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Weakly quasihomogeneous hypersurface singularities in positive characteristic

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1 Introduction

1971 *K*. Saito gave a characterization of hypersurface singularities admitting a good $\mathbb{C}^*$-action (cf. [S]):
Let $(X, 0) \subseteq (\mathbb{C}^n, 0)$ be the germ of an isolated hypersurface singularity defined by $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. The following conditions are equivalent:

(1) $(X, 0)$ admits a good $\mathbb{C}^*$- action[1]

(2) $f \in \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$

(3) $(X, 0)$ admits a non-nilpotent vectorfield.

We will generalize this characterization for hypersurface singularities defined over an algebraically closed field K of characteristic $p > 0$.

The following example shows that the theorem of Saito is wrong in the characteristic $p > 0$:

Let $f = x^5 + y^{11} + x^3 y^9$ and K be a field of characteristic 23. Then singularity defined by $f = 0$ is not quasihomogeneous, but $55f = 11x\frac{\partial f}{\partial x} + 5y\frac{\partial f}{\partial y}$. This fact requires to weaken the notion "quasihomogeneous" to obtain a similar characterization in the characteristic $p > 0$.

Let K be an algebraically closed field of the characteristic $p > 0$ and $\mathbb{F}_p$ its prime field. If there is no possibility of confusion, we will denote for an integer a its residue class $a \bmod p$ in $\mathbb{F}_p$ also by a and conversely for an element $a \in \mathbb{F}_p$ the representative $\tilde{a}$, $0 \leqq \tilde{a} < p$, also by a.

Let $x = (x_1, \ldots, x_n)$.

[1] $(X, 0)$ admits a good $\mathbb{C}^*$-action (also called quasihomogeneous) if $\hat{\mathcal{O}}_{X,0} \simeq \mathbb{C}[[x_1, \ldots, x_n]]/I$, $I = (p_1, \ldots, p_a)$ an ideal generated by quasihomogeneous polynomials p_i of positive degree d_i with respect to the positive weights $w_1, \ldots, w_n$ ($w_i, d_i \in \mathbb{Z}$), i.e. $p_i(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^{d_i} p_i$

Definition 1.1 $f \in K[[x]]$ is weakly quasihomogeneous with respect to the weight $(w_\circ : w_1 : \dots : w_n) \in \mathbb{P}^n_{\mathbb{F}_p}$ if one of the following equivalent conditions is satisfied:

(1) If $f = \sum a_{v_1 \dots v_n} x_1^{v_1} \cdot \ldots \cdot x_n^{v_n}$ and $a_{v_1 \dots v_n} \neq 0$, then $w_1 v_1 + \cdots + w_n v_n = w_\circ$ (in $\mathbb{F}_p$).

(2) $w_\circ f = \sum w_i x_i \dfrac{\partial f}{\partial x_i}$

(3) $f = \sum\limits_{v \geqq 0} f_{w_\circ + vp}$, $f_{w_0 + vp}$ is a quasihomogeneous polynomial of degree $w_\circ + vp$ with respect to the weights $w_1, \dots, w_n$.

Definition 1.2 The local ring $A = K[[x]]/f$ is weakly quasihomogeneous if there is an isomorphism $A \simeq K[[x]]/g$ and if g is weakly quasihomogeneous with respect to the weight $(w_\circ : w_1 : \dots w_n) \in \mathbb{P}^n_{\mathbb{F}_p}$.

$(w_\circ : \dots w_n)$ is called a weight of A. If A has a weight $(w_\circ : 1 : \dots : 1)$, then it is called weakly homogeneous.

Theorem 1.3 *Let $A = K[[x]]/f$ be the complete local ring of an isolated singularity and m its maximal ideal. The following conditions are equivalent:*

(1) *A is weakly quasihomogeneous*
(2) *there is a derivation $\delta \in \mathrm{Der}_K\, m$ which is not nilpotent*[2]

If, furthermore, the multiplicity of A is at least 3 and $f \in (x)\left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$, then the weight $(w_\circ : w_1 : \dots w_n)$ is uniquely determined (modulo permutation of $w_1, \dots, w_n$) and $w_\circ \neq 0$.

If $f \notin (x)\left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$ and $(w_\circ : w_1 : \dots w_n)$ is a weight for A, then $w_\circ = 0$ and A may have more than one weight.

The following examples show that the weight of A is not uniquely determined in general:

(1) For instance, $K[[x_1, x_2]]/x_1x_2$ has weights $(1 : a : 1 - a)$, $(0 : a : p - a)$ for $a \in \mathbb{F}_p$.
(2) $K[[x_1, x_2, x_3]]/x_1x_2x_3 + (x_1 + x_2 + x_3)^p$ has the weights $(0 : 1 : p - 1 : 0)$, $(0 : 1 : 0 : p - 1)$, $(0 : 1 : 1 : p - 2)$ for instance.

Remark 1.4 In characteristic $p > 0$ it is possible that there is a $\delta \in \mathrm{Der}_K A$ and $\delta(m) \nsubseteq m : f = x_1^p + x_2^a$ and $\delta = \dfrac{\partial}{\partial x_1}$.

This is not possible in characteristic 0 (cf. [S]). Such singularities are always weakly quasihomogeneous: If $\delta(x_i) \notin m$, then $x_i \delta \in \mathrm{Der}_K\, m$ and not nilpotent.

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[2] $\delta \in \mathrm{Der}_K\, m$ is nilpotent (resp. semi-simple) if for all i the endomorphism of m/m^{i+1} defined by δ is nilpotent (resp. semi-simple). In characteristic 0 the existence of such a derivation is equivalent to $f \in \left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$. This is not true in characteristic $p > 0$, cf. Remark 1.4

Definition 1.5 The local ring A is exceptional if A is not smooth and there is a $\delta \in \mathrm{Der}_K A$ and $\delta(m) \nsubseteq m$.

In Chapter 3 we apply Theorem 1.3. to characterize the characteristic exponents of an irreducible plane curve sincularity (for definition and properties cf. [C]).

Theorem 1.6 *Let $\beta_\circ, \ldots, \beta_g$ be the characteristic exponents of an irreducible weakly quasihomogeneous plane curve singularity. Let $t \geqq 1$ be minimal such that $gcd(\beta_\circ, \ldots, \beta_t) \not\equiv 0 \bmod p$, then $\beta_i \equiv \beta_{i+1} \bmod p$ for $i \geqq t$.*

This theorem corresponds to the fact that for characteristic 0 there are only two characteristic exponents if the singularity is quasihomogeneous. In case of exceptional singularities this theorem was already proved by A. Campillo (cf. [C1]). Exceptional curves are called strange branches in Campillo's paper.
In Chapter 4 we compute the Tjurina number $\tau(A)$ of a weakly quasihomogeneous plane curve singularity with local ring A in terms of the multiplicity sequence.

Definition 1.7 Let A be the local ring of a plane curve singularity with r branches, let δ be the length of A in its normalization and m the multiplicity. The Milnor number $\mu(A)$ is defined by $\mu(A) = 2\delta + 1 - r$ if A is not exceptional, and by $\mu(A) = 2\delta + m - r$ if A is exceptional.

In characteristic 0 one has $\mu(A) = \tau(A)$ iff A is quasihomogeneous (cf. [G-M-P]).

Theorem 1.8 *Let A be the local ring of a weakly quasihomogeneous irreducible plane curve singularity, then $\tau(A) \geqq \mu(A)$. If the multiplicity of A is smaller than p, then $\tau(A) = \mu(A)$. $\tau(A) - \mu(A)$ can be expressed in terms of the multiplicity sequence of the resolution of A.*

The proof is based on the following lemma:

Lemma 1.9 *Let A be as in Theorem 1.8, $\tilde{A}$ be the blowing up of A, m (resp. $\tilde{m}$) the multiplicity of A (resp. $\tilde{A}$) then*

$$(1) \quad \tau(A) = \mu(A) + \tau(\tilde{A}) - \mu(\tilde{A}) + \begin{cases} \tilde{m} - m & \text{if } A \text{ is weakly homogeneous} \\ \tilde{m} & \text{if } \tilde{A} \text{ is exceptional and } A \\ & \text{is not weakly homogeneous} \\ 0 & \text{else} \end{cases}$$

if $\tilde{m} \geqq 2$.

$$(2) \qquad \tau(A) = \mu(A) + \begin{cases} 1 & m \equiv 0 \bmod p \\ 0 & \text{else} \end{cases}$$

if $\tilde{m} = 1$.

Corollary 1.10 *Let A be as in Theorem 1.8. If $p \geqq \tau(A) - \beta_\circ - \beta_1$, then A is quasihomogeneous.*

The case of a reducible singularity can be reduced to Theorem 1.8:

Theorem 1.11 *Let A be the local ring of a weakly quasihomogeneous plane curve singularity and $A_1, \dots, A_r$ the local rings of the branches of A, then*

(1) $\mu(A) + \sum_i \left(\tau(A_i) - \mu(A_i)\right) \geqq \tau(A) \geqq \mu(A)$

(2) *If the branches of A are not exceptional or if A is exceptional, then* $\tau(A) = \mu(A) + \sum_i \left(\tau(A_i) - \mu(A_i)\right)$

(3) *If $p > m(A)$, then $\tau(A) = \mu(A)$.*

2 Characterization of weakly quasihomogeneous singularities

In this chapter we will prove Theorem 1.3.:
The proof is based on the following lemma (cf. [S-W]):

Lemma 2.1 *Let $A = K[[x]]$ and m be the maximal ideal.*

(1) *For $\delta \in \mathrm{Der}_K\, m$ there are $\delta_s, \delta_m \in \mathrm{Der}_K\, m$ with the following properties:*
(1.1) *δ_s is semi-simple δ_n is nilpotent*
(1.2) *$\delta = \delta_s + \delta_n$ and $[\delta_s, \delta_n] = 0$.*

(2) *$\delta \in \mathrm{Der}_K(m)$ is semi-simple iff there is a coordinate system $y = (y_1, \dots, y_n)$ of A such that $\delta(y_i) = \alpha_i y_i$, $\alpha_1, \dots, \alpha_n$ the eigenvalues of the endomorphism of m/m^2 defined by δ.*

(3) *If $\delta(f) = uf$ for $u, f \in K[[x]]$, then there is a $u_1 \in K[[x]]$ such that $\delta_s(f) = u_1 f$. u_1 is a unit iff u is a unit.*

Remark 2.2 This lemma is essentially the same as Lemma 1.5 in [S-W]. In [S-W] the characteristic of the field K is supposed to be 0. But Lemma 1.5 and its proof hold also in characteristic $p > 0$.

Proof of Theorem 1.3 (1) implies (2) is obvious by definition. To prove (2) implies (1) we consider $\delta \in \mathrm{Der}_K K[[x]]$, δ not nilpotent and $\delta(f) = uf$, $u \in K[[x]]$. We may assume that $\delta(m) \subseteq m$. Namely, if $\delta(x_i) \notin m$ for some i, then $x_i\delta$ is again not nilpotent. If u is a unit, we may assume furthermore $u = 1$. Now we can apply 2.1. to restrict our situation to the following two cases:

Let $\delta = \sum \overline{w}_i x_i \frac{\partial}{\partial x_i}$, $\overline{w}_i \in K$, and $\delta(f) = uf$.

Case 1 $u = 1$

Case 2 $u \in m$.

The second case is only possible in characteristic $p > 0$. We have to prove in the first case that there are $w_1, \dots, w_n \in \mathbb{F}_p$ such that $\sum w_i x_i \frac{\partial f}{\partial x_i} = f$, with uniquely determined $w_1, \dots, w_n$ if the multiplicity is at least 3 and $\sum w_i x_i \frac{\partial (vf(\varphi))}{\partial x_i} = 0$ for a suitable unit $v \in K[[x]]$ and a suitable automorphism φ in the second case.

Case 1 Let $f = \sum a_{v_1 \dots v_n} x_1^{v_1} \cdot \dots \cdot x_n^{v_n}$, then $\sum \overline{w}_i x_i \frac{\partial f}{\partial x_i} = f$ implies $\sum v_i \overline{w}_i = 1$ for all $v_1, \dots, v_n$ with $a_{v_1 \dots v_n} \neq 0$. This system of linear equations, which has the solution $(\overline{w}_1, \dots, \overline{w}_n) \in K^n$, has also a solution $(w_1, \dots, w_n) \in \mathbb{F}_p^n$, i.e. $\sum w_i x_i \frac{\partial f}{\partial x_i} = f$, which means that f is weakly quasihomogeneous.

Now assume that $f \in (x)^3$ and let $\overline{\delta} \in \mathrm{Der}_K\, m$ be semi-simple and $\overline{\delta}(f) = \overline{u} f$. We have to show that $\overline{u}$ is a unit and that the endomorphism of m/m^2 defined by $\overline{\delta}$ has $\overline{u}(0)w_1, \dots, \overline{u}(0)w_n$ as eigenvalues. Because $f \in \left(\frac{\partial f}{\partial x_i}, \dots, \frac{\partial f}{\partial x_n}\right)$ and f defines an isolated singularity, $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ is a regular sequence.

Now we have $\sum (\overline{w}_i x_i \overline{u} - \overline{\delta}(x_i)) \frac{\partial f}{\partial x_i} = 0$. This implies $\overline{w}_i x_i \overline{u} - \overline{\delta}(x_i) \in (\frac{\partial f}{\partial x_i}, \dots, \frac{\partial f}{\partial x_n})$. Because of $f \in (x)^3$ we get $\overline{w}_i x_i \overline{u} - \delta(x_i) \in (x)^2$, i.e. the endomorphism of m/m^2 defined by $\overline{\delta}$ has the eigenvalues $\overline{u}(0)w_1, \dots, \overline{u}(0)w_n$. This proves the theorem in the first case.

Case 2 Without restriction of generality we may assume that $\overline{w}_1 = 1$. Suppose $\overline{w}_1, \dots, \overline{w}_r \in \mathbb{F}_p$ for some $r \geqq 1$ and $\overline{w}_{r+1}, \dots, \overline{w}_n \notin \mathbb{F}_p$. If $r < n$, we conclude

$$(\delta - \delta^p)(f) = \sum_{i=r+1}^{n} (\overline{w}_i - \overline{w}_i^p) x_i \frac{\partial f}{\partial x_i} = u_1 f$$

for a suitable $u_1 \in m$.

Using this procedure one can deduce by induction that there are $w_1, \dots, w_n \in \mathbb{F}_p$ and $u \in m$ such that $\sum w_i x_i \frac{\partial f}{\partial x_i} = uf$. Let $\overline{\delta} := \sum w_i x_i \frac{\partial}{\partial x_i}$. We will prove now that there is a unit $v \in K[[x]]$ and an automorphism φ of $K[[x]]$ such that for $g = vf(\varphi)$ we have

$$\overline{\delta}(g) = 0.$$

Let $u = \sum_{v \geqq r} u^{(v)}$ and $f = \sum_{v \geqq s} f^{(v)}$ such that $\overline{\delta}(u^{(v)}) = v \cdot u^{(v)}$ and $\overline{\delta}(f^{(v)}) = v f^{(v)}$. Because of $u \in m$ it follows from $\overline{\delta}(f) = uf$ that $r > 0$ and $r \not\equiv 0 \bmod p$. Let $e = 1 + \frac{1}{r} u^{(r)}$ and $f_1 = e^{-1} f$ and $u_1 = u - e^{-1} \overline{\delta}(e)$ then $\overline{\delta}(f_1) = u_1 f_1$ and $u_1 = \sum_{v \geqq r+1} u_1^{(v)}$, $\overline{\delta}(u_1^{(v)}) = v u_1^{(v)}$. Let k be the index of determinancy of f, then we may assume, using the argument above, that $\sum_{v \geqq s+r} f^{(v)} \subseteq m^k$.

Now $\overline{\delta}(f) = u \cdot f$ implies

$$(s+r)f^{(s+r)} = u^{(r)} f^{(s)}$$
$$vf^{(\nu)} = 0 \qquad s \leqq \nu \leqq s+r-1 .$$

This implies that for $g = \sum_{\nu=s}^{s+r-1} f^{(\nu)}$ we have $\overline{\delta}(g) = 0$ and g is contact equivalent to f, i.e. $K[[x]]/f \simeq K[[x]]/g$. This proves the second step of the theorem.

3 Behaviour of the characteristic exponents

The aim of this chapter is to prove Theorem 1.6.

To begin with we study the blowing up of a weakly quasihomogeneous singularity. Let $\widetilde{A} = K[[x, y]]/\widetilde{f}$ be the blowing up of A and $m(A)$ (resp. $m(\widetilde{A})$) the multiplicity of A (resp. $\widetilde{A}$). Let us assume that the multiplicity of A is at least 2. Because f is irreducible we may assume $f = (x + \beta y)^m + \sum_{i+j>m} a_{ij} x^i y^i$ and $w_\circ f = w_1 x \frac{\partial f}{\partial x} + w_2 y \frac{\partial f}{\partial y}$, $(w_\circ : w_1 : w_2) \in \mathbb{P}^2_{\mathbb{F}_p}$ and $\widetilde{f} = \frac{1}{y^m} f(xy - \beta y, y)$.

Lemma 3.1 *$\widetilde{A}$ is weakly quasihomogeneous. Moreover, the following holds:*

(1) *A is exceptional iff $w_1 w_2 = 0$;*

(2) *if $w_1 = w_2$ or $w_2 = 0$ or $\beta = 0$, then $\widetilde{f}$ is weakly quasihomogeneous with weight $(w_\circ - m w_2 : w_1 - w_2 : w_2)$;*

(3) *if $w_1 \neq w_2$ and $w_2 \neq 0$ and $\beta \neq 0$, then $\widetilde{f}(x, (x-\beta)^r y)$ is weakly quasihomogeneous with weight $(0 : 1 : 0)$ for a suitable integer r with $(r+1)w_2 = r w_1$;*

(4) *if $w_1 = w_2$, i.e. A is weakly homogeneous, then $\widetilde{A}$ is exceptional;*

(5) *if $w_2 = 0$, then $\widetilde{A}$ is exceptional or smooth and $m \equiv 0 \bmod p$;*

(6) *if $w_1 = 0$ and $\beta = 0$, then $\widetilde{A}$ is not exceptional.*

Proof. Because A is irreducible, $w_1 w_2 = 0$ implies that A is exceptional. Let A be exceptional and $\delta \in \mathrm{Der}_K K[[x, y]]$, $u \in K[[x, y]]$ such that $\delta(f) = uf$ and $\delta((x, y)) \nsubseteq (x, y)$.

We may assume $\delta(x) = 1$. If $w_1 w_2 \neq 0$, then we have

$$(w_\circ - w_1 x u) f = (w_2 y - w_1 x \delta(y)) \frac{\partial f}{\partial y} .$$

Because f is irreducible, $m(A) \geqq 2$, and $w_1 w_2 \neq 0$, this implies $w_\circ = 0$ and $s(w_2 y - w_1 x \delta(y)) = u$ for a suitable $s \in K[[x, y]]$ i.e. $-w_1 x s f = \frac{\partial f}{\partial y}$. On the other hand, we have $w_1 x \frac{\partial f}{\partial x} = -w_2 y \frac{\partial f}{\partial y}$. This implies $w_2 y s f = \frac{\partial f}{\partial x}$, which contradicts the fact that f defines an isolated singularity and proves (1). Now $\tilde{f} = x^m + \sum_{i+j>m} a_{ij}(x-\beta)^i y^{i+j-m}$ implies

$$(w_\circ - w_2 m)\tilde{f} = (w_1 - w_2)(x-\beta)\frac{\partial \tilde{f}}{\partial y}.$$

This implies (2) and because of (1) also (4), (5), (6). Let $w_1 \neq w_2$, $w_2 \neq 0$, $\beta \neq 0$ and choose a positive integer r such that $(r+1)w_2 \equiv r w_1 \bmod p$. Then $\tilde{f}(x, (x-\beta)^r y) = x^m + \sum_{i+j>m} a_{ij}(x-\beta)^{(r+1)i+rj-rm} y^{i+j-m}$. Now $w_1 \neq w_2$ and $\beta \neq 0$ implies $m \equiv 0 \bmod p$, and $a_{ij} \neq 0$ implies $w_1 i + w_2 j \equiv 0 \bmod p$, i.e. because of $w_2 \neq 0$ and the choice of r, $(r+1)i + rj \equiv 0 \bmod p$. This implies $\frac{\partial}{\partial x}\tilde{f}(x, (x-\beta)^r y) = 0$ and proves (3).

For the proof of Theorem 1.6. we also need some information about the behaviour of the characteristic exponents in the blowing up and their relations to the multiplicity sequence, which is studied in [C].

Let $(m_1, r_1), \ldots, (m_l, r_l)$ be the multiplicity sequence of the resolution of A, i.e. in the sequence of blowing ups in the resolution of A, the multiplicity m_i occurs r_i times. Let $\beta_\circ, \ldots, \beta_g$ be the characteristic exponents of A and $\tilde{\beta}_0, \ldots, \tilde{\beta}_{\tilde{g}}$ be the characteristic exponents of $\tilde{A}$. Then the following holds (cf. [C]):

Lemma 3.2 (1) $\beta_\circ = m_1$, $\beta_1 = r_1 m_1 + m_2$.

(2) *Let $i_1, \ldots, i_{g+1}, g$ be inductively defined by $i_1 = 1$, $gcd(m_{i_{k-1}}, m_{i_{k-1}+1}) = m_{i_k}$ and $i_{g+1} = 1$, then $\beta_k - \beta_{k-1} = r_{i_k} m_{i_k} + m_{i_k+1} - m_{i_k-1}$, $k = 2, \ldots, g$. Furthermore, $r_v m_v + m_{v+1} - m_{v-1} = 0$ for $v \notin \{i_1, \ldots, i_g\}$,*

(3) *If $g = \tilde{g}$, then $\tilde{\beta}_i - \tilde{\beta}_{i-1} = \beta_i - \beta_{i-1}$ for $i \geqq 2$; and $gcd(\beta_\circ, \ldots, \beta_t) = gcd(\tilde{\beta}_\circ, \ldots, \beta_t)$ for $t \geqq 2$. If $g = \tilde{g}+1$, then $\tilde{\beta}_i - \tilde{\beta}_{i-1} = \beta_{i+1} - \beta_i$ for $i \geqq 2$; and $gcd(\beta_\circ, \ldots, \beta_t) = gcd(\tilde{\beta}_\circ, \ldots, \tilde{\beta}_{t-1})$ for $t \geqq 2$.*

Proof of Theorem 1.6 We prove the theorem using induction on $\sum r_i$.

We may assume the theorem to be true for the blowing up $\tilde{A}$ of A. We may further assume that $\tilde{A}$ is not smooth, otherwise the theorem is clear. If $t > 1$ or $t = 1$ and $g = \tilde{g}$, the theorem is a consequence of (3) of Lemma 3.2. If $t = 1$ and $g = \tilde{g} + 1$, then (3) of Lemma 3.2. implies $\beta_i \equiv \beta_{i+1} \bmod p$ for $i \geqq 2$. It remains to prove that $\beta_2 \equiv \beta_1 \bmod p$, i.e. $r_2 m_2 + m_3 - m_1 = \tilde{\beta}_1 - m_1 \equiv 0 \bmod p$ because of (2) of Lemma 3.2.

We may assume now that $A = K[[x, y]]/f$, $f = (x + \beta y)^{m_1} + \sum_{i+j>m_1} a_{ij} x^i y^i$ and $w_\circ f = w_1 x \frac{\partial f}{\partial x} + w_2 y \frac{\partial f}{\partial y}$. Because of $g = \tilde{g} + 1$ we know that m_2 divides m_1 (Lemma 3.2.(2)). Because of $t = 1$ this implies that $m_2 \not\equiv 0 \bmod p$. If $\beta \neq 0$ and $w_1 \neq w_2$, then $m_1 \equiv 0 \bmod p$. On the other hand, using Lemma 3.1. and the fact that $m_2 \not\equiv 0 \bmod p$ and $g = \tilde{g} + 1$ we may assume that $\tilde{A} \cong K[[x, y]]/h$,

$$h = y^{m_2} + \sum_{i+j>m_2} h_{ij} x^i y^i \quad \text{and} \quad \frac{\partial h}{\partial x} = 0 .$$

This implies $m_1 = \operatorname{ord}_x h(x, 0) \bmod p$ in particular. If $w_1 = w_2$, we may assume $\beta = 0$. If $\beta = 0$ and $\tilde{f} = \frac{1}{y^{m_1}} f(xy, y)$, $\tilde{A} = K[[x, y]]/\tilde{f}$, then $m_1 = \operatorname{ord}_x \tilde{f}(x, 0)$.

It remains to prove the following lemma and to apply it to $\tilde{f}$:

Lemma 3.3 *Let* $f = y^{m_1} + \sum_{i+j>m_1} a_{ij} x^i y^i$ *be irreducible and weakly quasi-homogeneous with characteristic exponents* $\beta_o, \ldots, \beta_g, g \geqq 1$. *Suppose that* $m_1 \not\equiv 0 \bmod p$, *then* $\beta_1 \equiv \operatorname{ord}_x f(x, 0) \bmod p$.

Proof. Let $(m_1, r_1), \ldots, (m_l, r_l)$ be the corresponding multiplicity sequence. We prove the lemma using induction on r_1. If $r_1 = 1$, then $\beta_1 = m_1 + m_2 = \operatorname{ord}_x f(x, 0)$. If $r_1 > 1$, we consider $\tilde{f} = \frac{1}{x^{m_1}} f(x, xy)$ the blowing up of f. The homogeneous part of degree m_1 of $\tilde{f}$ is $(\alpha x + y)^{m_1}$ because $\tilde{f}$ is irreducible. If $\alpha = 0$, then $\operatorname{ord}_x \tilde{f}(x, 0) + m_1 = \operatorname{ord}_x f(x, 0)$ and the lemma is proved by the induction hypothesis

$$\tilde{\beta}_1 = (r_1 - 1)m_1 + m_2 \equiv \operatorname{ord}_x \tilde{f}(x, 0) \bmod p .$$

If $\alpha \neq 0$, we use the fact the $\tilde{f}$ is weakly quasihomogeneous (Lemma 3.1.). Because of $\alpha \neq 0$ and $m_1 \not\equiv 0 \bmod p$ we know that $\tilde{f}$ is weakly homogeneous, i.e. there is a $w \in \mathbb{F}_p$ such that $\tilde{f} = wx\frac{\partial f}{\partial x} + wy\frac{\partial f}{\partial y}$.

Then $g := \tilde{f}(x, y - \alpha x)$ also satisfies $g = wx\frac{\partial g}{\partial x} + wy\frac{\partial g}{\partial y}$. Using the induction hypothesis we obtain

$$(r_1 - 1)m_1 + m_2 \equiv \operatorname{ord}_x g(x, 0) \bmod p .$$

Because g is weakly homogeneous we have

$$m_1 \equiv \operatorname{ord}_x (g(x, 0) \bmod p .$$

Now let $m_o = \operatorname{ord}_x f(x, 0)$, then $x^{m_o - m_1}$ occurs in $\tilde{f}$ as a monomial with a nonzero coefficient, i.e. $m_o - m_1 \equiv m_1 \bmod p$ because $\tilde{f}$ is weakly homogeneous. This implies $\operatorname{ord}_x f(x, 0) \equiv \operatorname{ord}_x g(x, 0) \cdot m_1 \bmod p$ and proves the lemma.

4 The Tjurina number of weakly quasihomogeneous plane curve singularities

In the first part of this chapter we will consider irreducible plane curve singularities and prove Theorem 1.8.

Let $A = K[[x, y]]/f$ be the local ring of an irreducible weakly quasihomogeneous plane curve singularity with multiplicity $m(A)$. Let $A \to \overline{A} = K[[t]]$ be the normalization. Then, for the Tjurina number of A $\tau(A) = \dim_K K[[x, y]]/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ the following holds:

Lemma 4.1

$$\tau(A) = \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - \begin{cases} 0 & \textit{if } A \textit{ is exceptional} \\ m(A) - 1 & \textit{else}. \end{cases}$$

Proof. Using Theorem 1.3. we may assume that

$$w_\circ f = w_1 x \frac{\partial f}{\partial x} + w_2 y \frac{\partial f}{\partial y} \quad \text{for} \quad (w_\circ : w_1 : w_2) \in \mathbb{P}^2_{\mathbb{F}_p}.$$

We may also assume that $m(A) \geqq 2$ otherwise the lemma is trivial. Furthermore, we know that A is exceptional iff $w_1 w_2 = 0$ (Lemma 3.1.(1)). Changing coordinates, if necessary, we may assume that $\frac{\partial f}{\partial x} \neq 0$ and $\operatorname{ord}_t \frac{\partial f}{\partial x} \leqq \operatorname{ord}_t \frac{\partial f}{\partial y}$.

By the definition of $\tau(A)$ we have $\tau(A) = \dim_K A/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Now let us consider

$$\overline{A} \supset \frac{\partial f}{\partial x}\overline{A} \supset \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) A$$

and

$$\overline{A} \supset A \supset \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) A.$$

We obtain

$$\begin{aligned}\tau(A) + \delta(A) &= \operatorname{ord}_t \frac{\partial f}{\partial x} + \dim_K \frac{\partial f}{\partial x}\overline{A}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) A \\ &= \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) + \dim_K \overline{A}/\left(1, \frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x}\right) A,\end{aligned}$$

with $\delta(A) = \dim_K \overline{A}/A$.

Now, in A we have $w_1 x \frac{\partial f}{\partial x} = -w_2 y \frac{\partial f}{\partial y}$ and by assumption $\frac{\partial f}{\partial x} \neq 0$. If $\frac{\partial f}{\partial y} = 0$, i.e. A is exceptional, we have $\dim_K \frac{\partial f}{\partial x}\overline{A}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \delta(A)$, i.e. $\tau(A) = \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. If $\frac{\partial f}{\partial y} \neq 0$, i.e. A is not exceptional, then $\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} = -\frac{w_1}{w_2} \cdot \frac{x}{y}$, i.e.

$$\begin{aligned}\dim_K \overline{A}\left(1, \frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x}\right)A &= \dim_K \overline{A}/\left(1, \frac{x}{y}\right)A \\ &= \dim_K y\overline{A}/(x, y)A \\ &= \delta(A) + 1 - \operatorname{ord}_t y \\ &= \delta(A) + 1 - m(A)\end{aligned}$$

because $\operatorname{ord}_t y \leqq \operatorname{ord}_t x$, since by assumption $\operatorname{ord}_t \frac{\partial f}{\partial x} \leqq \operatorname{ord}_t \frac{\partial f}{\partial y}$. The lemma is proved.

Now we want to study the behaviour of the Tjurina number under blowing up.

Definition 4.2

$$\tilde{\tau}(A) = \tau(A) + \begin{cases} 1 & \text{if } A \text{ is exceptional} \\ 0 & \text{else.}\end{cases}$$

Let $\widetilde{A}$ be the blowing up to A and assume $m(A) \geqq 2$.

Lemma 4.3 (1)

$$\tilde{\tau}(\widetilde{A}) = \tilde{\tau}(A) - m(A)(m(A)-1) - \begin{cases} m(A) & \textit{if } A \textit{ is exceptional} \\ -m(A) & \textit{if } A \textit{ is weakly homogeneous} \\ 0 & \textit{else.}\end{cases}$$

if $m(\widetilde{A}) \geqq 2$

(2) *If* $m(\widetilde{A}) = 1$, *then*

$$\tilde{\tau}(A) = m(A)(m(A)-1) + \begin{cases} m(A)+1, & A \textit{ exceptional}, m(A) \equiv 0 \bmod p \\ m(A), & A \textit{ exceptional}, m(A) \not\equiv \bmod p \\ 1, & A \textit{ not exceptional}, m \equiv 0 \bmod p \\ 0, & \textit{else}\end{cases}$$

Proof. Let $A = K[[x, y]]/f, f = (x + \beta y)^m + \sum_{i+j>m} a_{ij} x^i y^j$ and $m = \operatorname{ord}_t y \leqq \operatorname{ord}_t x$. Let $w_1 x \frac{\partial f}{\partial x} + w_2 y \frac{\partial f}{\partial y} = w_\circ f$ for $(w_\circ : w_1 : w_2) \in \mathbb{P}^2_{\mathbb{F}_p}$. Then we know

$$(1) \quad \tilde{\tau}(A) = \operatorname{ord}_t\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) + 1 - \begin{cases} 0 & \text{if } A \text{ is exceptional} \\ m(A) & \text{else}\end{cases}$$

$$(2) \quad \tilde{\tau}(\widetilde{A}) = \operatorname{ord}_t\left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial x}\right) + 1 - \begin{cases} 0 & \text{if } \widetilde{A} \text{ is exceptional} \\ m(\widetilde{A}) & \text{else}\end{cases}$$

here the embedding of $\tilde{A} \to K[[t]]$ is given by $\tilde{x}(t) = \dfrac{x(t)+\beta y(t)}{y(t)}$, $\tilde{y}(t) = y(t)$ if $\tilde{A} = K[[x, y]]/\tilde{f}$ and $\tilde{f} = \dfrac{1}{y^m} f(xy - \beta y, y)$.

(3) $\dfrac{\partial \tilde{f}}{\partial x} = \dfrac{1}{y^{m-1}} \dfrac{\partial f}{\partial x}(xy - \beta y, y)$

$y\dfrac{\partial \tilde{f}}{\partial y} = \dfrac{1}{y^m}((xy - \beta y)\dfrac{\partial f}{\partial x}(xy - \beta y, y) + y\dfrac{\partial f}{\partial y}(xy - \beta y, y)) - m\tilde{f}$ which implies

$$\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial x} = \operatorname{ord}_t \frac{\partial f}{\partial x} - m(m-1)$$

$$\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial y} = \operatorname{ord}_t \left(\frac{x}{y}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) - m^2 .$$

Now for the proof of the lemma we distinguish between three cases:

Case 1 $w_1 = w_2$. In this case A is weakly homogeneous and $\tilde{A}$ is exceptional (cf. Lemma 3.1). Because of $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 0$ and (3), we obtain

$$\begin{aligned}\operatorname{ord}_t \left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t \frac{\partial \tilde{f}}{\partial x} &= \operatorname{ord}_t \frac{\partial f}{\partial x} - m(m-1)\\ &= \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m(m-1).\end{aligned}$$

Using (1) and (2) we obtain the required result. Notice that in this case $m(\tilde{A}) \geqq 2$.

Case 2 $w_2 = 0$ or $w_1 w_2 \neq 0$ and $w_1 \neq w_2$ and $\beta \neq 0$. This implies $m(A) \equiv 0 \bmod p$. Using Lemma 3.1. we conclude that $\tilde{A}$ is exceptional or smooth. If $w_2 = 0$, then A is exceptional. If $w_1 w_2 \neq 0$, then A is not exceptional. Now $w_2 = 0$ implies $\dfrac{\partial f}{\partial x} = 0$ and this implies $\dfrac{\partial \tilde{f}}{\partial x} = 0$, i.e.

$$\operatorname{ord}_t \left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m^2 \quad \text{(using (3))}.$$

If $w_1 w_2 \neq 0$ and $w_1 \neq w_2$, then $w_1\left(x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}\right) = (w_1 - w_2)y\dfrac{\partial f}{\partial y}$ holds in A, i.e. $\operatorname{ord}_t \dfrac{\partial \tilde{f}}{\partial y} = \operatorname{ord}_t \dfrac{\partial f}{\partial y} - m^2$. Because of $\beta \neq 0$ we have $\dfrac{\partial \tilde{f}}{\partial x} \in \left(\dfrac{\partial \tilde{f}}{\partial y}\right)$ (cf. proof of 3.1.) and $\operatorname{ord}_t \dfrac{\partial f}{\partial x} = \operatorname{ord}_t \dfrac{\partial f}{\partial y}$. This implies

$$\operatorname{ord}_t \left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t \frac{\partial f}{\partial y} - m^2 = \operatorname{ord}_t \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m^2 .$$

Consequently, in the second case we have

$$\operatorname{ord}_t\left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m^2 .$$

Using (1) and (2) we obtain the lemma in the second case.

Case 3 $\beta = 0$ and $w_1 = 0$ or $w_1 w_2 \neq 0$ and $w_1 \neq w_2$. Using Lemma 3.1.1 we have that $\tilde{A}$ is not exceptional. If $\tilde{A}$ is smooth then $m(A) \not\equiv 0 \bmod p$. If $w_1 w_2 \neq 0$ and $w_1 \neq w_2$, then $w_2\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = (w_2 - w_1)x\frac{\partial f}{\partial x}$ holds in A, i.e.

$$\begin{aligned}\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial y} &= \operatorname{ord}_t \frac{x}{y}\frac{\partial f}{\partial x} \quad \text{(using (3))}\\ &= \operatorname{ord}_t \frac{\partial f}{\partial x} - m^2 + \operatorname{ord}_t \frac{x}{y}\\ &= \operatorname{ord}_t \frac{\partial f}{\partial x} - m(m-1) + \operatorname{ord}_t \tilde{x} - m\end{aligned}$$

If $w_1 = 0$, then $\frac{\partial f}{\partial y} = 0$ and we deduce also, using (3),

$$\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial y} = \operatorname{ord}_t \frac{x}{y}\frac{\partial f}{\partial x} = \operatorname{ord}_t \frac{\partial f}{\partial x} - m(m-1) + \operatorname{ord}_t \tilde{x} - m .$$

If $\operatorname{ord}_t \tilde{x} \geqq \operatorname{ord}_t \tilde{y} = m$, then $m(\tilde{A}) = m(A) = m$ and $\operatorname{ord}_t\left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m(m-1)$, because $\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial y} \geqq \operatorname{ord}_t \frac{\partial \tilde{f}}{\partial x}$.
If $\operatorname{ord}_t \tilde{x} < \operatorname{ord}_t \tilde{y} = m$, then $\operatorname{ord}_t \tilde{x} = m(\tilde{A})$ and

$$\operatorname{ord}_t\left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m(m-1) + m(\tilde{A}) - m,$$

because $\operatorname{ord}_t \frac{\partial \tilde{f}}{\partial y} < \operatorname{ord}_t \frac{\partial \tilde{f}}{\partial x}$. Finally we have

$$\operatorname{ord}_t\left(\frac{\partial \tilde{f}}{\partial x}, \frac{\partial \tilde{f}}{\partial y}\right) = \operatorname{ord}_t\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - m(A)(m(A) - 1) + m(\tilde{A}) - m(A)$$

in both cases. Again, using (1) and (2) we can prove the lemma in the third case.

Using the definition of the Milnor number (Definition 1.7.) and Lemma 4.3. we obtain the following corollary (Lemma 1.9.):

Corollary 4.4

(1) $\tau(A) = \mu(A) + \tau(\widetilde{A}) - \mu(\widetilde{A})$

$$+\begin{cases} m(\widetilde{A}) - m(A) & \textit{if } A \textit{ is weakly homogeneous} \\ m(\widetilde{A}) & \textit{if } A \textit{ is not weakly homogeneous} \\ & \textit{and } \widetilde{A} \textit{ exceptional} \\ 0 & \textit{else} \end{cases}$$

if $m(\widetilde{A}) \geqq 2$.

(2) $\tau(A) = \mu(A) + \begin{cases} 1 & m(A) \equiv 0 \bmod p \\ 0 & \textit{else} \end{cases}$

if $m(\widetilde{A}) = 1$.

Proof. By definition we have

$$\mu(A) = 2\delta(A) + \begin{cases} m(A) - 1 & \text{if } A \text{ is exceptional} \\ 0 & \text{else} \end{cases}$$

On the other hand

$$\tilde{\tau}(A) = \tau(A) + \begin{cases} 1 & \text{if } A \text{ is exceptional} \\ 0 & \text{else} \end{cases}$$

Now $2\delta(A) = 2\delta(\widetilde{A}) - m(A)(m(A) - 1)$ describes the behaviour of δ under blowing up (cf. [C]). Using Lemma 4.3. one gets the formulae required in the corollary.

To prove Theorem 1.8 we need to study the exceptional local rings occurring in the resolution process of A.

Lemma 4.5 (1) *Let $m(A) = m(\widetilde{A})$. If A is weakly homogeneous, then $\widetilde{A}$ is exceptional and the blowing up of $\widetilde{A}$ is not exceptional. If A is not weakly homogeneous and $\widetilde{A}$ is exceptional, then the blowing up of $\widetilde{A}$ is exceptional or smooth.*
(2) *Let $m(A) > m(\widetilde{A})$. If A is weakly homogeneous, then $\widetilde{A}$ is exceptional and the blowing up of $\widetilde{A}$ is exceptional or smooth. If A is not weakly homogeneous and $\widetilde{A}$ is exceptional, then the blowing up of $\widetilde{A}$ is not exceptional.*

Proof. The proof is a consequence of Lemma 3.1. We use the notations of Lemma 3.1.

Let $m(A) = m(\widetilde{A})$. If A is weakly homogeneous, we use (2) and (6) of Lemma 3.1. to obtain the required result. If A is not weakly homogeneous and $\widetilde{A}$ is exceptional, then either $w_2 = 0$ or $w_1 \neq w_2$ and $w_2 \neq 0$ and $\beta \neq 0$, and we can use (2) resp. (3) and (5) to get the required result.

Let $m(A) > m(\widetilde{A})$. If A is weakly homogeneous we use (2) and (5), if A is not weakly homogeneous we use (2) resp. (3) and (6), as before, to get the required result. Notice that if the multiplicity changes and $(0 : \widetilde{w}_1 : \widetilde{w}_2)$ is a weight of $\widetilde{A}$ obtained by (2) or (3) of Lemma 3.1., we have to exchange x and y to apply Lemma 3.1. to $\widetilde{A}$.

Definition 4.6 Let $A = A_1^{(1)}, \ldots, A_{r_1}^{(1)}, \ldots, A_{r_{i-1}}^{(i-1)}, A_1^{(i)}, \ldots, A_{r_i}^{(i)}, A_1^{(i+1)}, \ldots, A_{r_l}^{(l)}$ be the local rings occurring in the resolution process of A, $m(A_j^{(i)}) = m_i$. Let $I_A \subseteq \{1, \ldots, l-1\}$ be defined by the following property:

(1) $l-1 \in I_A$ iff $m_{l-1} \equiv 0 \bmod p$.

(2) If for $i \leqq l-2$, $A_{r_i}^{(i)}$ is not is not weakly homogenous and $A_{i+1}^{(1)}$ is exceptional, then $i \in I_A$. For $i \in I_A$ let $d_A(i) = m_{i+1}$ if $A_{r_i}^{(i)}$ is not exceptional. If $A_{r_i}^{(i)}$ is exceptional, then let s_i be minimal such that $A_{s_i}^{(i)}, \ldots, A_{r_i}^{(i)}$ are exceptional and

$$d_A(i) = \begin{cases} (r_i - s_i + 1)m_i + m_{i+1} & \text{if } s_i > 1 \\ r_i m_i + m_{i+1} - m_{i-1} & \text{if } s_i = 1 \text{ and } i > 1 \\ (r_1 - 1)m_1 + m_2 & \text{if } s_i = i = 1 \end{cases}$$

Lemma 4.7 *For $k \in I_A$ we have $d_A(k) \geqq 0$. Moreover, if $s_k = 1$ and $k \notin \{i_2, \ldots, i_g\}$, then $d_A(k) = 0$. If $k \in \{i_2, \ldots, i_g\}$, $k = i_\nu$, then $d_A(k) = \beta_\nu - \beta_{\nu-1}$.*[3]

Proof. Lemma 3.2.(2) implies $d_A(i_\nu) = \beta_\nu - \beta_{\nu-1}$ resp. $d_A(k) = 0$ if $k \notin \{i_1, \ldots, i_g\}$. We obtain $d_A(k) \geqq 0$.

Now we are prepared to prove a stronger version of theorem 1.8:

Theorem 4.8 (1) $\tau(A) = \mu(A) + \sum_{i \in I_A} d_A(i)$.

(2) *If $p > m(A)$, then $\tau(A) = \mu(A)$.*

Proof. (2) is a consequence of (1) because $i \in I_A$ implies $m_i \equiv 0 \bmod p$ (Lemma 3.1. and Definition 4.6.).

To prove (1) we use induction on the length of the multiplicity sequence. As before, let $\tilde{A}$ be the blowing up of A. If $\tilde{A}$ is smooth, then the theorem is a consequence of Corollary 4.4.(2). Assume now that $m(\tilde{A}) \geqq 2$.

Case 1 $r > 1$ In this case we have $I_A = I_{\tilde{A}}$, $d_A(i) = d_{\tilde{A}}(i)$ for $i > 1$ and if $1 \in I_A$, then

$$d_A(1) = d_{\tilde{A}}(1) + \begin{cases} m(\tilde{A}) & \text{if } A \text{ is not weakly homogeneous and } \tilde{A} \text{ is exceptional} \\ 0 & \text{else} \end{cases}$$

(Lemma 4.5.(1) and Definition 4.6).
By the induction hypothesis we have

$$\tau(\tilde{A}) = \mu(\tilde{A}) + \sum_{i \in I_A} d_{\tilde{A}}(i).$$

[3] For the definition of $i_1, \ldots, i_g$ c.f. Lemma 3.2

Using Corollary 4.4.(1):

$$\tau(A) = \mu(A) + \tau(\widetilde{A}) - \mu(\widetilde{A}) + \begin{cases} m(\widetilde{A}) & \textit{if } A \textit{ is not weakly homogeneous and} \\ & \widetilde{A} \textit{ exceptional} \\ 0 & \textit{else} \end{cases}$$

(because of $m(A) = m(\widetilde{A})$ we obtain

$$\tau(A) = \mu(A) + \sum_{i \in I_A} d_A(i).$$

Case 2 $r_1 = 1$. If $\widetilde{A}$ is not exceptional, then $I_A = \{i+1, i \in I_{\widetilde{A}}\}$ and $d_A(i+1) = d_{\widetilde{A}}(i)$. Because of Lemma 3.1. A cannot be weakly homogeneous, i.e. $\tau(A) - \mu(A) = \tau(\widetilde{A}) - \mu(\widetilde{A})$ (Corollary 4.4.). If A is weakly homogeneous, then $\widetilde{A}$ is exceptional and

$$I_A = \{i+1, i \in I_{\widetilde{A}}\}, \quad d_A(i+1) = d_{\widetilde{A}}(i), \quad i > 1,$$

$$d_{\widetilde{A}}(1) = (r_2 - 1)m_2 + m_3 \quad \text{and} \quad d_A(2) = r_2 m_2 + m_3 - m_1 = d_{\widetilde{A}}(1) + m_2 - m_1.$$

Corollary 4.4. yields

$$\tau(A) - \mu(A) = \tau(\widetilde{A}) - \mu(\widetilde{A}) + m(\widetilde{A}) - m(A).$$

If A is not weakly homogeneous and $\widetilde{A}$ is exceptional, then $1 \in I_A$ (Definition 4.6.) and $d_A(1) = m_2$.

$$I_A = \{1\} \cup \{i+1, i \in I_{\widetilde{A}}\} \quad \text{and} \quad d_A(i+1) = d_{\widetilde{A}}(i), \quad i > 1.$$

Corollary 4.4. yields

$$\tau(A) - \mu(A) = \tau(\widetilde{A}) - \mu(\widetilde{A}) + m(\widetilde{A}).$$

Using the induction hypothesis, we obtain the required result.

Proof of Corollary 1.10 Let $\beta_o, \ldots, \beta_g$ be the characteristic exponents of A. We prove that $g = 1$. Assume that $g \geqq 2$, i.e. $gcd(\beta_o, \beta_1) > 1$.

Using Theorem 1.6. we know that either $gcd(\beta_o, \beta_1) \equiv 0 \bmod p$ or $\beta_2 \equiv \beta_1 \bmod p$. On the other hand,

$$\tau(A) \geqq \mu(A) \geqq 2\delta(A) = \sum r_i m_i (m_i - 1) \quad \text{(Theorem 1.8.)}.$$

This implies by assumption

$$p \geqq \sum_{i=1}^{l} r_i m_i (m_i - 1) - m_1 - r_1 m_1 - m_2.$$

If $g \geqq 2$, then $m_1 \geqq 4$, which implies $\sum r_i m_i(m_i - 1) \geqq (r_1 + 2)m_1 + m_2$ and equality holds iff $m_1 = 4$, $m_2 = 2$, $r_1 = r_2 = 1$, i.e. $\beta_\circ = 4$, $\beta_1 = 6$. Similarly we obtain

$$\sum r_i m_i(m_i - 1) > (r_1 + 1)m_1 + m_2 + r_{i_2} m_{i_2} + m_{i_2+1} - m_{i_2-1}$$

if $gcd(m_1, m_2) = m_{i_2} > 1$.

Using the fact that $\beta_\circ = m_1$, $\beta_1 = r_1 m_1 + m_2$ and $\beta_2 - \beta_1 = r_{i_2} m_{i_2} + m_{i_2+1} - m_{i_2-1}$ (cf. Lemma 3.2.) we have

$$p \geqq \tau(A) - \beta_\circ - \beta_1 > \beta_\circ, \quad \text{i.e.} \quad p > \beta_\circ,$$

and

$$p \geqq \tau(A) - \beta_\circ - \beta_1 > \beta_2 - \beta_1 .$$

This gives a contradiction to the fact either $gcd(\beta_\circ, \beta_1) \equiv 0 \bmod p$ or $\beta_2 - \beta_1 \equiv 0 \bmod p$.

We have proved hat $g = 1$. In this case we have (cf. [C]) $2\delta(A) = (\beta_\circ - 1)(\beta_1 - 1)$. Because of $p \geqq \tau(A) - \beta_\circ - \beta_1$ we obtain $p > \beta_\circ\beta_1 - 2\beta_\circ - 2\beta_1$.

Now A is weakly quasihomogeneous and $m(A) \not\equiv 0 \bmod p$. We may assume that $A = K[[x, y]]/f$ and

$$f = x^{\beta_\circ} + y^{\beta_1} + \sum_{i+j>\beta_\circ} a_{ij} x^i y^j$$

and $a_{ij} \neq 0$ implies $i\beta_1 + j\beta_\circ \equiv \beta_\circ\beta_1 \bmod p$ and $i\beta_1 + j\beta_\circ > \beta_\circ\beta_1$. This implies

$$i\beta_1 + j\beta_\circ - \beta_\circ\beta_1 \geqq p > \beta_\circ\beta_1 - 2\beta_\circ - 2\beta_1, \quad \text{i.e.}$$
$$i\beta_1 + j\beta_\circ > 2\beta_\circ\beta_1 - 2\beta_\circ - 2\beta_1 .$$

Then there exists an automorphism φ of $K[[x, y]]$ such that $f(\varphi) = x^{\beta_\circ} + y^{\beta_1}$, i.e. A is quasihomogeneous.

Example 4.9 (1) Let $A = K[[x, y]]/f$, $f = (x + y^4)^p + xy^{5p-1}$. If $p = 3r + 1$, then $(p, 5)$, $(3, r)$, $(1, 3)$ is the multiplicity sequence. If $p = 3r + 2$, then $(p, 5)$, $(3, r)$, $(2, 1)$, $(1, 2)$ is the multiplicity sequence.

$\tau(A) = 5p^2 - 2p + 1$, $\mu(A) = 2\delta(A) = 5p^2 - 3p - 2$, $I_A = \{1\}$, $d_A(1) = p + 3$ because $A_5^{(1)}$ and $A_1^{(2)}$ are exceptional.

(2) Let $A = K[[x, y]]/f$, $f = x^5 + y^{11} + x^3y^9$, $p = 23$. A is weakly quasihomogeneous but not quasihomogeneous. $\tau = 40$, $\beta_\circ = 5$, $\beta_1 = 11$.

In the second part of this chapter we will consider reducible curve singularities and prove Theorem 1.11. Let $A = K[[x, y]]/f$ be the local ring of a reducible weakly quasihomogeneous plane curve singularity with multiplicity $m(A)$, and $A_1, \ldots, A_r$ the local rings of its irreducible components corresponding to the decomposition $f = f_1 \cdot \ldots \cdot f_r$.

To prove Theorem 1.11. we have to distinguish between the following situations:

(1) A is not exceptional and all branches are not exceptional.
(2) A is not exceptional, but some of the branches are exceptional.
(3) A is exceptional.

Definition 4.10 *A is almost exceptional if A is not exceptional, but the union of some of the branches is exceptional.*

Lemma 4.11 *If A is almost exceptional, then all branches are either exceptional or smooth. Moreover, the curve defined by A is a union of a uniquely determined exceptional component and a smooth component.*

The proof of the lemma is an immediate consequence of the following technical lemma.

Lemma 4.12 *Let $A = K[[x, y]]/f$ be the local ring of an isolated singularity. Let $w_\circ f = w_1 x\frac{\partial f}{\partial x} + w_2 y\frac{\partial f}{\partial y}$ for suitable $(w_\circ : w_1 : w_2) \in \mathbb{P}^2_{\mathbb{F}_p}$ and $f = gh, g, h \in (x, y)$. Let $A_1 = K[[x, y]]/g$ and $A_2 = K[[x, y]]/h$. The following holds:*

(1) *A_1 and A_2 are weakly quasihomogeneous.*
(2) *A is exceptional iff $w_\circ = w_1 w_2 = 0$.*
(3) *If A is exceptional, then A_1 and A_2 are exceptional or smooth.*
(4) *If $w_1 w_2 \neq 0$, then A_1 and A_2 are not exceptional.*
(5) *If $w_\circ \neq 0$ and $w_1 = 0$, then $f = y\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y^2} = 0$. Furthermore, if $\frac{\partial f}{\partial y} = h_1 \cdot h_2$, then the singularity defined by yh_1 is not exceptional.*
(6) *A is almost exceptional iff $w_\circ \neq 0$ and $w_1 w_2 = 0$ and $m(A) \geqq 3$.*

Proof. Let $\delta = w_1 x\frac{\partial}{\partial x} + w_2 y\frac{\partial}{\partial y}$, then $\delta f = w_\circ f$ implies $\delta g = ug$ and $\delta h = (w_\circ - u)h$ for a suitable $u \in K[[x, y]]$ because of $gcd(g, h) = 1$. Using Theorem 1.3. we obtain (1).
If $w_\circ = w_1 w_2 = 0$, then A is exceptional by definition. Let A be exceptional. We may assume that there is a $\bar{\delta} \in \mathrm{Der}_K K[[x, y]]$ with $\bar{\delta}(x) = 1$ and $\bar{\delta}(f) = uf$ for a suitable $u \in K[[x, y]]$. This implies $\frac{\partial f}{\partial x} \in \left(f, \frac{\partial f}{\partial y}\right)$. On the other hand, we obtain

$$(w_\circ - w_1 xu)f = (w_2 y - w_1 x\bar{\delta}(y))\frac{\partial f}{\partial y}\ .$$

If $w_\circ \neq 0$, then $f \in \left(\frac{\partial f}{\partial y}\right)$, i.e. $\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial y}\right)$. This is not possible because f defines an isolated singularity and $f \in (x, y)^2$. This implies $w_\circ = 0$.
Assume now $w_1 w_2 \neq 0$, i.e. $x\frac{\partial f}{\partial x} = wy\frac{\partial f}{\partial y}$, $w = -\frac{w_2}{w_1} \neq 0$.

We obtain $(wy + x\bar{\delta}(y))\frac{\partial f}{\partial y} = xuf$.

$wy + y\bar{\delta}(y)$ is irreducible and $\frac{\partial f}{\partial y} \notin (f)$ because f defines an isolated singularity and $\frac{\partial f}{\partial x} \in \left(f, \frac{\partial f}{\partial y}\right)$. This implies $f = (wy + x\bar{\delta}(y)) \cdot l$ for a suitable $l \in (x, y)$. Then

$$xul = \frac{\partial f}{\partial y} = \left(w + x\frac{\partial \bar{\delta}(y)}{\partial y}\right)l + (wy + x\bar{\delta}(y))\frac{\partial l}{\partial y}$$

and, because of $w \neq 0$, $wy + x\bar{\delta}(y)$ is a factor of l. This is a contradiction to the fact that f defines an isolated singularity and implies $w_1 w_2 = 0$.

To prove (3) we may assume (using (2)) that $\frac{\partial f}{\partial y} = 0$. Then $\frac{\partial g}{\partial y} = u \cdot g$ and $\frac{\partial h}{\partial y} = -u \cdot h$ for a suitable $u \in K[[x, y]]$ because of $gcd(g, h) = 1$, i.e. A_1 and A_2 are exceptional. To prove (4) we may assume that A_1 is not smooth and $\delta g = ug$ for a suitable $u \in K[[x, y]]$. If $u \in (x, y)$, then $A_1 \simeq K[[x, y]]/\bar{g}$ and $\delta \bar{g} = 0$ (second case of the proof of Theorem 1.3). Because of (2) A_1 is not exceptional. If $u \notin (x, y)$ and $u(0)^{-1} =: a \in K$, then $\bar{\delta} = u^{-1}\delta$ is semisimple with eigenvalues aw_1, aw_2, i.e. $A_1 \simeq K[[x, y]]/\bar{g}$ and $w_1 ax\frac{\partial \bar{g}}{\partial x} + w_2 ay\frac{\partial \bar{g}}{\partial y} = \bar{g}$ (first case of the proof of Theorem 1.3.). If $a \notin \mathbb{F}_p$, then there are $\bar{w}_1, \bar{w}_2 \in \mathbb{F}_p$ and

$$\bar{w}_1 x\frac{\partial \bar{g}}{\partial x} + \bar{w}_2 y\frac{\partial \bar{g}}{\partial y} = \bar{g}\,.$$

Because of (2) A_1 is not exceptional.

To prove (5) we differentiate the equation $w_\circ f = w_2 y\frac{\partial f}{\partial y}$: $w_\circ \frac{\partial f}{\partial y} = w_2\frac{\partial f}{\partial y} + w_2 y\frac{\partial^2 f}{\partial y^2}$. If $w_\circ \neq w_2$, then y is a factor of $\frac{\partial f}{\partial y}$, i.e. y^2 is a factor of f, which is a contradiction to the fact that f defines an isolated singularity. This yields $f = y\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y^2} = 0$.

Let $\frac{\partial f}{\partial y} = h_1 \cdot h_2$, h_1, $h_2 \in (x, y)$ and assume that yh_1 defines an exceptional singularity.

$\frac{\partial^2 f}{\partial y^2} = 0$ implies $\frac{\partial h_1}{\partial y} = uh_1$ for a suitable $u \in K[[x, y]]$ and $y\frac{\partial yh_1}{\partial y} = (1 + yu)yh_1$. Now $\bar{\delta} = (1 + yu)^{-1}y\frac{\partial}{\partial y}$ is semisimple with eigenvalues 0 and 1 and $\bar{\delta}(yh_1) = yh_1$. Consequently $K[[x, y]]/yh_1 \simeq K[[x, y]]/l$ and $l = y\frac{\partial l}{\partial y}$ (first case of the proof of Theorem 1.3.). Using (2) we obtain that $K[[x, y]]/yh_1$ is not exceptional. Now (6) is a consequence of (2) and (4).

Definition 4.13 Let A be almost exceptional and E (resp. L) the local ring of the maximal exceptional (resp. smooth) component of A. Let $d(A) := i(E, L) - m(E)$, where $i(E, L)$ is the intersection multiplicity of E and L and $m(E)$ the multiplicity of E.

Now we are prepared to give a stronger version of Theorem 1.11.:

Theorem 4.14 (1) *If A is not almost exceptional, then*

$$\tau(A) = \mu(A) + \sum_{i=1}^{r}(\tau(A_i) - \mu(A_i)) .$$

(2) *If A is almost exceptional, then*

$$\tau(A) = \mu(A) + \sum_{i=1}^{r}(\tau(A_i) - \mu(A_i)) - d(A)$$

and

$$\sum_{i=1}^{r}(\tau(A_i) - \mu(A_i)) \geqq d(A) .$$

(3) *If $p > m(A)$, then $\tau(A) = \mu(A)$.*

To prove the theorem we use the following lemmata:

Lemma 4.15 *Let A be almost exceptional and E (resp. L) the local ring of the maximal exceptional (resp. smooth) component of A, then*

(1) $\tau(A) = \tau(E) + i(E, L)$
(2) $\mu(A) = \mu(E) + i(E, L) + d(A)$.
(3) *If $p \geqq m(A)$, then $d(A) = 0$.*
(4) *Let $A_1, \ldots, A_{r-1}$ be the branches of E, then $d(A) = \sum_{i=1}^{r-1} (i(A_i, L) - m(A_i))$ and $\tau(A_i) - \mu(A_i) \geqq i(A_i, L) - m(A_i)$.*

Lemma 4.16 *Let A be not almost exceptional and assume that A splits into two (not necessarily irreducible) components with local rings B and C, then*

(1) $\tau(A) = \tau(B) + \tau(C) + 2i(B, C) - \begin{cases} 0 & A \text{ exceptional} \\ 1 & \text{else} \end{cases}$

(2) $\mu(A) = \mu(B) + \mu(C) + 2i(B, C) - \begin{cases} 0 & A \text{ exceptional} \\ 1 & \text{else} \end{cases}$

We will prove Lemma 4.15 and Lemma 4.16 at the end of this chapter.

Proof of Theorem 4.14 (3) is a consequence of Lemma 4.15(3), Theorem 4.8. and (1) resp. (2) of the theorem. (2) is a consequence of (1) using Lemma 4.15.(1) and (2). (1) is a consequence of Lemma 4.16. using induction on the number of branches of A.

Proof of Lemma 4.15 Using Lemma 4.12. we may assume that $A = K[[x, y]]/f$, $f = yh$ and $\frac{\partial h}{\partial y} = 0$, $E = K[[x, y]]/h$ and $L = K[[x, y]]/y$. Now

$$\tau(A) = \dim_K K[[x, y]]/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \dim_K K[[x, y]]/\left(h, y\frac{\partial h}{\partial x}\right)$$

and

$$\tau(E) = \dim_K K[[x, y]]/\left(h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = \dim_K K[[x, y]]/\left(h, \frac{\partial h}{\partial x}\right).$$

This implies

$$\begin{aligned}\tau(A) - \tau(E) &= \dim_K \left(h, \frac{\partial h}{\partial x}\right)\Big/\left(h, y\frac{\partial h}{\partial x}\right)\\ &= \dim_K K[[x, y]]/(y, h)\\ &= i(E, L)\end{aligned}$$

because of $gcd\left(h, \frac{\partial h}{\partial x}\right) = 1$. This proves (1)

(2) is a consequence of the fact that

$$\delta(A) = \delta(E) + \delta(L) + i(E, L)$$

always and the definition of the Milnor number (1.7) $\mu(A) = 2\delta(A) + 1 - r$, $\mu(E) = 2\delta(E) + m(E) - (r - 1)$, and $\delta(L) = 0$. If $p \geqq m(A)$, then $p > m(E)$. Because of $\frac{\partial h}{\partial y} = 0$ we obtain $h = \alpha x^{m(E)} + \sum_{i+j>m(E)} a_{ij} x^i y^i$, $\alpha \in K$ and $\alpha \neq 0$. This implies $i(E, L) = \dim_K K[[x, y]]/(y, h) = m(E)$, i.e. $d(A) = 0$, which proves (3).

To prove (4) let B be a branch of $E, B = K[[x, y]]/g$, $h = l \cdot g$. Then $i(B, L) = \operatorname{ord}_x g(x, 0)$ and $\frac{\partial g}{\partial y} = ug$ for a suitable $u \in K[[x, y]]$. If $d(A) > 0$, then $g = \beta y^{m(B)} + \sum_{i+j>m(B)} a_{ij} x^i y^i$ and because of $\frac{\partial g}{\partial y} = ug$ we have $m(B) \equiv 0 \bmod p$. The blowing up of B is again exceptional or smooth, i.e. $1 \in I_B$ (cf. Definition 4.6. and Lemma 4.5.). Let $(m_1, r_1), (m_2, r_2), \ldots$ be the multiplicity sequence of B, $m_1 = m(B)$, then Theorem 4.8., Lemma 4.7. and Definition 4.6 yield

$$\tau(B) \geqq \mu(B) + (r_1 - 1)m_1 + m_2.$$

On the other hand, $\operatorname{ord}_x g(x, 0) \leqq r_1 m_1 + m_2$. This implies $\tau(B) - \mu(B) - i(B, L) + m(B) = \tau(B) - \mu(B) - \operatorname{ord}_x g(x, 0) + m_1 \geqq 0$, which proves (4).

Proof of Lemma 4.16 (2) is a consequence of the fact that $\delta(A) = \delta(B) + \delta(C) + i(B, C)$ and the definition of the Milnor number (cf. 1.7):
If A is exceptional, then
$\mu(A) = 2\delta(A) + m(A) - r(A)$, $\mu(B) = 2\delta(B) + m(B) - r(B)$,
$\mu(C) = 2\delta(C) + m(C) - r(C)$ and $r(A) = r(B) + r(C)$,
$m(A) = m(B) + m(C)$.
If A is not exceptional and thus, by assumption, not almost exceptional, then $\mu(A) = 2\delta(A) + 1 - r(A)$, $\mu(B) = 2\delta(B) + 1 - r(B)$, $\mu(C) = 2\delta(C) + 1 - r(C)$.
To prove (1) we may assume that $A = K[[x, y]]/f$, $f = g \cdot h$, $g, h \in (x, y)$, $B = K[[x, y]]/g$, $C = K[[x, y]]/h$ and $w_0 f = w_1 x\frac{\partial f}{\partial x} + w_2 y\frac{\partial f}{\partial y}$, $(w_0 : w_1 : w_2) \in \mathbb{P}^2_{\mathbb{F}p}$. Furthermore, using Lemma 4.12. we may assume that either $w_o = w_1 = 0$ (A is exceptional) or $w_1 w_2 \neq 0$.
Now it is not difficult to see that the following sequence is exact:

$$0 \to K[[x, y]]/\left(h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) \oplus K[[x, y]]/\left(g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

$$\xrightarrow{i} K[[x, y]]/\left(gh, g\frac{\partial h}{\partial x}, g\frac{\partial h}{\partial y}, h\frac{\partial g}{\partial x}, h\frac{\partial y}{\partial y}\right)$$

$$\downarrow p$$

$$K[[x, y]]/(g, h) \to 0$$

$$i(\bar{a}, \bar{b}) = \overline{ag + bh}, \quad p(\bar{a}) = \bar{a}\,.$$

This implies

$$\tau(B) + \tau(C) + i(B, C) = \dim_K K[[x, y]]/\left(gh, g\frac{\partial h}{\partial x}, g\frac{\partial h}{\partial y}, h\frac{\partial g}{\partial x}, h\frac{\partial g}{\partial y}\right)\,.$$

On the other hand,

$$\tau(A) = \dim_K K[[x, y]]/\left(gh, \frac{\partial g}{\partial x}h + g\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}h + g\frac{\partial h}{\partial y}\right)\,.$$

Let $\Gamma := \left(gh, \frac{\partial g}{\partial x}h + g\frac{\partial h}{\partial x}, \frac{\partial g}{\partial y}h + g\frac{\partial h}{\partial y}\right)$ then $\left(gh, g\frac{\partial h}{\partial x}, g\frac{\partial h}{\partial y}, h\frac{\partial g}{\partial x}, h\frac{\partial g}{\partial y}\right) = h\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) + \Gamma$ and $\dim_K h(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}) + \Gamma/\Gamma = \tau(A) - \tau(B) - \tau(C) - i(B, C)$.
We have to prove that

$$\dim_K h\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) + \Gamma/\Gamma = i(B, C) - \begin{cases} 0 & \text{if } w_0 = w_1 = 0 \\ 1 & \text{if } w_1 w_2 \neq 0\,. \end{cases}$$

To prove this we use the following isomorphisms:

$$\varphi : h\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) + \Gamma/\Gamma \xrightarrow{\sim} (x, y)K[[x, y]]/(g, h)$$

$$\varphi\left(h\left(a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y}\right)\right) = w_2 ya - w_1 xb$$

if $w_1 w_2 \neq 0$.

If $w_0 = w_1 = 0$, then $h\frac{\partial g}{\partial y} = -g\frac{\partial h}{\partial y}$, i.e. $\frac{\partial h}{\partial y} = uh$ and $\frac{\partial g}{\partial y} = -ug$ for a suitable $u \in K[[x, y]]$. This implies $h\frac{\partial g}{\partial y} \in \Gamma$ and

$$\psi : h\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) + \Gamma/\Gamma \xrightarrow{\sim} K[[x, y]]/(g, h)$$

$$\psi\left(ah\frac{\partial g}{\partial x}\right) = a$$

defines an isomorphism.

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