RECOGNITION OF UNIMODAL MAP GERMS FROM THE PLANE TO THE PLANE BY INVARIANTS

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Abstract. In this article we characterize the classification of unimodal maps from the plane to the plane with respect to $\mathcal{A}$-equivalence given by Rieger in terms of invariants. We recall the classification over an algebraically closed field of characteristic 0. On the basis of this characterization we present an algorithm to compute the type of the unimodal maps from the plane to the plane without computing the normal form and also give its implementation in the computer algebra system Singular.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $\mathcal{M} =< x, y > \mathbb{K}[[x, y]]$. Let $A =\text{Aut}_{\mathbb{K}}(\mathbb{K}[[x, y]]) \times \text{Aut}_{\mathbb{K}}(\mathbb{K}[[x, y]])$ acting on $\mathcal{M}$ by

$$A \times \mathcal{M} \to \mathcal{M}$$

such that

$$((\varphi, \psi), f) \mapsto \varphi^{-1} \circ f \circ \psi.$$

This is equivalent to consider the set of map germs $(\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ under the action of the group $A =\text{Aut}_{\mathbb{K}}(\mathbb{K}^2, 0) \times \text{Aut}_{\mathbb{K}}(\mathbb{K}^2, 0)$.
The map germs $f, g \in \mathcal{M}$ are called $\mathcal{A}$-equivalent ($f \sim_{\mathcal{A}} g$) if they are in the same orbit under the action of A. In the classification of map germs with respect to the action of the group A the tangent spaces to the orbit under the action of this group and their codimension play an important role (cf. [1]). Given $f \in \mathcal{M}$ the orbit map $\theta_f : A \to \mathcal{M}$ is defined by $\theta_f(\varphi, \psi) = \varphi^{-1} \circ f \circ \psi$. The corresponding tangent map has as image the tangent space to the orbit at $f = (f_1, f_2)$:

$$T_{\theta_f,id} =< x, y >< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >_{\mathbb{K}[[x,y]]} + < f_1, f_2 > \mathbb{K}[[f_1, f_2]]^2.$$

In this article we characterize the classification of unimodal maps from the plane to the plane given by Rieger, in terms of invariants [1] and on the basis of this characterization we give the implementation of this classification in the computer algebra system Singular [4], [5]. Rieger achieved the classification of all simple and unimodal map germs from the plane to the plane of corank at most 1 with respect to $\mathcal{A}$-equivalence in [8], [9]. A characterization of Rieger's classification of simple map germs from the plane to the plane in terms of invariants is given in [3].

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[1] Rieger gave the classification over $\mathbb{R}$ which can be easily extended to a classification over an algebraically closed field of characteristic 0.

Rieger's classification is based on explicit coordinate changes, Mather's Lemma (cf. Lemma 3.1 in [7]) and complete transversals (Theorem 2.9 in [2]). A crucial step is the calculation of the determinacy degree. Rieger computed the extended codimension of the tangent space to the orbit, the cusp number and the double fold number of the map germ as invariants.
In his paper [6], Kabata gave a characterization of Rieger's classification in terms of $\lambda = \frac{\partial(f_1,f_2)}{\partial(x,y)}$ and $\eta = \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$ spanning $\ker(df)$.
We use the following invariants for our characterization:
A map germ $f \in \mathcal{M}$ of corank at most 1 is always $\mathcal{A}$-equivalent to $(x, g(x,y))$, for suitable g with $g(x,0) = 0$. Let $f(x,y) = (x, g(x,y))$ then the codimension of the tangent space, $\mathrm{codim}(f) = \dim_{\mathbb{K}} \frac{<x,y>\mathbb{K}[[x,y]]}{T_{\theta_f,id}}$, is one of the invariants used in the classification. Algorithms to compute the codimension are implemented in Singular (cf. [1]). The other invariants are the Milnor number $\mu(\Sigma) = \mu(\frac{\partial g}{\partial y})$ of the critical set Σ of the map germ f, the multiplicity $m(f) = \dim_{\mathbb{K}} \frac{\mathbb{K}[[x,y]]}{<x,g>}$ and the double fold numbers [2] $d(f) = \frac{1}{2}\dim_{\mathbb{K}} \frac{\mathbb{K}[[x,y,t]]}{I}$, where

$$I =< g_y(x,y), h = t^{-2}(g(x,y+t) - g(x,y) - t g_y(x,y)), \frac{\partial h}{\partial t} > .$$

Let us recall the definition of modality: The modality of $f \in \mathcal{M}$ is the smallest integer m such that a sufficiently small neighborhood of f can be covered by a finite number of $m-$parameter families of orbits under the action of A on $\mathcal{M}$. Maps of modality 0 (resp. 1) are called simple (resp. unimodal).

Table-1 contains all unimodal map germs from the $\mathbb{K}$-plane to the $\mathbb{K}$-plane obtained from Rieger's classification over the real numbers (cf. [8]).

Table 1

Normal form	codim(f)	$m(f)$	$\mu(\Sigma)$	$d(f)$
$(x, xy + y^6)$	8	6	0	6
$(x, xy + y^6 + y^{14})$	7	6	0	6
$(x, xy + y^6 + y^9)$	6	6	0	6
$(x, xy + y^6 + y^8 + \alpha y^9)$	6	6	0	6
$(x, xy^2 + y^6 + y^7 + \alpha y^9)$	7	6	1	8
$(x, y^4 + x^3y^2 + x^l y), l \geq 5$	$l + 4$	4	7	l
$(x, y^4 + x^k y + x^l y^2), k = 4, 5,\ k - 1 \leq l \leq 2k - 1$	$k + l + 1$	4	$2k - 2$	k
$(x, y^4 + x^2y^2 + x^k y), k \geq 4$	$k + 3$	4	4	k
$(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^k y), k \geq 6$	$k + 3$	4	$k + 1$	3
$(x, y^4 + x^3y + \alpha x^2y^2), \alpha \neq \frac{-3}{2}$	9	4	4	3
$(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^4y^2)$	8	4	6	3
$(x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2), \alpha \neq \frac{-3}{2}$	8	4	4	3
$(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^3y^2)$	7	4	5	3
$(x, y^4 + x^3y + \alpha x^2y^2 + x^3y^2), \alpha \neq \frac{-3}{2}$	7	4	4	3

[2]The double fold number of f is the number of double folds in a versal deformation of f.

2. Characterization of Uni-Modal Map Germs From the Plane to the Plane

In this section we give the characterization of unimodal map germs from the plane to the plane in terms of invariants.

Proposition 2.1. *Let $f(x,y)$ be a map germ from $(\mathbb{K}^2,0)\to(\mathbb{K}^2,0)$. Suppose $\mu(\sum)=0$ and $m(f)=6$ then $f\sim_{\mathcal{A}}(x,xy+y^6+\sum_{i>6}a_{0,i}y^i)$ and 14-determined. The possible values of* codim(f) *are 6, 7, 8. This implies especially that f is unimodal.*

Proof. We may assume that $f=(x,g(x,y))$ then $\sum=V(\frac{\partial g}{\partial y})$. If $\mu(\sum)=0$ and $m(f)=6$ then we have $g=a_{1,1}xy+a_{0,2}y^2+$ h.o.t. such that $a_{1,1}\neq 0$, the coefficients of y^2,y^3,y^4 and y^5 must be zero and the coefficient of y^6 is not equal to zero. We can take $a_{1,1}=1$. Then the transformation

$$y\to y-(a_{k,0}x^{k-1}+a_{k-1,1}x^{k-2}y+\cdots+a_{2,k-2}xy^{k-2}+a_{1,k-1}y^{k-1})$$

removes all the terms of degree k which are divisible by x at the level of the $k-$jet, where $k>2$, and it transforms g into $xy+y^6+\sum_{i>6}a_{0,i}y^i$. It is proved in [9] that f is 14-determined. Using the computer algebra system Singular and the library classifyMapGerms.lib [3] one can show that codim$(f)=6$ iff $a_{0,7}$, $a_{0,8}$, $a_{0,9}$ do not vanish simultaneously. codim$(f)=7$ iff $a_{0,7}=a_{0,8}=a_{0,9}=0$ and $a_{0,10},\dots,a_{0,14}$ do not vanish simultaneously. codim$(f)=8$ iff $a_{0,i}=0$ for $i=7,\dots,14$. □

Corollary 2.2.
(1) *If* codim$(f)=6$ *then* $f\sim_{\mathcal{A}}(x,xy+y^6+\alpha y^8+\beta y^9)$. *If $a_{0,7}\neq 0$ then $\alpha=\frac{5a_{0,8}-3a_{0,7}^2}{5a_{0,7}^2}$ and $\beta=\frac{25a_{0,9}+14a_{0,7}^3-35a_{0,7}a_{0,8}}{25a_{0,7}^3}$. If $a_{0,7}=0$ then $\alpha=a_{0,8}$ and $\beta=a_{0,9}$. If $\alpha=0$ then β is the modulus and if $\alpha\neq 0$, define η by $\eta^2=\frac{1}{\alpha}$ then $\eta^3\beta$ is the modulus.*
(2) *If* codim$(f)=7$ *then* $f\sim_{\mathcal{A}}(x,xy+y^6+y^{14})$.
(3) *If* codim$(f)=8$ *then* $f\sim_{\mathcal{A}}(x,xy+y^6)$.

Proof. (2) and (3) are immediate consequences of Proposition-2.1 and Rieger's classification. To prove (1) we give explicitly the $\mathcal{A}$-equivalence. If $a_{0,7}\neq 0$ then it is easy to see that $f\sim_{\mathcal{A}}(x,xy+y^6+y^7+\frac{a_{0,8}}{a_{0,7}^2}y^8+\frac{a_{0,9}}{a_{0,7}^3}y^9+\dots)=(x,g)$. Using the morphisms φ respectively ψ defined by $\varphi^{-1}(x)=x-\frac{1}{5}g$ and $\varphi^{-1}(y)=5\sum_{v=1}^{8}(\frac{1}{5}y)^v$ respectively $\psi(x)=x+\frac{1}{5}y$ and $\psi(y)=y$, we obtain $f\sim_{\mathcal{A}}(x,xy+y^6+\alpha y^8+\beta y^9)$, $\alpha=a-\frac{3}{5}$, $\beta=b-\frac{7a}{5}+\frac{14}{25}$, since $\varphi^{-1}(xy+y^6+\alpha y^8+\beta y^9)=g$. This can be checked with Singular as follows:

```
ring R=(0,a,b),(x,y),ds;
poly g=xy+y6+y7+a*y8+b*y9;
poly h=xy+y6+(a-3/5)*y8+(b-7/5*a+14/25)*y9;
map phi_invers=R,x-1/5*g,y+1/5*y2+1/25*y3+1/125*y4+1/625*y5+1/3125*y6
                +1/15625*y7+1/78125*y8;
jet(phi_invers(h),9);
xy+y6+y7+(a)*y8+(b)*y9.
```

[3]In the library classifyMapGerms.lib the computation of the codimension is based on the computation of a special standard basis (vStd). We consider a ring with parameters $a_{0,7},\dots,a_{0,14}$ and variables x,y and compute the standard basis of $T_{\theta_f,id}$ depending on the parameters. In this way, we obtain the conditions for codim(f) to be 6, 7 respectively 8.

□

Proposition 2.3. *Let $f(x,y)$ be a map germ from $(\mathbb{K}^2,0)\to(\mathbb{K}^2,0)$. Suppose $\mu(\sum)=1$ and $m(f)=6$ then $f\sim_{\mathcal{A}}(x,xy^2+y^6+\sum_{i>6}a_{0,i}y^i)$ and* $\operatorname{codim}(f)=k+7$ *if $2k+7$ is minimal with $a_{0,2k+7}\neq 0$. If $a_{0,7}\neq 0$ then f is 9-determined and unimodal. For $k\geq 1$, f is not unimodal.*

Proof. We may assume that $f=(x,g(x,y))$ then $\sum=V(\frac{\partial g}{\partial y})$. If $\mu(\sum)=1$ then we have $g=b_{2,1}x^2y+b_{1,2}xy^2+b_{0,3}y^3+$ h.o.t. such that $3b_{2,1}b_{0,3}-b_{1,2}^2\neq 0$ otherwise $\mu(\frac{\partial g}{\partial y})\neq 1$ and since $m(f)=6$ this implies the coefficients of y^3,y^4 and y^5 must be zero but the coefficient of y^6 is not equal to zero. Also $b_{1,2}\neq 0$ otherwise $\mu(\frac{\partial g}{\partial y})\neq 1$. Then by using the transformation

$$y\to y-\frac{b_{2,1}}{2b_{1,2}}x$$

we can transform g into xy^2+ h.o.t. Now the transformation

$$y\to y-\frac{1}{2}(a_{k+1,1}x^k+a_{k,2}x^{k-1}y+\cdots+a_{2,k}xy^{k-1}+a_{1,k+1}y^k)$$

removes all the terms of degree $k+1$ which are divisible by x at the level of the $(k+1)-$jet, $k\geq 2$ and it transforms g into $xy^2+y^6+\sum_{j>6}a_{0,j}y^j$. A monomial basis of $\frac{<x,y>\mathbb{K}[[x,y]]^2}{T_{\theta_f,id}}$ is

$$\begin{pmatrix}0\\xy\end{pmatrix},\begin{pmatrix}0\\y\end{pmatrix},\ldots,\begin{pmatrix}0\\y^5\end{pmatrix},\begin{pmatrix}0\\y^7\end{pmatrix},\begin{pmatrix}0\\y^9\end{pmatrix},\ldots,\begin{pmatrix}0\\y^{2k+5}\end{pmatrix},\begin{pmatrix}0\\y^{2k+9}\end{pmatrix}.$$

It is proved in [9] that f is 9-determined if $a_{0,7}\neq 0$ and as a consequence that f is unimodal. It follows from Rieger's classification that for $k\geq 1$, f is not unimodal. □

Corollary 2.4. *If* $\operatorname{codim}(f)=7$ *then $f\sim_{\mathcal{A}}(x,xy^2+y^6+y^7+\alpha y^9)$ with $\alpha=\frac{a_{0,9}}{a_{0,7}^3}-\frac{5a_{0,8}}{4a_{0,7}^2}$.*

Proof. This is an immediate consequence of Proposition-2.3 that

$$(x,g)=(x,xy^2+y^6+y^7+ay^8+by^9)\sim_{\mathcal{A}}(x,xy^2+y^6+y^7+(b-\frac{5}{4}a)y^9)=(x,h),$$

since $\varphi^{-1}(h)=g \mod <x,y>^9$ with $\varphi^{-1}(x)=x-\frac{1}{2}ag$ and $\varphi^{-1}(y)=y+\frac{1}{4}ay^3+\frac{3}{2}(\frac{a}{4})^2y^5+\frac{5}{2}(\frac{a}{4})^3y^7$ and $\psi(x)=x+\frac{1}{2}ay$, $\psi(y)=g$. This can be checked with Singular as follows:

```
ring R=(0,a,b),(x,y),ds;
poly g=xy2+y6+y7+a*y8+b*y9;
poly h=xy2+y6+y7+(b-5/4*a)*y9;
map phi_invers=R,x-1/2*a*g,y+1/4*a*y3+3/32*a2*y5+5/128*a3*y7;
jet(phi_invers(h),9);
xy2+y6+y7+(a)*y8+(b)*y9.
```

□

Proposition 2.5. *Let $f(x,y)$ be a map germ from $(\mathbb{K}^2,0)\to(\mathbb{K}^2,0)$. Suppose $\mu(\sum)\geq 4$ and $m(f)=4$ then $f\sim_{\mathcal{A}}(x,g)$ with $g=y^4+\beta x^2y^2+\gamma x^3y+h.o.t.$*

Proof. We may assume that $f = (x, g(x, y))$ then $\sum = V(\frac{\partial g}{\partial y})$. It is not difficult to see that $\mu(\sum) \geq 4$ and $m(f) = 4$ implies $j^3(f) = (x, 0)$. If $j^3(f) = (x, 0)$, $\mu(\sum) \geq 4$ then $g = a_{0,4}y^4 + a_{1,3}xy^3 + a_{2,2}x^2y^2 + a_{3,1}x^3y + h.o.t$ and $m(f) = 4$ gives $a_{0,4} \neq 0$, so we can take $a_{0,4} = 1$. Then the transformation

$$y \to y - \frac{a_{1,3}}{4}x$$

transform g into $y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t.$ □

Corollary 2.6. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = 4$ then*

(1) *if* $\operatorname{codim}(f) = 7$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + x^3y^2)$, $\alpha \neq -\frac{3}{2}$;
(2) *if* $\operatorname{codim}(f) = 8$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2)$, $\alpha \neq -\frac{3}{2}$;
(3) *if* $\operatorname{codim}(f) = 9$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2)$, $\alpha \neq -\frac{3}{2}$;
(4) *if* $\operatorname{codim}(f) = d(f) + 3$ *and* $d(f) \geq 4$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^2y^2 + x^{d(f)}y)$.

Proof. $\mu(\sum) = 4$ implies $8\beta^3 + 27\gamma^2 \neq 0$.
If $\gamma \neq 0$, we obtain for $\alpha = \frac{\beta}{\xi^2}$ with $\xi^3 = \gamma$

$$f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + h.o.t.)$$

with $\alpha \neq -\frac{3}{2}$. In this case $y^4 + x^3y + \alpha x^2y^2 + h.o.t.$ is 6-determined (see [9]). It is not difficult to see that

$$(x, y^4 + x^3y + \alpha x^2y^2 + h.o.t.) \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + \lambda x^3y^2 + h.o.t).$$

$\lambda \neq 0$ iff $\operatorname{codim}(f) = 7$. If $\lambda = 0$, we obtain that

$$f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + \delta x^4y^2 + h.o.t).$$

$\delta \neq 0$ iff $\operatorname{codim}(f) = 8$, in this case $f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2)$, since f is 6-determined. If $\delta = 0$ then $f \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2)$, since f is 6-determined. In this case $\operatorname{codim}(f) = 9$. If $\gamma \neq 0$ then $d(f) = 3$.
If $\gamma = 0$ then $d(f) = k$, if $\operatorname{codim}(f) = k + 3$, $k \geq 4$ then $f \sim_{\mathcal{A}} (x, y^4 + x^2y^2 + x^ky)$, since f is $(k + 1)$-determined (see [9]). □

Corollary 2.7. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = 5$ then if* $\operatorname{codim}(f) = 7$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^3y^2)$.

Corollary 2.8. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = 6$ then if* $\operatorname{codim}(f) = 8$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^4y^2)$.

Corollary 2.9. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = 2k - 2$, where $k = 4, 5$ then if* $\operatorname{codim}(f) = k + l + 1$ *and* $d(f) = k$, $k - 1 \leq l \leq 2k - 1$, *then*[4] $f \sim_{\mathcal{A}} (x, y^4 + x^ky + x^ly^2)$.

Corollary 2.10. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = 7$ then if* $\operatorname{codim}(f) = l + 4$ *and* $d(f) = l$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y^2 + x^ly)$, *where* $l \geq 5$.

Corollary 2.11. *Let $f \sim_{\mathcal{A}} (x, g)$ with $g = y^4 + \beta x^2y^2 + \gamma x^3y + h.o.t$ and $\mu(\sum) = k + 1$ then if* $\operatorname{codim}(f) = k + 3$ *and* $d(f) = 3$ *then* $f \sim_{\mathcal{A}} (x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^ky)$.

[4] if $l = 2k - 1$ we can even obtain $f \sim_{\mathcal{A}} (x, y^4 + x^ky)$.

Corollary 2.12. *Let* $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ *be a map germ. Suppose* $\mu(\sum) \geq 4$, $m(f) = 4$ *and* $(\mathrm{codim}(f),\ 4,\ \mu(\sum),\ d(f))$ *are entries in Table* 1 *then* f *is unimodal.*

Proof of Corollaries 2.7 *to* 2.11*.* Corollaries 2.7 to 2.11 are an immediate consequence of Proposition-2.5 and similar to the proof of Corollary 2.6. Corollary 2.12 follows from Corollaries 2.7 to 2.11 and Rieger's classification. □

3. The Algorithm for the Classifier

The following algorithm is used for computing the type of the unimodal map germs from the plane to the plane.

Algorithm 1 ModulusA

Input: $g(x, y) = \sum_{i+j\geq 2} a_{ij}x^i y^j$ with non-degenerate $2-$jet, $\mathrm{ord}(g(0, y)) = 6$ and $\mathrm{codim}((x, g)) = 6$.
Output: (α, β) such that $(x, g) \sim_{\mathcal{A}} (x, xy + y^6 + \alpha y^8 + \beta y^9)$.

Choose $\varphi : \mathbb{K}[[x, y]] \to \mathbb{K}[[x, y]]$ an automorphism with $\varphi(x) = x$ such that $\varphi(g) = xy + y^6 + \sum_{7\leq i\leq 9} a_{0i}y^i \mod < x, y >^{10}$;
if $a_{07} \neq 0$ **then**
  $\alpha = \frac{5a_{08}-3a_{07}^2}{5a_{07}^2}$ and $\beta = \frac{25a_{09}+14a_{07}^3-35a_{07}a_{08}}{25a_{07}^3}$;
if $a_{07} = 0$ **then**
  $\alpha = a_{08}$ and $\beta = a_{09}$;
if $\alpha = 0$ **then**
  return $(0, 1)$.
Choose η such that $\eta^2 = \frac{1}{\alpha}$;
return $(1, \eta^3\beta)$.

Algorithm 2 ModulusB

Input: $g(x, y) = \sum_{i+j\geq 3} a_{ij}x^i y^j$, $\mathrm{ord}(g(0, y)) = 6$, $\mu(\sum) = 1$ and $\mathrm{codim}((x, g)) = 7$.
Output: α such that $(x, g) \sim_{\mathcal{A}} (x, xy^2 + y^6 + y^7 + \alpha y^9)$.

Choose $\varphi : \mathbb{K}[[x, y]] \to \mathbb{K}[[x, y]]$ an automorphism with $\varphi(x) = x$ such that $\varphi(g) = xy^2 + y^6 + \sum_{7\leq i\leq 9} a_{0i}y^i \mod < x, y >^{10}$;
return $(\frac{a_{09}}{a_{07}} - \frac{5a_{08}}{4a_{07}})$.

Algorithm 3 ModulusC

Input: $g(x, y) = \sum_{i+j\geq 4} a_{ij}x^i y^j$, $\mu(\sum) = 4$ and $\mathrm{codim}((x, g)) \geq 7$.
Output: α such that $(x, g) \sim_{\mathcal{A}} (x, y^4 + x^3y + \alpha x^2y^2 + h.o.t.)$.

Choose a linear automorphism $\varphi : \mathbb{K}[[x, y]] \to \mathbb{K}[[x, y]]$ with $\varphi(x) = x$ such that the $4-$th jet of $\varphi(g)$ is $y^4 + \beta x^2y^2 + \gamma x^3y$;
Choose ξ with $\xi^3 = \gamma$;
return $(\frac{\beta}{\xi^2})$.

Algorithm 4 UnimodalMaps

Input: A germ $f(x,y) = (f_1(x,y), f_2(x,y))$ from the plane to the plane .
Output: $(x, g(x,y))$, the type or 0 if f is not unimodal.

if $\mathrm{ord}(f_1) > 1$ and $\mathrm{ord}(f_2) > 1$ **then**
    return 0;
Compute $c = \mathrm{codim}(f)$, the codimension of f.
Transform f into $(x, g(x,y)) \bmod < x, y >^{c+8}$.
Compute $m(f)$, the multiplicity of f, $\mu(\Sigma)$, the Milnor number of critical set and $d(f)$, the double fold number.
if $\mu(\Sigma) = 0$ and $m(f) = 6$ **then**
    if $\mathrm{codim}(f) = 6$ **then**
        Compute $(\alpha, \beta) = \mathrm{modulusA}(g)$;
        return $(x, xy + y^6 + \alpha y^8 + \beta y^9)$;
    if $\mathrm{codim}(f) = 7$ **then**
        return $(x, xy + y^6 + y^{14})$;
    if $\mathrm{codim}(f) = 8$ **then**
        return $(x, xy + y^6)$;
if $\mu(\Sigma) = 1$ and $m(f) = 6$ **then**
    if $\mathrm{codim}(f) = 7$ **then**
        Compute $\alpha = \mathrm{modulusB}(g)$;
        return $(x, xy^2 + y^6 + y^7 + \alpha y^9)$;
if $\mu(\Sigma) = 4$ and $m(f) = 4$ **then**
    if $\mathrm{codim}(f) = 7$ and $d(f) = 3$ **then**
        Compute $\alpha = \mathrm{modulusC}(g)$;
        return $(x, y^4 + x^3y + \alpha x^2y^2 + x^3y^2)$;
    if $\mathrm{codim}(f) = 8$ and $d(f) = 3$ **then**
        Compute $\alpha = \mathrm{modulusC}(g)$;
        return $(x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2)$;
    if $\mathrm{codim}(f) = 9$ and $d(f) = 3$ **then**
        Compute $\alpha = \mathrm{modulusC}(g)$;
        return $(x, y^4 + x^3y + \alpha x^2y^2)$;
    if $\mathrm{codim}(f) = k + 3$, $d(f) = k$ and $k \geq 4$ **then**
        return $(x, y^4 + x^2y^2 + x^ky)$;
if $\mu(\Sigma) = 5$, $m(f) = 4$, $\mathrm{codim}(f) = 7$ and $d(f) = 3$ **then**
    return $(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^3y^2)$;
if $\mu(\Sigma) = 6$, $m(f) = 4$, $\mathrm{codim}(f) = 8$ and $d(f) = 3$ **then**
    return $(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^4y^2)$;
if $\mu(\Sigma) = 7$, $m(f) = 4$, $d(f) \geq 5$, $\mathrm{codim}(f) = d(f) + 4$ **then**
    return $(x, y^4 + x^3y^2 + x^{d(f)}y)$;
if $\mu(\Sigma) \geq 7$, $m(f) = 4$, $\mathrm{codim}(f) = \mu(\Sigma) + 2$, $d(f) = 3$ **then**
    return $(x, y^4 + x^3y - \frac{3}{2}x^2y^2 + x^{\mu(\Sigma)-1}y)$;
if $\mu(\Sigma) = 2d(f) - 2$, $m(f) = 4$, $\mathrm{codim}(f) = d(f) + l - 1$, $d(f) = 4$ or 5 **then**
    return $(x, y^4 + x^{d(f)}y + x^ly^2)$;
return 0.

4. Singular Examples

The algorithms described in Section 3 are implemented in Singular in the library classifyMapGerms.lib. We give some examples.

```
LIB"classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
ideal I=x+y+x2y+2xy2+y3+x2y2+2xy3+y4+xy6+y7+6xy7+6y8+15xy8
+15y9+21xy9+21y10+24xy10+24y11+42xy11+42y12+85xy12+85y13
+126xy13+126y14+126xy14+126y15+84xy15+84y16+36xy16+36y17
+9xy17+9y18+xy18+y19,
x2+3xy+2y2+xy2+y3+y6+6y7+15y8+21y9+24y10+42y11+85y12+126y13
+126y14+84y15+36y16+9y17+y18;

classifyUnimodalMaps(I);
_[1]=x
_[2]=xy+y6+y9

I=x+y, xy2+y3+2xy3+2y4+xy4+y5+y6+7y7+22y8+118y9+743y10+2813y11+6490y12
       +9709y13+9703y14+6468y15+2772y16+693y17+77y18

classifyUnimodalMaps(I);
_[1]=x
_[2]=xy2+y6+y7+77y9
```

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