C(f)) (L(h')/L(f)) f.
```

This algorithm terminates because of the well ordering.

Buchberger's algorithm runs as follows:

We start with a set F of generators of an ideal $\mathfrak{a}$ and get a Gröbner base G using the following algorithm:

```
G := F
B := {(f1, f2) | f1, f2 ∈ G , f1 ≠ f2}
while B ≠ ∅ do
      (f1, f2) := a pair in B
         B := B \ {(f1, f2)}
         h := Spoly (f1, f2)
         h' := NF(G, h)
         if h' ≠ 0 then
            B := B ∪ {(g, h'), g ∈ G}
            G := G ∪ {h'}.
```

The point in which we need a well ordering in this algorithm is in the NF-algorithm.

Let us consider an example:

Consider $K[x]$ with the ordering $1 > x > x^2 > \ldots$.

Let $G = \{x + x^2\}$ and $h = x$. Then $NF(G, h)$ will produce $-x^2, x^3, -x^4, \ldots$ and this is perfectly o.k. because we know that in $K[[X]]$ $x = \sum_{\nu=0}^{\infty}(-1)^{\nu}x^{\nu}(x + x^2)$.

But NF does not terminate.

In $K[[x]]$ resp. $K[X]_{(X)}$ it is not important whether we reduce some h ...

If we had started with $(1 + X) \cdot X$ instead of X everything would have been O.K.

How to produce this unit with the computer?

We consider the reduction of x by $x + x^2$ to x^2. After this step we extend the set of elements which are used for the reduction process by x and the algorithm terminates. This is the general concept for the local normal form algorithm (NFL):

```
h' := NFL(G, h)
h' := h
T := ∅
while there exists f ∈ G ∪ T such that L(f)/L(h') do
      choose a suitable (what is suitable will be explained later)
         f ∈ G ∪ T such that L(f)|L(h')
      T := T ∪ {h'}
      h' := h' - (C(h')/C(f)) (L(h')/L(f)) f
```

This means expecially $NFL(G, h) = NF(G, e \cdot h)$ for a suitable $e \in K[x_1, \ldots, x_n]$ which is a unit in $K[[x_1, \ldots, x_n]]$.

What means suitable?

This will depende on the order (cf. [MPT]).

If we have the ordering given by

$$\begin{pmatrix} -1 & -1 & \ldots & -1 & -1 \\ 0 & -1 & \ldots & -1 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & -1 \end{pmatrix}$$

and we define for any $f \in K[x_1, \ldots, x_n]$ $d(f) := \deg(f) - \mathrm{ord}(f)$ (degree of f as a polynomial, order of f as a power series), we have to choose $f \in G \cup T$ such that $L(f)/L(h')$ and $\max\{d(h'), d(f)\}$ is minimal.

The basic idea of T. Mora tangent cone algorithm is to replace the procedure NF in Buchberger's algorithm by NFL.

The algorithm works for orderings with the following properties (tangent cone orderings (cf. [MPT]): let the ordering be defined by the matrix $\mathcal{U} = \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix}$; then there is a k such that

a) $u_1, \ldots, u_k \in \mathbb{Z}^n$.
b) For all $d_1, \ldots, d_k \in \mathbb{Z}$ the set $\{\alpha \in T \mid \alpha u_1 = d_1, \ldots, \alpha u_k = d_k\}$ is well ordered with respect to the order defined by $\mathcal{U}$.

These properties are always satisfied if $\mathcal{U} \in M_{n,s}(\mathbb{Q})$ or $u_1 = (-1, \ldots, -1)$ or $\mathcal{U}$ defines a well ordering.

The algorithm produces for a given ideal $\mathfrak{a} \subset K[x_1 \ldots x_n]$ a standard base of $\mathfrak{a}\mathrm{Loc}(K[x_1, \ldots, x_n])$, $\mathrm{Loc}(K[x_1, \ldots, x_n]) = \{(1 + g)^{-1} f,\ f, g \in$ ...

Notice that if $1 > x^\alpha$ for all $\alpha \neq 0$ then $\text{Loc}(K[x_1,\ldots,x_n]) = K[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$.

If additionally the order satisfies

a) $1 > x^\alpha$

b) if x^β and a sequence $x^{\alpha_1} > \ldots > x^{\alpha_s} > \ldots$ are given, then there is a k such that $x^\beta > x^{\alpha_k}$

then it produces a standard base of $aK[[x_1,\ldots,x_n]]$.

The additional properties are satisfied if the entries in $\mathcal{U}$ are all negative integers.

A slight modification of the algorithm produces also a standard base of a submodule $U \subseteq R^s$.

Especially if $\dim_k R^s/\mathcal{U} < \infty$ we get

$$\dim_k(gr_m R)^s/L(\mathcal{U}) = \dim_k R^s/U.$$

As well as Buchberger's algorithm, the algorithm of Mora is much more sophisticated (cf. [MPT]):

- The criteria can be used as in Buchberger's algorithm.
- In NFL not all h' have to be added to T: if $f \in G \cup T$ and $L(f)/L(h')$ and $d(f) \leq d(h')$ then h' need not to be added to T. If there is a $g \in T$ and $L(h')/L(g)$ and $d(h') \leq d(g)$ then g can be removed by adding h'.
- There is an early termination test if the ideal is zero dimensional: if we have $g_1,\ldots,g_n \in G$ such that $L(g_i) = x_i^{\alpha_i}$ then one can compute a β such that all $x^\alpha < x^\beta$ are already in $\mathfrak{a}K[x]_{(x)}$. Knowing this we can use NF instead of NFL and stop the reductions if we come to x^β. In some applications we even know x^β in advance and can immediately switch to NF.
- There is a "lazy" version of the algorithm which is usually faster than the "classical" one because it makes after every single reduction step a decision to postpone further reductions to the normal form and starts reducings other elements which look at that level more efficient to reduce.

3. Furhter applications

In the applications it is useful to work first in characteristic p (181 or 32003 are nice primes to work with) which turns to be often 100 times faster, and work later with the interesting examples in characteristic 0. Besides the two applications mentioned already we often use (as already many others did),

(1) the test $\mu = \tau$ to decide whether a singularity is quasihomogeneous.

(2) for a family $F(\mathbf{x},t)$ the test $\partial F/\partial t \in (F, \partial F/\partial x_1,\ldots,\partial F/\partial x_n)$ to decide whether it is analytically trivial.

(3) to compare $\dim K[[x_1,\ldots,x_n]]/(\partial F/\partial x_1,\ldots,\partial F/\partial x_n) = \mu(t)$ with $\dim K[x_1,\ldots,x_n]/(\partial F/\partial x_1,\ldots,\partial F/\partial x_n) = \mu_{\text{glob}}(t)$ to decide whether

(4) We found an example against the conjecture that τ-constant in a family of special singularities implies μ-constant (cf. [**LaP**]):

$$F(x,y,t) = x^{11} + y^5 + x^7y^2 + 2tx^2y^4 + t^2x^4y^3$$

$$\tau = 34, \quad \mu(F(t=0)) = 40, \quad \mu(F(t \neq 0) = 39.$$

(5) The way in which the algorithm is organized could be used to simplify theoretical proofs (cf. [LuP]).

(6) We are working from time to time on the Zariski conjecture: μ-constant in a family of hypersurfaces implies multiplicity is constant.

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