# MODULI PARAMETERS OF COMPLEX SINGULARITIES WITH NON-DEGENERATE NEWTON BOUNDARY 

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#### Abstract

Our recent extension of Arnold's classification includes all singularities of corank $\leq 2$ equivalent to a germ with a non-degenerate Newton boundary, thus broadening the classification's scope significantly by a class which is unbounded with respect to modality and Milnor number. This method is based on proving that all right-equivalence classes within a $\mu$-constant stratum can be represented by a single normal form derived from a regular basis of a suitably selected special fiber. While both Arnold's and our preceding work on normal forms addresses the determination of a normal form family containing the given germ, this paper takes the next natural step: We present an algorithm for computing for a given germ the values of the moduli parameters in its normal form family, that is, a normal form equation in its stable equivalence class. This algorithm will be crucial for understanding the moduli stacks of such singularities. The implementation of this algorithm, along with the foundational classification techniques, is implemented in the library arnold.lib for the computer algebra system Singular.


## 1. Introduction

Our recent extension Böhm, Marais, and Pfister (2020) of Arnold's classification of isolated hypersurface singularities (see Arnold (1976); Arnold et al. (1985)) includes all singularities of corank $\leq 2$ which are equivalent to a germ with non-degenerate Newton boundary in the sense of Kouchnirenko. This broadens the scope of the classification by a class of singularities, which is unbounded both in terms of modality and Milnor number. We establish that there is a single polynomial normal form that contains representatives (at least one, but only finitely many) of all right equivalence classes within a given $\mu$-constant stratum of a given input germ. Based on this result, and algorithmic methods developed in Böhm, Marais, and Pfister (2016ab), an algorithm for effectively determining a normal form from a regular basis of a suitably chosen special fiber is given.

While both Arnold's and our preceding work on normal forms only addresses the determination of a normal form family containing the given germ, this continuation takes the next natural step: determining the moduli parameters in the normal form families associated with these input germs. That is, we find for a given input germ an element in the normal form associated to its $\mu$-constant stratum such that this element is right equivalent to the input germ, hence determining exactly its stable and right equivalence class. While one could argue that, with the normal form known, such a germ could be found using an Ansatz for the right equivalence by taking finite determinacy into account, this is practically inefficient and will not yield a result except for trivial input. Taking our

[^0]clue from some case-by-case studies in Marais and Steenpaß (2015, 2016); Böhm, Marais and Steenpaß (2019); Böhm, Marais, and Pfister (2016a) we thus develop an iterative method for eliminating the terms of the germ which are not present in the normal form family. This methods has some similarities the algorithm finding the normal form. The main challenge arises here from the fact that singularities with non-degenerate Newton boundary in general do not satisfy Condition A. In an iterative process we thus have to control higher order contributions in the binomial expansion when applying a right equivalence to the germ (since those can be of lower piecewise degree). Applications of our result could occur in the context of the study of Baikov polynomials, see Böhm et al. (2018); Lee and Pomeransky (2013).

Our paper is structured as follows:
In Section 2 we give a review of the foundational concepts and preliminary results on singularities and classification.

In Section 3, we recall the main results on the determination of normal forms for singularities of corank $\leq 2$ equivalent to a germ with non-degenerate Newton boundary. We also recall the algorithmic framework determining the normal form.

In Section 4, we address the determination of the moduli parameters in the normal form corresponding to a given input germ.

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## 2. Definitions and Preliminary Results

In this section, fundamental definitions, theorems, and notations relevant to our discussion are presented. We use $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ to denote the ring of convergent power series, that is, power series that converge in open neighborhoods of the point $(0, \ldots, 0)$. We use $\mathfrak{m}$ for the maximal ideal of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$.

Notation 2.1. We denote by $\operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$ the monoid of monomials in $x_{1}, \ldots, x_{n}$. For $f \in \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ and a monomial $m \in \operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$, the coefficient of $m$ in $f$ is denoted by coeff $(f, m)$.

Definition 2.2. If $w=\left(c_{1}, \ldots, c_{n}\right) \in \mathbb{N}^{n}$ is a weight for the variables $\left(x_{1}, \ldots, x_{n}\right)$, the $w$-weighted degree on $\operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$ is defined by the expression

$$
w-\operatorname{deg}\left(\prod_{i=1}^{n} x_{i}^{s_{i}}\right)=\sum_{i=1}^{n} c_{i} s_{i} .
$$

In the case that the weight of all variables is one, the weighted degree of a monomial $m$ is called its standard degree, denoted by $\operatorname{deg}(m)$. This notation is also used for terms in polynomials.

Definition 2.3. Consider a finite family of weights $w=\left(w_{1}, \ldots, w_{s}\right) \in\left(\mathbb{N}^{n}\right)^{s}$ for $\left(x_{1}, \ldots, x_{n}\right)$. For a term $m \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$, its piecewise weight with respect to $w$ is defined as

$$
w-\operatorname{deg}(m):=\min _{i=1, \ldots, s} w_{i}-\operatorname{deg}(m)
$$

Definition 2.4. Fix a (piecewise) weight $w$ on $\operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$.
(1) Suppose

$$
f=\sum_{i=0}^{\infty} f_{i}
$$

is the decomposition of $f \in \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ into weighted homogeneous components $f_{i}$ with $w$-degree of $i$. In the case that $f_{i}=0$ for $i>d$ and $f_{d} \neq 0$ is the lowest non-zero component, we set $w-\operatorname{deg}(f)=d$. The (piecewise) weighted j-jet of $f$, denoted by $w-\operatorname{jet}(f, j)$, is given by

$$
w-\operatorname{jet}(f, j):=\sum_{i=0}^{j} f_{i}
$$

The sum of terms of $f$ with the lowest $w$-degree is called the principal part of $f$ with respect to $w$. The order of $f$ with respect to $w$ is defined as the degree of its principal part, and is denoted by $w-\operatorname{ord}(f)$.
(2) A power series in $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ is said to have filtration $d \in \mathbb{N}$ with respect to $w$ if all its monomials have a w-weighted degree $\geq d$. By $E_{d}^{w}$ we denote the subvector space of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ of power series of filtration $d$ with respect to $w$. The sub-spaces $E_{d}^{w}$, for varying $d \in \mathbb{N}$, form a filtration on $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$.

Definition 2.5. A piecewise homogeneous germ $f_{0}$ of degree d satisfies Condition $A$, if for every germ $g$ of filtration $d+\delta>d$ in the ideal spanned by the derivatives of $f_{0}$, there is a decomposition

$$
g=\sum_{i} \frac{\partial f_{0}}{\partial x_{i}} v_{i}+g^{\prime}
$$

where the vector field $v$ has filtration $\delta$ and $g^{\prime}$ has filtration bigger than $d+\delta$.
Definition 2.6. We say that $f \in \mathfrak{m}^{2} \subset \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ is $k$-determined if

$$
f \sim \operatorname{jet}(f, k)+g \quad \text { for all } g \in E_{k+1}
$$

with respect to right-equivalence. The determinacy of $f$, denoted by $\operatorname{dt}(f)$, is the smallest integer $k$ for which $f$ is $k$-determined.

Definition 2.7. For $f \in \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$, the Jacobian ideal

$$
\operatorname{Jac}(f)=\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle
$$

is the ideal of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ generated by the partial derivatives of $f$. The local algebra

$$
Q_{f}=\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\} / \operatorname{Jac}(f)
$$

of $f$ is the quotient of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ by the Jacobian ideal. The Milnor number of $f$ is the dimension of $Q_{f}$ as a $\mathbb{C}$-vector space.

Remark 2.8. If the germ $f$ defines an isolated singularity, then $f$ is $k$-determined if $k \geq \mu(f)+1$, hence $f$ is finitely determined. So an isolated singularity can be represented by a polynomial.

Definition 2.9. The annihilator of a germ $f$, denoted by $\operatorname{ann}(f)$, is the ideal of all elements of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ that yield zero when multiplied with $f$.

Definition 2.10. Suppose that $\phi$ is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$, and $w$ is a single weight on $\operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$.
(1) For any positive integer $j$, the automorphism $w-\operatorname{jet}(\phi, j):=\phi_{j}^{w}$, is defined by

$$
\phi_{j}^{w}\left(x_{i}\right):=w-\operatorname{jet}\left(\phi\left(x_{i}\right), w-\operatorname{deg}\left(x_{i}\right)+j\right) \quad \text { for } i=1, \ldots, n
$$

If $w=(1, \ldots, 1)$, we use the notation $\phi_{j}$ for $\phi_{j}^{w}$.
(2) We say that $\phi$ has filtration $d$ if

$$
(\phi-\mathrm{id}) E_{\lambda}^{w} \subseteq E_{\lambda+d}^{w}
$$

for all $\lambda \in \mathbb{N}$.
Remark 2.11. We note that $\phi_{0}\left(x_{i}\right)=\operatorname{jet}\left(\phi\left(x_{i}\right), 1\right)$ for $i=1, \ldots, n$. Moreover, note that $\phi_{0}^{w}$ has filtration $\leq 0$. For $j>0, \phi_{j}^{w}$ has filtration $j$ if $\phi_{j-1}^{w}=\mathrm{id}$.

Definition 2.12. For $f=\sum_{i_{1}, \ldots, i_{n}} a_{i_{1}, \ldots, i_{n}} x_{1}^{i_{1}} \cdots x_{n}^{i_{n}} \in \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$, write

$$
\operatorname{mon}(f):=\left\{x_{1}^{i_{1}} \cdots x_{n}^{i_{n}} \mid a_{i_{1}, \ldots, i_{n}} \neq 0\right\}
$$

for the set of monomials of $f$, and

$$
\sup (f):=\left\{i_{1} \cdots i_{n} \mid a_{i_{1}, \ldots, i_{n}} \neq 0\right\}
$$

for the support of $f$. We set

$$
\Gamma_{+}(f):=\bigcup_{x_{1}^{i_{1} \ldots x_{n}^{i_{n}} \in \operatorname{supp}(f)}}\left(\left(i_{1}, \ldots, i_{n}\right)+\mathbb{R}_{+}^{n}\right)
$$

and define $\Gamma(f)$ as the boundary of the convex hull of $\Gamma_{+}(f)$ in $\mathbb{R}_{+}^{n}$. The set $\Gamma(f)$ is called the Newton boundary of $f$. Then:
(1) The compact segments of $\Gamma(f)$ are referred to as facets. ${ }^{1}$ If $\Delta$ is a facet, we write $\operatorname{supp}(f, \Delta)$ for the set of monomials of $f$ with exponent vector on $\Delta$. The sum of the terms lying on $\Delta$ is denoted by $\operatorname{jet}(f, \Delta)$. Moreover, we write $\operatorname{supp}(\Delta)$ for the set of monomials corresponding to the lattice points of $\Delta$. Considering the monomials lying on a union of facets, we use the same terminology for a set of facets.
(2) To a facet $\Delta$ we associate a weight $w(\Delta)$ on the monomials in $\operatorname{mon}\left(x_{1}, \ldots, x_{n}\right)$ as follows: If $-\left(w_{x_{1}}, \ldots, w_{x_{n}}\right)$ is the normal vector of $\Delta$ in lowest terms with integers $w_{x_{1}}, \ldots, w_{x_{n}}>0$, we define

$$
w(\Delta)-\operatorname{deg}\left(x_{1}\right)=w_{x_{1}}, \ldots, w(\Delta)-\operatorname{deg}\left(x_{n}\right)=w_{x_{n}}
$$

(3) Now suppose that $w_{1}, \ldots, w_{s}$ are the weight vectors of the facets of $\Gamma(f)$ ordered by increasing slope. Then there are uniquely determined minimal integers $\lambda_{1}, \ldots, \lambda_{s} \geq$ 1 with the property that the piecewise weight with respect to

$$
w(f):=\left(\lambda_{1} w_{1}, \ldots, \lambda_{s} w_{s}\right)
$$

is constant on the Newton boundary $\Gamma(f)$. We refer to this constant by $d(f)$.
(4) Suppose that $\Delta_{1}$ and $\Delta_{2}$ are adjacent facets with weights $w_{1}$ and $w_{2}$, respectively, $w$ is the piecewise weight defined by $w_{1}$ and $w_{2}$, and d is the $w$-degree of the monomials on $\Delta_{1}$ and $\Delta_{2}$. We then write $\operatorname{span}\left(\Delta_{1}, \Delta_{2}\right)$ for the Newton polygon associated to the sum of all monomials of $\left(w_{1}, w_{2}\right)$-degree $d$.

[^1](5) If $\Gamma(f)$ has at least one facet, we say that a monomial $m$ is strictly below, on or above $\Gamma(f)$, if the $w(f)$-degree of $m$ is less than, equal to or larger than $d(f)$, respectively.
(6) We write jet $(f, \Gamma(f))$ for the expansion of $f$ up to $w(f)$-order $d(f)$.

Definition 2.13. Assume that $f$ has finite Milnor number. A basis $\left\{e_{1}, \ldots, e_{\mu}\right\}$ of the local algebra of $f$ consisting out of homogeneous elements is regular with respect to the filtration given the piecewise weight $w$, if for each $D \in \mathbb{N}$, the basis elements of degree $D$ with respect to $w$ are independent modulo the sum vector space $\operatorname{Jac}(f)+E_{>D}^{w}$ of germs of filtration larger than $D$.

Remark 2.14. Arnold has proven in Arnold (1974) that for each germ $f \in \mathbb{C}\left\{x, \ldots, x_{n}\right\}$ there exists a regular basis consisting out of monomials.

Definition 2.15. For a union of right-equivalence classes $K \subset \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}$ a normal form for $K$ is a smooth map

$$
\Phi: \mathcal{B} \longrightarrow \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] \subset \mathbb{C}\left\{x_{1}, \ldots, x_{n}\right\}
$$

of a finite-dimensional $\mathbb{C}$-linear space $\mathcal{B}$ into the space of polynomials such that:
(1) $\Phi(\mathcal{B})$ intersects all equivalence classes of $K$,
(2) for each equivalence class the inverse image in $\mathcal{B}$ is finite
(3) $\Phi^{-1}(\Phi(\mathcal{B}) \backslash K)$ is contained in a proper hypersurface in $\mathcal{B}$.

We denote the elements of the image of $\Phi$ as normal form equations. A normal form is called a polynomial normal form if the map $\Phi$ is polynomial.
Example 2.16. For the germ $f=x^{4}+y^{4}$ of Arnold's type $X_{9}$, the $\mu$-constant stratum of $f$ is covered by the normal form $\Phi: \mathbb{C} \rightarrow \mathbb{C}[x, y], \Phi(a)=x^{4}+a x^{2} y^{2}+y^{4}$. For instance, the function $g=x^{4}+\epsilon x^{3} y+y^{4}$, with a fixed value of $\epsilon$, lies in the same $\mu$-constant stratum as $f$. Hence, there is a $\mathbb{C}$-algebra isomorphism $\phi_{1}$ transforming $g$ into $x^{4}+a x^{2} y^{2}+y^{4}$ with some $a$ in $\mathbb{C}$. As a result, there is also a $\mathbb{C}$-algebra isomorphism $\phi_{2}$ that maps $g$ to $x^{4}-a x^{2} y^{2}+y^{4}$.
Definition 2.17. (Arnold et al. (1985)) Let $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ and let $k$ be an upper bound for the determinacy of $f$. The modality of a germ $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ is the least number such that a sufficiently small neighborhood of $\operatorname{jet}(f, k)$ in the $k$-jet space can be covered by a finite number of m-parameter families of orbits under the right-equivalence action.

Definition 2.18. Arnold, 1974) Let $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ be a germ with a non-degenerate Newton boundary. The inner modality is the number of monomials in a regular basis for $Q_{f}$ lying on or above $\Gamma(f)$.
Remark 2.19. In the subsequent section, we will recall that the inner modality of a germ $f \in \mathbb{C}\{x, y\}$ is equal to the number of parameters in the normal form of the germ. Moreover it is shown in Böhm, Marais, and Pfister (2020), using results from Gabriélov (1974), that the inner modality and modality of a germ with a non-degenerate Newton boundary coincide.

## 3. Classification of Singularities with Non-Degenerate Newton Boundary

In this section we recall the results from Böhm, Marais, and Pfister (2020), where it is, in particular, shown that
(1) the $\mu$-constant stratum of a germ $f \in \mathbb{C}\{x, y\}$ with a non-degenerate Newton boundary can be covered, up to right-equivalence, by a single normal form.
(2) there is a normalization condition for the Newton boundary of a germ with a non-degenerate Newton boundary, that ensures the following: In a $\mu$-constant stratum which contains a germ with non-degenerate Newton boundary, all germs with normalized non-degenerate Newton boundary have the same Newton polygon. Hence, the Newton polygon can be considered as a name of the $\mu$-constant stratum, replacing Arnolds notation of a type.
(3) there is an effective algorithm to compute the normal form (satisfying the normalization condition) for a given input germ $f \in \mathbb{C}\{x, y\}$ which is equivalent to germ a non-degenerate Newton boundary.
Note that, in this section, we only consider germs in the bivariante convergent power series ring $\mathbb{C}\{x, y\}$.

The following result from Greuel et al. (2007), Corollary 2.71, and Böhm, Marais, and Pfister (2020), Theorem 3.9, gives a local description of the $\mu$-constant stratum of a germ with a non-degenerate Newton boundary.

Theorem 3.1. Let $f \in \mathbb{C}\{x, y\}$ be a germ with a non-degenerate Newton boundary at the origin. A miniversal, equisingular unfolding is given by

$$
F(x, y, t)=f+\sum_{i=1}^{m} t_{i} g_{i},
$$

where $m$ is the modality of $f$, and $g_{1}, \ldots, g_{m}$ represent a regular basis for $Q_{f}$ on and above $\Gamma(f)$.

A first step to find a normal form for the entire $\mu$-constant stratum of a germ with a non-degenerate Newton boundary is to investigate how a regular basis of the germs in the stratum change, while moving through the stratum.

Proposition 3.2. Böhm, Marais, and Pfister (2020), Proposition 3.12) Let $f_{0}$ be a germ with a non-degenerate Newton boundary $\Gamma\left(f_{0}\right)$ and let $f$ be a germ with the same Newton polygon as $f_{0}$ and non-degenerate Newton boundary. Then for $f$ sufficiently close to $f_{0}$ with respect to the Euclidean distance in the $(\mu+1)$-jet space, the monomials in $\operatorname{mon}(x, y)$ representing a regular basis for $f_{0}$ with respect to the filtration defined by $\Gamma\left(f_{0}\right)=\Gamma(f)$ also represent a regular basis for $f$ with respect to the same filtration.

Next, it is important to observe that all the germs in the $\mu$-constant stratum of a germ with a non-degenerate Newton boundary have the same topological type (see Böhm, Marais, and Pfister (2020), Remark 3.16). Since germs with a non-degenerate Newton boundary has the same topological type if and only if their characteristic exponents and intersection numbers coincide (see Brieskorn, Knörrer (1986), Theorem 15), and the characteristic exponents and intersection numbers of a germ in $\mathbb{C}\{x, y\}$ determines the non-degenerate Newton boundaries of a germ that is equivalent to a germ with a nondegenerate Newton boundary (see Böhm, Marais, and Pfister (2020), Proposition 4.17 and Corollary 4.18), the next result follows:

Theorem 3.3. Böhm, Marais, and Pfister (2020), Theorem 3.18) Suppose $f \in \mathbb{C}\{x, y\}$ is a convenient germ with non-degenerate Newton boundary $\Gamma$. Then all the germs in the $\mu$-constant stratum of $f$ are equivalent to a germ with the same Newton polygon $\Gamma$ and non-degenerate Newton boundary.

Let $f$ be a germ with a non-degenerate Newton boundary $\Gamma$. Taking the previous result into account, the next result shows that there exists a set of monomials that is a regular basis for at least one germ, a germ with a non-degenerate Newton boundary and a Newton polygon that coincide with that of $\Gamma$, in each right-equivalence class of the germs in the $\mu$-constant strantum of $f$.
Lemma 3.4. Böhm, Marais, and Pfister (2020), Lemma 3.20) Let $f$ be a convenient germ with non-degenerate Newton boundary. Define $f_{0}$ as the sum of the monomials of $f$ lying on the vertex points of $\Gamma(f)$. Then any regular basis of $f_{0}$ is also a regular basis for every germ with a non-degenerate Newton boundary in the $\mu$-constant stratum of $f$.

Corollary 3.5. Böhm, Marais, and Pfister (2020), Corollary 3.21) Suppose $f$ is a convenient germ with non-degenerate Newton boundary. Define $f_{0}$ as the sum of the monomials of $f$ lying on the vertex points of $\Gamma(f)$, and $f_{0}^{\prime}$ as the sum of the terms of $f$ on the vertex points of $\Gamma(f)$. Then any regular basis for $f_{0}$ is also a regular basis for $f_{0}^{\prime}$ and for $f$.

By the next theorem, every germ in the $\mu$-constant with non-degenerate Newton boundary can be written in terms of its Newton boundary and a regular basis of the germ.

Proposition 3.6. (Boubakri et al. (2011), Corollary 4.6) Let $f \in \mathbb{C}\{x, y\}$ be a convenient germ with a non-degenerate Newton boundary. Let $f_{0}$ be the principal part of $f$ and let $\left\{e_{1}, \ldots, e_{n}\right\}$ be the set of all monomials in a regular basis for $f_{0}$ lying above $\Gamma\left(f_{0}\right)$. Then there are $\alpha_{i}$ such that

$$
f \sim f_{0}+\sum_{i=1}^{n} \alpha_{i} e_{i} .
$$

Using Theorem 3.1, Proposition 3.2, Theorem 3.3, Lemma 3.4 and Proposition 3.6 the following theorem can be proved (see Böhm, Marais, and Pfister (2020), Theorem 3.22).
Theorem 3.7. Suppose $f$ is a convenient germ with non-degenerate Newton boundary. Define $f_{0}$ as the sum of the monomials of $f$ lying on the vertex points of $\Gamma(f)$, and let $\left\{e_{1}, \ldots, e_{n}\right\}$ be the set of all monomials in a regular basis for $f_{0}$ lying on or above $\Gamma(f)$. Then the family

$$
f_{0}+\sum_{i=1}^{n} \alpha_{i} e_{i}
$$

defines a normal form of the $\mu$-constant stratum containing $f$. Restricting the parameters $\alpha_{1}, \ldots, \alpha_{n}$ to values such that every germ $f_{0}+\sum_{i=1}^{n} \alpha_{i} e_{i}$ has a non-degenerate Newton boundary and the same Newton polygon as that of $f$, we obtain all germs in the $\mu$-constant stratum of $f$.

Remark 3.8. Using Theorem 3.7 a normal form can be constructed for any germ $f \in$ $\mathbb{C}\{c, y\}$ with a non-degenerate Newton boundary. Note that the Newton polygon of $f$ fixes the Newton polygon of all the germs in the constructed normal form. Since the normal form constructed in Theorem 3.7 is a normal form for the full $\mu$-constant stratum, it follows that if $f^{\prime}$ is a germ with a non-degenerate Newton boundary and a different Newton polygon than $f$, then the normal form constructed for $f^{\prime}$ describes the same $\mu$-constant stratum as that for $f$. In fact, by Theorem 3.3, $f$ is equivalent to a germ with a non-degenerate Newton boundary and same Newton polygon as $f^{\prime}$. Hence, the normal form for the $\mu$-constant stratum of $f$ as constructed using Theorem 3.7, depends
on the choice of non-degenerate Newton boundary of germs in the equivalence class of $f$ and the choice of regular basis for the chosen $f_{0}$. In his lists of normal forms, Arnold associate a type $T$ to each $\mu$-constant stratum. He then fixes a Newton polygon and a choice of moduli monomials (which boils down to the choice of regular basis for $f_{0}$ in Theorem 3.7). For distinguishing between different types, it is sufficient to know the Newton polygon of the normal form.

To achieve uniqueness of the Newton polygon associated to a fixed type (in order to label types by Newton polygons), a normalization condition on the Newton boundary of a germ with a non-degenerate Newton boundary is needed. Such a condition ensures that the same Newton polygon for any germ in the $\mu$-constant stratum is consistently chosen in order to construct a normal form by using Theorem 3.7.

It is important to distinguish between smooth and non-smooth facets:
Definition 3.9. A facet of the Newton polygon of a germ is called a smooth if the saturation of its jet is smooth.

Definition 3.10. Suppose $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ is a convenient germ with non-degenerate Newton boundary. Let $\Delta$ be a facet of $\Gamma(f)$, and write $w=w(\Delta)$. Then $\operatorname{jet}(f, \Delta)$ factorizes in $\mathbb{C}[x, y]$ as

$$
\operatorname{jet}(f, \Delta)=x^{a} \cdot y^{b} \cdot g_{1} \cdots g_{n} \cdot \widetilde{g}
$$

where $a, b$ are integers, $g_{1}, \ldots, g_{n}$ are linear homogeneous polynomials not associated to $x$ or $y$, and $\widetilde{g}$ is a product of non-associated irreducible non-linear homogeneous polynomials. We say that $f$ is normalized with respect to the facet $\Delta$, if

$$
\begin{array}{lll}
w(x)=w(y) & & \Longrightarrow \quad a, b \neq 0 \\
w(x)>w(y) \quad \text { and } \quad a=0 \quad \Longrightarrow \quad n=0 \\
w(x)<w(y) \quad \text { and } \quad b=0 \quad \Longrightarrow \quad n=0
\end{array}
$$

A germ for which all the facets satisfy the above normalization condition is not necessarily convenient. We can transform such a germ to a convenient germ in the same right-equivalence class by adding the terms $x^{d}$ or $y^{d}$ with $d=\mu(f)+2$, if needed. This will create non-normalized smooth facets that cut the coordinate axes. We address this in the following definition:

Definition 3.11. A germ $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ satisfies the normalization condition if all of its facets, except smooth facets cutting the coordinate axes, are normalized, and each of its smooth facets that cut a coordinate axis, cut the axis in standard degree d, where $d=\mu(f)+1$.

It directly follows from Theorem 3.7 that two germs with the same normalized Newton boundary have the same normal form and hence lie in the same $\mu$-constant stratum. The following theorem states that the normalization condition is reasonable:

Theorem 3.12. In a $\mu$-constant stratum which contains a germ with a non-degenerate Newton boundary, every right-equivalence class contains a normalized germ, and all germs in the $\mu$-constant stratum satisfying the normalization condition have the same Newton polygon (up to permutation of the variables).

In Böhm, Marais, and Pfister (2020), algorithms are given to determine a normal form for the $\mu$-constant stratum of a germ which is equivalent to a germ with a non-degenerate Newton boundary. This algorithm also detects if the given input germ is not equivalent to one with a non-degenerate Newton boundary.

As a first step, this algorithm uses Algorithm 4 of Böhm, Marais, and Pfister (2020) to transform an input germ $f \in \mathfrak{m}^{2} \subset \mathbb{C}\{x, y\}$ to a germ which is right-equivalent and has a normalized non-degenerate Newton boundary (this step will detect non-degeneracy).

Knowing the normalized non-degenerate Newton boundary, Algorithm 6 of Böhm, Marais, and Pfister (2020) is used to find a regular basis for the sum of the vertex monomials of the non-degenerate Newton polygon.

Finally, Algorithm 7 applies Theorem 3.7 to construct a normal form for the $\mu$-constant stratum of the input germ $f$.

Building on this construction and its implementation in Böhm, Marais, and Pfister (2024), the subsequent section will address finding the moduli parameters corresponding to the given input germ in the normal form.

## 4. Determining the Values of the Moduli Parameters in the Normal From of a germ with a Non-Degenerate Newton Boundary

After finding the normal form as discussed in Section 3, the values of the moduli parameters can, in theory, be computed via an Ansatz for the right equivalence mapping the given germ under consideration to an element of the normal form (making use of finite determinacy). However, this is not practicable except for very small examples. In this section, we discuss an efficient algorithm for this problem. For brevity of presentation, we introduce the following shorthand notation: if $\Delta$ is a set of facets of the Newton polygon of $f$, we write $f_{\Delta}=\operatorname{jet}(f, \Delta)$ as introduced in Definition 2.12,

Now, let $f \in \mathfrak{m}^{2}$ be a polynomial with a normalized non-degenerate Newton boundary, with $w(f)=\left(w_{1}, \ldots, w_{n}\right)$ the induced weight on the Newton polygon. Let $f_{0}$ be the sum of the vertex monomials of $\Gamma(f)$ and write $f_{0}^{\prime}=w(f)-\operatorname{jet}(f, d(f))$ for the sum of the terms of $f$ on $\Gamma(f)$.

Recall from Lemma 3.4 and Corollary 3.5 that a regular basis $B$ for $f_{0}$ is also a regular basis for $f_{0}^{\prime}$ and for $f$. Assume that $f$ has a term above $\Gamma(f)$ not in $B$, and let $d^{\prime}$ be the lowest $w(f)$-degree occurring among these terms. Let $t$ be a term of piecewise degree $d^{\prime}$ in $f$. Note that by the properties of a regular basis we can write

$$
\begin{equation*}
t=g \frac{\partial f}{\partial x}+h \frac{\partial f}{\partial y}+\text { terms of } w(f) \text {-degree } d^{\prime} \text { in } B+\text { terms of } w(f) \text {-degree }>d^{\prime}, \tag{1}
\end{equation*}
$$

where $g, h \in \mathbb{C}[x, y]$. Define the right equivalence $\phi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ by

$$
\begin{equation*}
\phi(x)=x-g, \phi(y)=y-h, \tag{2}
\end{equation*}
$$

where $g, h$ are as in equation (1). Note that

$$
\begin{equation*}
\phi(f)=f-\underbrace{\left(\frac{\partial f}{\partial x} g+\frac{\partial f}{\partial y} h\right)}_{\text {first order of the binomial expansion of } \phi(f)}+\frac{1}{2} \underbrace{\left(\frac{\partial^{2} f}{\partial^{2} y} h^{2}+\frac{\partial^{2} f}{\partial x \partial y} g h+\frac{\partial^{2} f}{\partial^{2} x} g^{2}\right)+\cdots}_{\text {higher order of the binomial expansion of } \phi(f)} \tag{3}
\end{equation*}
$$

A germ with a non-degenerate Newton boundary does not necessarily satisfy Condition A. Hence we cannot be certain that terms in the higher-order terms in the binomial expansion of $\phi(f)$ are of $w(f)$-degree larger than $d^{\prime}$. Thus the method introduced in Böhm, Marais, and Pfister (2020) for finding a normal form equation in general cannot be applied. Algorithm 1 provides a method applicable to any germ with a non-degenerate Newton boundary.

We rely on the following lemma to formulate the algorithm. Fix a set $\Gamma^{\prime}$ of connected facets of the Newton polygon of $f$.

Lemma 4.1. Suppose that $a$ and $b$ are the maximal exponents such that the monomial $x^{a} y^{b}$ divides both $\frac{\partial f_{\Gamma^{\prime}}}{\partial x}$ and $\frac{\partial f_{\Gamma^{\prime}}}{\partial y}$, and define the exponents $a^{\prime}$ and $b^{\prime}$ by $f_{\Gamma^{\prime}}=x^{a^{\prime}} y^{b^{\prime}} \operatorname{sat}\left(f_{\Gamma^{\prime}}\right)$. If $0 \neq g, h \in \mathbb{C}[x, y]$ satisfy

$$
g \cdot \frac{\partial f_{\Gamma^{\prime}}}{\partial x}+h \cdot \frac{\partial f_{\Gamma^{\prime}}}{\partial y}=0
$$

then

$$
\left(\frac{g}{x^{s}}, \frac{h}{y^{t}}\right)
$$

is a syzygy of

$$
\left(x^{i} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial y}\right)\right)
$$

where $s, i, t, \nu$ are given by

$$
\begin{aligned}
& s=\left\{\begin{array}{ll}
\max \left\{\alpha \mid x^{\alpha} \text { divides } \frac{\left.\partial f_{\Gamma^{\prime}}\right\}}{\partial y}\right\} & \text { if } a=0, \\
1 & \text { if } a \neq 0,
\end{array} \quad i= \begin{cases}\max \left\{\alpha \mid x^{\alpha} \text { divides } \frac{\left.\partial f_{\Gamma^{\prime}}\right\}}{\partial x}\right\} & \text { if } a=0, \\
0 & \text { if } a \neq 0,\end{cases} \right. \\
& t=\left\{\begin{array}{ll}
\max \left\{\beta \mid y^{\beta} \text { divides } \frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right\} & \text { if } b=0, \\
1 & \text { if } b \neq 0,
\end{array} \quad \nu= \begin{cases}\max \left\{\beta \mid y^{\beta} \text { divides } \frac{\left.\partial f_{\Gamma^{\prime}}\right\}}{\partial y}\right\} & \text { if } b=0, \\
0 & \text { if } b \neq 0,\end{cases} \right.
\end{aligned}
$$

Moreover, the vector $\left(\frac{g}{x^{s}}, \frac{h}{y^{t}}\right)$ is a polynomial multiple of the Koszul syzygy of

$$
\left(x^{i} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial y}\right)\right) .
$$

We postpone the proof of the lemma to the end of this section. We only remark at the current point that $x^{i}$ sat $\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right), y^{\nu}$ sat $\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial y}\right)$ is a regular sequence, hence the vector $\left(\frac{g}{x^{s}}, \frac{h}{y^{t}}\right)$ is a polynomial multiple of the mentioned Koszul syzygy.
Definition 4.2. Let $f \in \mathbb{C}[x, y]$ and suppose $m=\frac{m_{1}}{m_{2}} \in \operatorname{Quot}(\mathbb{C}[x, y])$ with monomial $m_{1}, m_{2} \in \mathbb{C}[x, y]$, that is, $m$ is a Laurent monomial in $x, y$. Then $\bar{f}^{m}$ is defined as the sum of all terms $t$ of $f$ such that $m \cdot t \in \mathbb{C}[x, y]$.

Using the next result, we will be able to show in the proof of Algorithm 1 that higher order terms $t^{\prime}$ of the binomial expansion of $\phi(f)$, where $\phi$ is defined in $\sqrt{22}$, can be written as

$$
t^{\prime}=g^{\prime} \frac{\partial f}{\partial x}+h^{\prime} \frac{\partial f}{\partial y}+\text { terms of } w(f) \text {-degree } d^{\prime} \text { in } B+\text { terms of } w(f) \text {-degree }>d^{\prime}
$$

where $g^{\prime}, h^{\prime} \in \mathbb{C}[x, y]$. Moreover we will show that the transformation $\phi_{\text {new }}: \mathbb{C}[x, y] \rightarrow$ $\mathbb{C}[x, y]$ defined by $\phi_{\text {new }}(x)=x-g^{\prime}, \phi_{\text {new }}(y)=y-h^{\prime}$ has a higher filtration than $\phi$.

Theorem 4.3. Let $I$ be the ideal generated by all the monomials of $w$-degree $d^{\prime}$ with $d^{\prime} \geq d(f)$ fixed. If $(a, b)$ is a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ over $\mathbb{C}[x, y] / I$, then there exist Laurent monomials $z_{j}$ in $x, y$ such that

$$
a-\sum_{j=1}^{k} z_{j}{\frac{\overline{\partial f}^{z_{j}}}{\partial y}}^{\in \operatorname{Ann}\left(\frac{\partial f}{\partial x}\right) \quad \text { and } \quad b+\sum_{j=1}^{k} z_{j} \overline{\partial f}^{z_{j}}} \quad \in \operatorname{Ann}\left(\frac{\partial f}{\partial y}\right)
$$

over $\mathbb{C}[x, y] / I$.

Furthermore if no $y^{\alpha}, \alpha>0$, is a monomial of $\frac{\overline{\partial f}^{z_{j}}}{\partial y}$, then $x$ divides all terms of $z_{j} \frac{\overline{\partial f}^{z_{j}}}{\partial y}$ that are not in $\operatorname{Ann}\left(\frac{\partial f}{\partial x}\right)$ and do not get cancelled in the sum $\sum_{j=1}^{k} z_{j} \frac{\overline{\partial f}^{z_{j}}}{\partial y}$. Similarly, if no $x^{\beta}, \beta>0$, is a monomial of $\frac{\overline{\partial f}^{z_{j}}}{\partial x}$, then $y$ divides all terms of $z_{j} \frac{\partial^{z^{z}}}{}{ }^{2}$ that are not in $\operatorname{Ann}\left(\frac{\partial f}{\partial y}\right)$ and do not get cancelled in the sum $\sum_{j=1}^{k} z_{j} \frac{\overline{\partial f}^{z x}}{\partial j}$.

Proof. Let $(a, b)$ be a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y] / I$, where $I$ is the ideal generated by all the monomials of $w$-degree $d^{\prime}$. If the syzygy equation $g_{1}=a \frac{\partial f}{\partial x}+b \frac{\partial f}{\partial y}=0$ has no cancellation below $w$-degree $d^{\prime}$ then we are finished. Now suppose $g_{1}$ has cancellation below degree $d^{\prime}$. Suppose in particular $g_{1}$ has cancellation below $w_{1}$-degree $d^{\prime}$. Let

$$
l_{1}=\min \left(w_{1}-\operatorname{ord}\left(a \frac{\partial f}{\partial x}\right), w_{1}-\operatorname{ord}\left(b \frac{\partial f}{\partial y}\right)\right) .
$$

For a face $\Delta$ of the Newton polygon, we write

$$
f_{\Delta, x}=\frac{\partial \operatorname{jet}(f, \Delta)}{\partial x} \quad \text { and } \quad f_{\Delta, y}=\frac{\partial \operatorname{jet}(f, \Delta)}{\partial y} .
$$

Then

$$
w_{1}-\operatorname{jet}\left(g_{1}, l_{1}\right)=m^{(1)}(\underbrace{x^{s_{1}} y^{\nu_{1}} \operatorname{sat}\left(f_{\Delta_{1}, y}\right)}_{w_{1}-\operatorname{jet}\left(a, l_{1}-d(f)\right)} f_{\Delta_{1}, x} \underbrace{-y^{t_{1}} x^{i_{1}} \operatorname{sat}\left(f_{\Delta_{1}, x}\right)}_{w_{1}-\operatorname{jet}\left(b, l_{1}-d(f)\right)} f_{\Delta_{1}, y}),
$$

where $m^{(1)}$ is a monomial, $\Delta_{1}$ is the face with the smallest slope of $\Gamma(f)$, and $i_{1}, \nu_{1}, s_{1}$ and $t_{1}$ is as in Lemma 4.1.

Now consider the syzygy $\left(\frac{\partial f}{\partial y},-\frac{\partial f}{\partial x}\right)$ of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y]$, with syzygy equation $g_{0}=\frac{\partial f}{\partial y} \frac{\partial f}{\partial x}-\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}=0$. Let $l_{0}=w_{1}-\operatorname{ord}\left(\frac{\partial f}{\partial y} \frac{\partial f}{\partial x}\right)$, then

$$
w_{1}-\operatorname{jet}\left(g_{0}, l_{0}\right)=m_{w_{1}}\left(x^{s_{1}} y^{\nu_{1}} \operatorname{sat}\left(f_{\Delta_{1}, y}\right) f_{\Delta_{1}, x}-y^{t_{1}} x^{i_{1}} \operatorname{sat}\left(f_{\Delta_{1}, x}\right) f_{\Delta_{1}, y}\right),
$$

where $m_{w_{1}}$ is the product of the maximal power of $x$ and the maximal power of $y$ dividing both $f_{\Delta_{1}, x}$ and $f_{\Delta_{1}, y}$. Let $n^{(1)}=\operatorname{lcm}\left(m_{w_{1}}, m^{(1)}\right)$. Then the lowest nonzero $w_{1}$-jet of $\frac{n^{(1)}}{m^{(1)}}(a, b)$ and $\frac{n^{(1)}}{m_{w_{1}}}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)$ coincide.

Then

$$
\left(a^{(1)}, b^{(1)}\right)=\left(\frac{n^{(1)}}{m^{(1)}} a-\frac{n^{(1)}}{m_{w_{1}}} \frac{\partial f}{\partial y}, \frac{n^{(1)}}{m^{(1)}} b+\frac{n^{(1)}}{m_{w_{1}}} \frac{\partial f}{\partial x}\right)
$$

is a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y] / I^{(1)}$, where $I^{(1)}=\frac{n^{(1)}}{m^{(1)}} I$.
Let $d_{2}^{\prime}=d^{\prime}\left(w_{1}-\operatorname{deg}\left(\frac{n^{(1)}}{m^{(1)}}\right)\right)$. Now, if the equation $g_{2}=a^{(1)} \frac{\partial f}{\partial x}+b^{(1)} \frac{\partial f}{\partial y}$ has any terms of $w_{1}$-degree less than $d_{2}^{\prime}$, then the lowest non-zero $w_{1}$-jet of $\left(a^{(1)}, b^{(1)}\right)$ is

$$
m^{(2)}\left(x^{s_{2}} y^{\nu_{2}} \operatorname{sat}\left(f_{\Delta_{1}, y}\right),-y^{t_{2}} x^{i_{2}} \operatorname{sat}\left(f_{\Delta_{1}, x}\right)\right) .
$$

Let $n^{(2)}=\operatorname{lcm}\left(m^{(2)}, m_{w_{1}}\right)$, then the lowest non-zero $w_{1}$-jet of the syzygies $\frac{n^{(2)}}{m^{(2)}}\left(a^{(1)}, b^{(1)}\right)$ and $\frac{n^{(2)}}{m_{w_{1}}}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)$ coincide. Similarly, as before, we can create the syzygy

$$
\left(a^{(2)}, b^{(2)}\right)=\left(\frac{n^{(2)}}{m^{(2)}} a^{(1)}-\frac{n^{(2)}}{m_{w_{1}}} \frac{\partial f}{\partial y}, \frac{n^{(2)}}{m^{(2)}} b^{(1)}+\frac{n^{(2)}}{m_{w_{1}}} \frac{\partial f}{\partial x}\right)
$$

in $\mathbb{C}[x, y] / I^{(2)}$, where $I^{(2)}=\frac{n^{(2)}}{m^{(2)}} I^{(1)}$.
We can go on with this process until the syzygy equation $g_{k}=a^{(k-1)} \frac{\partial f}{\partial x}-b^{(k-1)} \frac{\partial f}{\partial y}$ has no terms of $w_{1}$-degree less than $w_{1}$-degree $d_{k}^{\prime}=d_{(k-1)}^{\prime}\left(w_{1}-\operatorname{deg}\left(\frac{n^{(k-1)}}{m^{(k-1)}}\right)\right)$. We show now that this will eventually happen. Now, $n^{(j)}=\operatorname{gcd}(c, d, e, h)$ and $m^{(j+1)}$ be the the monomial with the maximal $x$ - and $y$-power dividing $c-d$ and $e-h$, where $c=\frac{n^{(j+1)}}{m^{(j+1)}} a^{(j)}, d=\frac{n^{(j+1)}}{m_{w_{1}}} \frac{\partial f}{\partial y}, e=\frac{n^{(j+1)}}{m^{(j+1)}} b^{(j)}$ and $h=\frac{n^{(j+1)}}{m_{w_{1}}} \frac{\partial f}{\partial x}$. Since the lowest non-zero $w_{1}$-jet of $c$ and $d$, and $e$ and $h$, cancel, respectively, in $c-d$ and $e-h$, it follows that $w_{1}-\operatorname{deg}\left(m^{(j+1)}\right)>w_{1}-\operatorname{deg}\left(n^{(j)}\right)$. Now,

$$
\begin{aligned}
& w_{1}-\operatorname{deg}\left(\frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1}-\operatorname{deg}\left(\frac{\partial f}{\partial y}\right)-d_{j+1}^{\prime} \\
= & w_{1}-\operatorname{deg}\left(\frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1}-\operatorname{deg}\left(\frac{\partial f}{\partial y}\right)-w_{1}-\operatorname{deg}\left(\frac{n^{(j+1)}}{m^{(j+1)}}\right) d_{j}^{\prime} \\
= & w_{1}-\operatorname{deg}\left(\frac{m^{(j+1)}}{n^{(j+1)}} \frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1}-\operatorname{deg}\left(\frac{\partial f}{\partial y}\right)-d_{j}^{\prime} \\
= & w_{1}-\operatorname{deg}\left(\frac{m^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1}-\operatorname{deg}\left(\frac{\partial f}{\partial y}\right)-d_{j}^{\prime} \\
< & w_{1}-\operatorname{deg}\left(\frac{n^{(j)}}{m_{w_{1}}}\right) \cdot w_{1}-\operatorname{deg}\left(\frac{\partial f}{\partial y}\right)-d_{j}^{\prime} .
\end{aligned}
$$

This implies that the difference in the degree of the lowest non-zero $w_{1}$-jet of $\left(a^{(j)}, b^{(j)}\right)$ and the degree of the lowest order elements in $d_{k}^{\prime}$ becomes smaller. Hence eventually the $w_{1}$-degree of $\left(a^{(j)}, b^{(j)}\right)$ will be $\geq d_{j}^{\prime}$.

Suppose now that $\left(a^{(j-1)}, b^{(j-1)}\right)$ is such that $g_{j}$ has no terms below $w_{1}$-degree $d_{j-1}^{\prime}$. We now consider the $w_{2}$-degree of $\left(a^{(j-1)}, b^{(j-1)}\right)$. We follow the same strategy. Suppose $g_{j}$ has terms of $w_{2}$-degree less than $d_{j-1}^{(2)}$, where

$$
d_{k}^{(r)}=w_{r}-\operatorname{deg}\left(\frac{n^{(1)} n^{(2)} \cdots n^{(k)}}{m^{(1)} m^{(2)} \cdots m^{(k)}}\right) d^{\prime}
$$

then let

$$
l_{j}=\min \left(w_{2}-\operatorname{ord}\left(a^{(j-1)} \frac{\partial f}{\partial x}\right), w_{2}-\operatorname{ord}\left(b^{(j-1)} \frac{\partial f}{\partial y}\right)\right)
$$

Then

$$
w_{2}-\operatorname{jet}\left(g_{j}, l_{j}\right)=m^{(j)}\left(x^{s_{j}} y^{\nu_{j}} \operatorname{sat}\left(f_{\Delta_{2}, y}\right) f_{\Delta_{2}, x}-y^{t_{j}} x^{i_{j}} \operatorname{sat}\left(f_{\Delta_{2}, x}\right) f_{\Delta_{2}, y}\right)
$$

where $m^{(j)}$ is a monomial, $\Delta_{2}$ is the face with the second smallest slope of $\Gamma(f)$, and $s_{j}$, $\nu_{j}, i_{j}$ and $t_{j}$ is as in Lemma 4.1. Furthermore

$$
w_{2}-\operatorname{jet}\left(g_{0}, l_{0}\right)=m_{w_{2}}\left(x^{s_{j}} y^{\nu_{j}} \operatorname{sat}\left(f_{\Delta_{2}, y}\right) f_{\Delta_{2}, x}-y^{t_{j}} x^{i_{j}} \operatorname{sat}\left(f_{\Delta_{2}, x}\right) f_{\Delta_{2}, y}\right),
$$

where $m_{w_{2}}=\operatorname{gcd}\left(f_{\Delta_{2}, x}, f_{\Delta_{2}, y}\right)$. With

$$
n^{(j)}:=\operatorname{lcm}\left(m_{w_{2}}, m^{(j)}\right)
$$

the lowest nonzero $w_{2}$-jets of

$$
\frac{n^{(j)}}{m^{(j)}}\left(a^{(j-1)}, b^{(j-1)}\right) \text { and } \frac{n^{(j)}}{m^{w_{2}}}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)
$$

coincide. Similarly, as before, we can create the syzygy

$$
\left(a^{(j)}, b^{(j)}\right)=\left(\frac{n^{(j)}}{m^{(j)}} a^{(j-1)}-\frac{n^{(j)}}{m_{w_{j}}} \frac{\partial f}{\partial y}, \frac{n^{(j)}}{m^{(j)}} b^{(j-1)}+\frac{n^{(j)}}{m_{w_{j}}} \frac{\partial f}{\partial x}\right)
$$

in $\mathbb{C}[x, y] / I^{(j)}$, where

$$
I^{(j)}=\frac{n^{(j)}}{m^{(j)}} I^{(j-1)}
$$

Note that the syzygy equation $g_{j}$ has no terms below $w_{2}$-degree

$$
d_{j}^{\prime}:=w_{2}-\operatorname{deg}\left(\frac{n^{(j)}}{m^{(j)}}\right) d_{j-1}^{(2)}
$$

and $w_{1}$-degree

$$
d_{j}^{\prime \prime}:=\left(w_{1}-\operatorname{deg}\left(\frac{n^{(j)}}{m^{(j)}}\right)\right) \cdot d_{j-1}^{\prime} .
$$

If this would not be the case, this would imply that

$$
-\frac{n^{(j)}}{m_{w_{2}}} \frac{\partial f}{\partial y} \frac{\partial f}{\partial x}+\frac{n^{(j)}}{m_{w_{2}}} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}
$$

has terms below $w_{1}$-degree $d_{j}^{\prime \prime}$.
Let $l_{\left(x, w_{1}\right)}^{\prime}$ and $l_{\left(y, w_{1}\right)}^{\prime}$ be the lowest non-zero $w_{1}$-orders of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Let $l_{\left(x, w_{2}\right)}^{\prime}$ and $l_{\left(y, w_{2}\right)}^{\prime}$ be the lowest $w_{2}$-orders of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. We observe that

$$
w_{1}-\operatorname{jet}\left(\frac{\partial f}{\partial x}, l_{\left(x, w_{1}\right)}^{\prime}\right) \text { and } w_{2}-\operatorname{jet}\left(\frac{\partial f}{\partial x}, l_{\left(x, w_{2}\right)}^{\prime}\right)
$$

have coinciding terms at a vertex monomial. The same holds true for

$$
w_{1}-\operatorname{jet}\left(\frac{\partial f}{\partial y}, l_{\left(y, w_{1}\right)}^{\prime}\right) \text { and } w_{2}-\operatorname{jet}\left(\frac{\partial f}{\partial y}, l_{\left(y, w_{2}\right)}^{\prime}\right) .
$$

All the terms of

$$
w_{1}-\operatorname{jet}\left(\frac{\partial f}{\partial x}, l_{\left(x, w_{1}\right)}^{\prime}\right)
$$

have the same $w_{1}$-degree. The same holds true for

$$
w_{1}-\operatorname{jet}\left(\frac{\partial f}{\partial y}, l_{\left(y, w_{1}\right)}^{\prime}\right) .
$$

Similarly, all terms of

$$
w_{2}-\operatorname{jet}\left(\frac{\partial f}{\partial x}, l_{\left(x, w_{2}\right)}^{\prime}\right)
$$

have the same $w_{2}$-degree, and the same holds true for the terms of $w_{2}$-jet $\left(\frac{\partial f}{\partial y}, l_{\left(y, w_{2}\right)}^{\prime}\right)$. Hence, we conclude that

$$
w_{2}-\operatorname{jet}\left(-\frac{n^{(j)}}{m_{w_{2}}} \frac{\partial f}{\partial y} \frac{\partial f}{\partial x}+\frac{n^{(j)}}{m_{w_{2}}} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}, l_{j}\right)
$$

also has terms below $w_{1}$-degree $d_{j}^{\prime \prime}$, which means that

$$
w_{2}-\operatorname{jet}\left(\frac{n^{(j)}}{m^{(j)}} a^{(j-1)} \frac{\partial f}{\partial x}+\frac{n^{(j)}}{m^{(j)}} b^{(j-1)} \frac{\partial f}{\partial y}, l_{j}\right)
$$

has terms below $w_{1}$-degree $d_{j}^{\prime \prime}$. This again implies that $w_{2}-\operatorname{jet}\left(a^{(j-1)} \frac{\partial f}{\partial x}-b^{(j-1)} \frac{\partial f}{\partial y}, l_{j-1}\right)$ has terms below $w_{1}$-degree $d_{j-1}^{\prime}$. We conclude that $a^{(j-1)} \frac{\partial f}{\partial x}-b^{(j-1)} \frac{\partial f}{\partial y}$ has terms below $w_{1}$-degree $d_{j-1}^{\prime}$, a contradiction.

Continuing as above, we can construct a syzygy $\left(a^{(k)}, b^{(k)}\right)$ of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y] / I^{(k)}$, where the syzygy equation $g_{k}$ has no terms below $w_{i}$-degree $d_{k}^{(i)}$, for $i=1,2, \ldots, n$. Let $r_{0}=\frac{n^{(1) \ldots} n^{(k)}}{m^{(1) \ldots m^{(k)}}}$ and $r_{j}=\frac{n^{(j)} \ldots n^{(k)}}{m^{(j+1) \ldots m^{(k)} m_{w_{j}}}}$, for $j=\{1, \ldots, k-1\}$, and $r_{k}=\frac{n^{(k)}}{m_{w_{i_{k}}}}$. By construction

$$
\begin{equation*}
w_{i}-\operatorname{jet}\left(r_{0} a, d_{k}^{(i)}-\left(w_{i}-\operatorname{ord}\left(\frac{\partial f}{\partial x}\right)\right)\right)=w_{i}-\operatorname{jet}\left(\sum_{j=1}^{k} r_{j} \frac{\partial f}{\partial y}, d_{k}^{(i)}-\left(w_{i}-\operatorname{ord}\left(\frac{\partial f}{\partial x}\right)\right)\right) . \tag{4}
\end{equation*}
$$

Since $r_{0}$ divides all the terms on the right side of the equation, it follows that if $\frac{r_{j}}{r_{0}} t \notin$ $\mathbb{C}[x, y]$, for some term $t$ of $\frac{\partial f}{\partial y}$, then $t$ either gets cancelled on the righthand side of the equation or $r_{j} t$ is of $w_{i}$-degree higher than $d_{k}^{(i)}-\left(w_{i}-\operatorname{ord}\left(\frac{\partial f}{\partial x}\right)\right)$ for all $i$, that is $r_{j} t \frac{\partial f}{\partial x}$ is of $w_{i}$-order higher than $d^{\prime}$ for all $i$. Hence $r_{j} t \frac{\partial f}{\partial x}$ is contained in $I^{(k)}$. Therefore

$$
\begin{equation*}
w_{i}-\operatorname{jet}\left(r_{0} a, d_{k}^{(i)}-\left(w_{i}-\operatorname{ord}\left(\frac{\partial f}{\partial x}\right)\right)\right)=w_{i}-\operatorname{jet}\left(\sum_{j=1}^{k} r_{j} \frac{\overline{\partial f}^{z_{j}}}{\partial y}, d_{k}^{(i)}-\left(w_{i}-\operatorname{ord}\left(\frac{\partial f}{\partial x}\right)\right)\right), \tag{5}
\end{equation*}
$$

where

$$
z_{j}=\frac{m^{(1)} \cdots m^{(j)}}{n^{(1)} \cdots n^{(j-1)} m_{w_{j}}} .
$$

Therefore

$$
a-\sum_{j=1}^{k} z_{j} \overline{\partial f}_{\partial y}^{z_{j}} \in \operatorname{Ann}\left(\frac{\partial f}{\partial y}\right) \quad \text { and } \quad b+\sum_{j=1}^{k} z_{j} \frac{\overline{\partial f}^{z_{j}}}{\partial x} \in \operatorname{Ann}\left(\frac{\partial f}{\partial y}\right)
$$

over $\mathbb{C}[x, y] / I$.
We will now show that if $y^{\alpha}$ is not a term of $\frac{\overline{\partial f}^{z_{j}}}{\partial y}$, then $x$ divides all terms of $z_{j} \frac{\overline{\partial f}^{\partial y}}{}{ }^{z_{j}}$ that are not in $\operatorname{Ann}\left(\frac{\partial f}{\partial x}\right)$ and do not get cancelled in the sum $\sum_{j=1}^{k} z_{j} \frac{\overline{\partial f}^{z_{j}}}{\partial y}$.

So suppose that $\frac{\overline{\partial f}^{z_{j}}}{\partial y}$, has only mixed terms. Let $c x^{\alpha} y^{\beta}, c \in \mathbb{C}$ be such a term of $\frac{\overline{\partial f}^{z_{j}}}{\partial y}$ such that $z_{j} c x^{\alpha} y^{\beta} \notin \operatorname{Ann}\left(\frac{\partial f}{\partial x}\right)$ and $z_{j} c x^{\alpha} y^{\beta}$ is not cancelled in the sum

$$
\sum_{j=1}^{k} z_{j} \overline{\partial f}_{\partial y}^{z_{j}}
$$

Clearly $x^{\alpha-1} y^{\beta+1}$ is a monomial of $\frac{\partial f}{\partial x}$, and there exists a monomial $l x^{\eta} y^{\gamma}, l \in \mathbb{C}$ of $\frac{\partial f}{\partial x}$ such that

$$
z_{j} c x^{\alpha} y^{\beta} x^{\eta} y^{\gamma} \notin I .
$$

But then $l x^{\eta+1} y^{\gamma-1}$ is a monomial of $\frac{\partial f}{\partial y}$. Furthermore, notice that $r_{j} \frac{c \alpha}{\beta+1} x^{\alpha-1} y^{\beta+1}$ is cancelled in $\sum_{j=1}^{k} r_{j} \frac{\partial f}{\partial x}$ if and only if $r_{j} c x^{\alpha} y^{\beta}$ is cancelled in $\sum_{j=1}^{k} r_{j} \frac{\partial f}{\partial y}$. Since, in
addition, $z_{j} x^{\alpha-1} y^{\beta+1} x^{\eta+1} y^{\gamma-1} \notin I$, it follows that $x^{\alpha-1} y^{\beta+1}$ is a monomial of $\frac{\overline{\partial f}^{z_{j}}}{\partial x}$. On the other hand, since $f$ is convenient, it follows that $y^{\alpha}, \alpha>0$, is a monomial of $\frac{\partial f}{\partial y}$, for some $\alpha>0$. Furthermore, since $f$ has a non-degenerate Newton boundary, it follows that if $y^{\beta}$ is a monomial of $\frac{\partial f}{\partial y}, \beta<\alpha$. Hence, since $y^{\alpha}$ is not a monomial of $\frac{\overline{\partial f}^{z_{j}}}{}$, $z_{j} y^{\alpha} \notin \mathbb{C}[x, y]$. This implies that $z_{j} y^{\beta} \notin \mathbb{C}[x, y]$. Therefore $y^{\beta}$, for any $\beta>0$ is not a monomial of $\frac{\overline{\partial f}_{\partial x}^{z_{j}}}{}$. This again implies that $\alpha>1$. Suppose $z_{j}=\frac{a_{j}}{b_{j}}$, then $b_{j} \mid x^{\alpha} y^{\beta}$ and $b_{j} \mid x^{\alpha-1} y^{\beta}$. Hence $x \mid z_{j} x^{\alpha} y^{\beta}$. In a similar way, we can prove that if $x^{\beta} \notin \overline{\frac{\partial f}{x}}^{z_{j}}$, then $y \left\lvert\, z_{j} \overline{\frac{\partial f}{\partial x}}^{z_{j}}\right., \beta>0$.

Analogously we can show that, if $x^{\beta}$ is not a term of $\frac{\overline{\partial f}^{z_{j}}}{\partial x}$ for all $j$, then $y$ divides all terms of $z_{j} \overline{\frac{\partial f}{\partial x}}^{z_{j}}$ that are not in $\operatorname{Ann}\left(\frac{\partial f}{\partial y}\right)$ and do not get cancelled.

In line 20 of Algorithm 1, we rely on Lemma 4.4 below. Since its statement is quite obvious, we postpone the proof of the lemma to the end of the chapter.

Lemma 4.4. Let $\Gamma$ be the Newton polygon of a germ with normalized, non-degenerate Newton boundary. Let $f_{0}$ be the sum of the monomials corresponding to the vertices of $\Gamma$, and let $B$ be a regular basis for $f_{0}$. Then
(1) any monomial corresponding to a lattice point of $\Gamma$ is a monomial of $f_{0}$ or an element of B, and
(2) at most two monomials of $f_{0}$ of degree $\leq \operatorname{dt}\left(f_{0}\right)$ are not in $B$.

The above lemma shows that for a germ with a normalized non-degenerate Newton boundary every monomial corresponding to a lattice point of its Newton polygon is a monomial occurring in any normal form of the germ with the same Newton polygon. Furthermore it shows that only two vertex monomials under the determinacy and on the Newton boundary are not parameter monomials in any such normal form. This implies that the Newton boundary of the given germ can be transformed to the Newton boundary of any of its normal form equations by scaling $x$ and $y$, except for terms above the determinacy.

We now prove correctness and termination of Algorithm 1 .
Proof of Algorithm 1. In the transformation $\phi$ in line 8 we know that terms coming from first partial derivatives with degree $\leq d^{\prime}$, not in $B$, cancel, as described in (3). We now discuss the effect of higher order terms in the binomial expansion after applying $\phi$. Now, it follows from Theorem 4.3 that $\phi(x)=x+\sum_{i} z_{i} \overline{\partial f}_{\partial y}^{z_{i}}$ and $\phi(y)=y+\sum_{i} z_{i} \frac{\frac{\partial f}{x}^{z_{i}}}{}$. Note that, since applying any transformation of filtration $<2$ to $f$ will change its nondegenerate Newton boundary to a degenerate boundary, $\phi$ has filtration $\geq 2$. That is $\operatorname{ord}\left(z_{i} \overline{\partial f}^{z_{i}}\right) \geq 2$ and ord $\left(z_{i} \frac{\overline{\partial f}}{}^{z_{i}}\right) \geq 2$. We systematically consider the terms coming from the higher order binomial expansions of $\phi(f)$. In higher order binomial expansions the terms are of the following form:

$$
\begin{equation*}
\frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot\left(z_{i_{1}} \overline{\partial f}^{z_{i_{1}}}\right) \cdots\left(z_{i_{s}} \overline{\partial f}^{z_{i_{s}}}\right) \cdot\left(z_{i_{s+1}} \overline{\partial f}^{z^{z_{s+1}}}\right) \cdots\left(z_{i_{n}}^{\frac{\partial f}{\partial x}^{z_{i_{n}}}}\right) \tag{6}
\end{equation*}
$$

$\overline{\text { Algorithm } 1 \text { Determining the moduli parameters in the normal Form of a germ with a }}$ normalized non-degenerate Newton boundary
Input: A polynomial germ $f \in \mathbb{Q}[x, y], f \in \mathfrak{m}^{3}$ of corank 2 with a normalized nondegenerate Newton boundary; $f_{0}$, the sum of the vertex monomials of $\Gamma(f)$; a set of monomials $B=\left\{b_{1}, \ldots, b_{m}\right\}$ that is the set of all monomials of $w(f)$-degree $\geq d(f)$ in a regular basis for $f_{0}$
Output: A normal form of $f$ and a normal form equation equivalent to $f$ such that the normal form equation is a member of the normal form.
Let $d^{\prime \prime}$ be a determinacy bound for $f$.
Let $w:=\left(w_{1}, \ldots, w_{n}\right):=w(f)$.
$F:=f_{0}+\sum_{i=1}^{m} \alpha_{i} \cdot b_{i}$.
$S:=\operatorname{mon}(f-w(f)-\operatorname{jet}(f, d(f)))$.
while $S \not \subset B$ do
Let $d^{\prime}$ be the lowest $w(f)$-degree above $d(f)$ with non-zero terms in $f$.
Write the sum $q$ of the terms of $w(f)$-degree $d^{\prime}$ as
$q=g \frac{\partial f}{\partial x}+h \frac{\partial f}{\partial y}+$ terms in $B$ of $w(f)$-degree $d^{\prime}+$ terms of higher $w(f)$-degree than $d^{\prime}$.

$$
\begin{aligned}
& \text { Define } \phi: \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y], \phi(x)=x+g, \phi(y)=y+h . \\
& \text { Let } l \text { be the filtration of } \phi . \\
& q:=w(j)-\operatorname{jet}\left(\phi(f)-\left(f+g \frac{\partial f}{\partial x}+h \frac{\partial f}{\partial y}\right), d^{\prime}\right) . \\
& f:=\operatorname{jet}\left(\phi(f), d^{\prime \prime}\right) \text {. } \\
& \text { while } q \neq 0 \text { do } \\
& \text { Write } \\
& \qquad q=\phi_{x} \cdot \frac{\partial f}{\partial y}+\phi_{y} \cdot \frac{\partial f}{\partial x}, \text { where } \phi_{x}, \phi_{y} \in\langle x, y\rangle^{l+1} . \\
& \quad \phi_{x}=w-\operatorname{jet}\left(\phi_{x}, d^{\prime}\right), \quad \phi_{y}=w-\operatorname{jet}\left(\phi_{y}, d^{\prime}\right) \\
& \text { Define } \phi: \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y] \text { by } \\
& \quad \phi(x):=x-w-\operatorname{jet}\left(\phi_{x}, d^{\prime}\right), \quad \phi(y):=y-w-\operatorname{jet}\left(\phi_{y}, d^{\prime}\right) . \\
& \quad q:=\phi(f)-(f+q) . \\
& f:=\operatorname{jet}\left(\phi(f), d^{\prime \prime}\right) \text {. } \\
& l=\min \left(\operatorname{ord}\left(\phi_{x}\right), \operatorname{ord}\left(\phi_{y}\right)\right) . \\
& S:=\operatorname{mon}(f-w(f)-\operatorname{jet}(f, d(f))) .
\end{aligned}
$$

Normalize two terms of $f$ of $w(f)$-degree $d(f)$, of degree $\leq d^{\prime \prime}$, and not in $B$ to coefficient 1.
if $\Gamma(f)$ does not intersect the $x$-axis then $f:=f+x^{a}$ with $a=\mu(f)+2$.
if $\Gamma(f)$ does not intersect the $y$-axis then $f:=f+y^{a}$ with $a:=\mu(f)+2$.
return $F, f$
where $n>1, i_{j}$ 's $\in \mathbb{Z}$, as well as terms in $J$, where $J$ is the ideal generated by all monomials of degree $d^{\prime}+1$. Let $z_{i_{j}}=\frac{a_{i_{j}}}{b_{i_{j}}}$. We distinguish between the following types of $b_{i_{j}}$ 's:
(i) $b_{i_{j}}=c x^{\alpha} y^{\beta}, \alpha, \beta>0$;
(ii) $b_{i_{j}}=c y^{\beta}, \beta>0$;
(iii) $b_{i_{j}}=c x^{\alpha}, \alpha>0$;
(iv) $b_{i_{j}}=1$.

We consider the following cases:
(1) $b_{i_{j}}=1$ for some $j \leq s$;
(2) $b_{i_{j}}=1$ for some $j>s$;
(3) all the $b_{i_{j}}$ 's are of type (i), (ii) and (iii), and the number of $b_{i_{j}}$ 's of type (i) and (iii) is $<s$.
(4) all the $b_{i_{j}}$ 's are of type (i), (ii) and (iii), the number of $b_{i_{j}}$ 's of type (ii) $<n-s$, and the number of $b_{i_{j}}$ 's of type (iii) is $<s$.
(5) all the $b_{i_{j}}$ 's are of type (i), (ii) and (iii), and the number of $b_{i_{j}}$ 's of type (i) and (ii) is $<n-s$.

Note that if $z_{k}=\frac{a_{k}}{b_{k}}$ and $b_{k}=1$, then

$$
\begin{equation*}
\frac{\frac{a_{k}}{b_{k}}\left(\overline{x^{s} \cdot y^{n-(s+1)} \frac{\partial^{n} f}{\partial x^{n} \partial y^{n-s}}} z_{k}\right)}{x^{s} \cdot y^{n-(s+1)}}=\frac{a_{k}\left(x^{s} \cdot y^{n-(s+1)} \frac{\partial^{n} f}{\partial x^{s} y^{n-s}}\right)}{x^{s} \cdot y^{n-(s+1)}}=z_{k}\left({\overline{\frac{\partial}{}_{n}} z_{k}}_{\partial x^{s} \partial y^{n-s}}\right) . \tag{7}
\end{equation*}
$$

Similarly

$$
\frac{\frac{a_{k}}{b_{k}}\left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{\partial} \partial y^{n-s}}} z_{k}\right.}{x^{s-1} \cdot y^{n-s}}=\frac{a_{k}\left(x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}\right)}{x^{s-1} \cdot y^{n-s}}=z_{k}\left({\frac{\frac{\partial}{}_{n}}{z_{k}}}_{\partial x^{s} \partial y^{n-s}}\right) .
$$

Furthermore, note that the monomials of $\left({\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{\partial} \partial y^{n-s}}}{ }^{z}}_{k}\right) \in \mathbb{C}[x, y]$ is a subset of the monomials of $\frac{\overline{\partial f}^{z_{k}}}{\partial x}$. Noticing that for all terms $t$ in $\frac{\partial f}{\partial y}-\frac{\partial^{z_{k}}}{\partial y}$, it follows from the proof of Theorem 4.3 that $a_{k} \cdot t \in \operatorname{Ann}_{R / J^{\prime}}\left(\frac{\partial f}{\partial x}\right)$, where $J^{\prime}$ is the ideal generated by all the terms of $w$-degree $\left(d^{\prime}+w-\operatorname{deg}\left(b_{k}\right)+1\right)$, and $z_{k}=\frac{a_{k}}{b_{k}}$, it follows that

$$
\begin{equation*}
\left(x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}\right) \cdot z_{k} \overline{\partial f}_{\partial y}^{z_{k}}=z_{k}\left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}{ }^{z_{k}}\right) \cdot \frac{\partial f}{\partial y}+J . \tag{8}
\end{equation*}
$$

In case (1), it hence follows from (7) and (8) that

$$
\begin{aligned}
& \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial y}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial x}\right) \\
& =\frac{1}{x^{s-1} \cdot y^{n-s}} \cdot\left(x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}\right) \sum_{k=1}^{s}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial y}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}} \overline{\partial f}^{z^{z_{i_{k}}}}\right) \\
& =\underbrace{\left(z_{i_{j}}{\overline{\partial^{n} f}}^{z_{i_{j}}}\right) \cdot \sum_{k=1 \ldots s, k \neq j}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial y}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial x}\right)}_{\in\langle x, y\rangle^{l+1} \subset \mathbb{C}[x, y]} \cdot \frac{\partial f}{\partial y} .
\end{aligned}
$$

Note that $\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i}}}{\partial x}\right),\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i}}}{}{ }^{i_{k}}\right) \in\langle x, y\rangle^{l} \subset \mathbb{C}[x, y]$, for all $k=s+1, \ldots, n$, where $l \geq 2$.

In case (2) it similarly follows that

$$
\begin{aligned}
& \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s}\left(z_{i_{k}} \frac{\overline{\partial f}^{z_{i_{k}}}}{\partial y}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}} \overline{\partial f}^{z_{i_{k}}}\right) \\
= & \underbrace{\left(z_{i_{j}} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}\right.}_{\in\langle x, y\rangle^{l+1} \subset \mathbb{C}[x, y]}) \cdot \sum_{k=1}^{s}\left(z_{i_{k}} \overline{\partial f}_{\partial y}^{\partial z_{k}}\right) \cdot \sum_{k=s+1, \ldots n, k \neq j}\left(z_{i_{k}} \frac{\overline{\partial f}}{\partial x}\right)
\end{aligned} \frac{\partial f}{\partial x} .
$$

Next we consider case (3). Note that

$$
\begin{aligned}
\left(x^{s-1} y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}\right) \cdot z_{k} \frac{\overline{\partial f}_{\partial y}^{z}}{z_{k}} \cdot z_{l} \frac{\overline{\partial f}^{z_{l}}}{\partial x} & =z_{k}\left({\left.\overline{x^{s-1} y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}{ }^{z_{k}}\right) \cdot \frac{\partial f}{\partial y} \cdot z_{l} \overline{\partial f}_{\partial x}^{z_{l}}}=z_{k}\left({\overline{x^{s-1} y^{n-s}} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}^{z_{k}}\right) \cdot z_{l} \frac{\overline{\partial f}^{z_{l}}}{\partial y} \cdot \frac{\partial f}{\partial x}\right.
\end{aligned}
$$

Using the above method, it follows that

$$
\begin{aligned}
& \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s}\left(z _ { i _ { k } } { \overline { \frac { \overline { f } ^ { z _ { i } } } { \partial y } } ) \cdot \sum _ { k = s + 1 } ^ { n } ( z _ { i _ { k } } \frac { \overline { \partial f } ^ { z _ { i _ { k } } } } { \partial x } ) } _ { = } \frac { z _ { i _ { j } } } { x ^ { s - 1 } y ^ { n - s } } \left({\overline{x^{s-1} y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}} z_{i_{j}}} \sum_{k=1, \ldots, s, s \neq j}\left(z_{i_{k} \frac{f}{\partial y}}^{z_{i_{k}}}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}}{\overline{\frac{\partial f}{}} \bar{z}^{z_{i_{k}}}}_{\partial x}\right) \frac{\partial f}{\partial y},\right.\right.
\end{aligned}
$$

where all the $b_{i_{j}}$ 's are ordered that first, $b_{i_{j}}$ 's of type (i) and (iii) occur and then those of type (ii). In other words the last $n-s+1 b_{i_{j}}^{\prime}$ s are of type (ii). This means by Theorem 4.3 that

$$
\begin{aligned}
& \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s}\left(z_{i_{k}} \overline{\partial f}^{z_{i_{k}}}\right) \cdot \sum_{k=s+1}^{n}\left(z_{i_{k}} \overline{\partial f}^{z_{i_{k}}}\right) \\
= & \frac{1}{x^{s-1}} \cdot z_{i_{s}}\left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}{ }^{z_{i_{s}}}\right) \sum_{k=1, \ldots, s-1}\left(z_{i_{k}} \overline{\partial f}_{\partial y}^{z_{i}}\right) \cdot \sum_{k=s+1}^{n} \underbrace{\frac{\left(z_{i_{k}} \frac{\partial \bar{z}^{z_{i}}}{\partial x}\right)}{y}}_{\in \mathbb{C}[x, y]} \frac{\partial f}{\partial y},
\end{aligned}
$$

Since the number of $b_{i_{i}}^{\prime} s$ that are of type (ii) is $>n-s, b_{i_{s}}$ is of type (ii). But then,
 as

$$
\begin{equation*}
\underbrace{q^{\prime} \frac{\bar{z}_{k} \frac{\overline{\partial f}^{z_{k}}}{\partial x}}{y}}_{\in\langle x, y\rangle^{++1}} \frac{\partial f}{\partial y}, \quad \text { with } q^{\prime} \in\langle x, y\rangle^{2}, \tag{9}
\end{equation*}
$$

and $b_{k}$ is of type (ii). In case (4) it follows similarly that (6) can be expressed as in (9), where $b_{k}$ is of type (i) or (ii).
In case (5), exchanging $x$ and $y$ in (9), we express (6) as

$$
\underbrace{q^{\prime} \frac{z_{\frac{z^{\frac{\partial}{\partial y}}}{\partial y}}^{x}}{x}}_{\in\langle x, y\rangle^{l+1}} \frac{\partial f}{\partial x}, \quad \text { with } q^{\prime} \in\langle x, y\rangle^{2},
$$

where $b_{k}$ is of type (iii).
We conclude that $q$ can be expressed as

$$
\begin{equation*}
q=\phi_{x} \cdot \frac{\partial f}{\partial x}+\phi_{y} \frac{\partial f}{\partial y} \tag{10}
\end{equation*}
$$

where $\phi_{x}, \phi_{y} \in\langle x, y\rangle^{l+1}$. Then the higher orders of the binomial expansion of $\phi(f)$ of degree $\leq d^{\prime}$ can be removed by the first order terms of the transformation $\phi: \mathbb{C}[x, y] \rightarrow$ $\mathbb{C}[x, y]$ defined by

$$
\phi_{\text {new }}(x):=x-\phi_{x}, \phi_{\text {new }}(y):=y-\phi_{y} .
$$

In an analogous way one can show that the sum, $q_{\text {new }}$, of higher order terms of the binomial expansion of degree $\leq d^{\prime}$ of $\phi_{\text {new }}(\phi(f))$ can be written in terms of the same formulas, now in terms of $\phi_{\text {new }}(x)=x-\sum_{i} z_{i}^{\prime} \frac{\overline{\partial f}^{z^{\prime}}}{}{ }^{\prime}$ and $\phi_{\text {new }}(y)=y-\sum_{i} z_{i}^{\prime} \frac{\overline{\partial f}^{z^{\prime}}}{} z_{i}^{\prime}$, as above. Note that the filtration of $\phi_{\text {new }}$ is higher than that of $\phi$. Hence eventually there will be no terms, not in $B$, of $w$-degree $\leq d^{\prime}$.

For many examples the transformations arising from line 7 can be chosen in such a way that the higher orders of the binomial expansion of $\phi(f)$ are of degree larger than $d^{\prime}$. Choosing $g$ and $h$ in such a way, the while-loop from line 12 to 18 is redundant. The next example proves that this is unfortunately not in general the case.

Example 4.5. Let $f=y^{28}+x y^{7}+x^{2} y^{3}+11 x^{2} y^{4}+x^{22}$. Then $\frac{\partial f}{\partial x}=2 x y^{3}+22 x y^{4}+$ $y^{7}+22 x^{21}$ and $\frac{\partial f}{\partial y}=3 x^{2} y^{2}+44 x^{2} y^{3}+7 x y^{6}+28 y^{27}$, and a regular basis for $f$ is $x^{22} y, x^{21} y, x^{20} y, x^{19} y, x^{18} y, x^{17} y, x^{16} y, x^{2} y^{3}$. Note that $x^{2} y^{4}$ is the only monomial above the Newton Boundary, with $w(f)$-degree 1008. We can express $-11 x^{2} y^{4}$ as follows in terms of the first partial derivatives.

$$
-11 x^{2} y^{4}=-\left(7 x y+28 y^{22}\right) \frac{\partial f}{\partial x}+y^{2} \frac{\partial f}{\partial y}
$$

The corresponding transformation $\phi$ is given by

$$
\begin{aligned}
\phi(x) & =x-\left(7 x y+28 y^{22}\right)=x-\sum_{i} z_{i} \frac{\partial f}{\partial y}=x-\frac{1}{y^{5}} \frac{\overline{\partial f}}{\partial y}{ }^{\frac{1}{y^{5}}}+p_{x} \\
\phi(y) & =y+y^{2}=y-\sum_{i} z_{i} \frac{\partial f}{\partial x}=y-\left(-\frac{1}{y^{5}} \frac{\partial f}{\partial x}^{\frac{1}{y^{5}}}\right)+p_{y}
\end{aligned}
$$

where $p_{x} \frac{\partial f}{\partial x}$ and $p_{y} \frac{\partial f}{\partial y}$ are of $w(f)$-degree 1008 or higher. Note that the filtration of $\phi$ is $l=2$. Now let

$$
q=w-\operatorname{jet}\left(\phi(f)-f-\phi(x) \frac{\partial f}{\partial x}-\phi(y) \frac{\partial f}{\partial y}, 1008\right)
$$

Then $q$ is the contribution of the orders $>1$ in the binomial expansion of $\phi$ in the 1008-jet of $\phi(f)$.

$$
q=-28 x y^{9}+182 y^{30}
$$

To write $q$ as in line 13 in Algorithm 1, we are computing $\phi_{x}$ and $\phi_{y}$. We do this by forming an ideal $I$ generated by the set

$$
\left\{x^{i} y^{j} \frac{\partial f}{\partial x}, \left.x^{i} y^{j} \frac{\partial f}{\partial y} \right\rvert\, \quad i+j=l+1=3\right\}
$$

and all monomials of higher piecewise degree than 1008. We then write $q$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, where the coefficients are of order 3 or higher, using a Gröbner basis of $I$ with
a global ordering, and then lifting the result back to the original generators using the command liftstd() in Singular. It turns out that

$$
q=\left(\left(-28-364 y^{17}-4004 y^{18}\right) x y^{2}+182 y^{23}\right) \cdot \frac{\partial f}{\partial x}
$$

By abuse of notation, let $f=\phi(f)$. We now form a new $\phi$ as described in line 15 of Algorithm 1. We set

$$
\phi_{x}=28 x y^{2}-182 y^{23} \text { and } \phi_{y}=0
$$

We define $\phi: \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$ by $\phi(x)=x-\phi_{x}$ and $\phi(y)=y-\phi_{y}=y$. Hence the first order terms of the binomial expansion of $\phi(f)$ up to degree 1008 cancel $q$. In this case higher order terms of the binomial expansion of $\phi(f)$ does not have terms of $w(f)$-degree 1008 or lower. In fact $w-\operatorname{jet}(\phi(f), 1008)=y^{28}+x y^{7}+x^{2} y^{3}+x^{22}$.

In the next example more than one iteration of the while-loop in Algorithm 1 is needed to transform the germ $f$ to its normalform equation up to $w(f)$-degree 700 .

Example 4.6. For $f=x^{2} y^{4}+x^{4} y^{2}+x^{20}+y^{40}+60 x^{21} y^{14}$, we have $\frac{\partial f}{\partial x}=4 x^{3} y^{2}+2 x y^{4}+$ $20 x^{19}+1260 x^{20} y^{14}$ and $\frac{\partial f}{\partial y}=2 x^{4} y+4 x^{2} y^{3}+840 x^{21} y^{13}+40 y^{39}$, and a regular basis is $x^{4} y^{4}, x^{4} y^{3}, x^{3} y^{4}, x^{4} y^{2}, x^{3} y^{3}, x^{2} y^{4}$. Note that $x^{21} y^{14}$ is the only monomial above the Newton boundary with $w(f)$-degree 700 . Now $60 x^{21} y^{14}$ can be written as

$$
60 x^{21} y^{14}=\left(2 x^{4} y^{12}+4 x^{2} y^{14}+40 y^{50}-10 x^{20} y^{10}\right) \frac{\partial f}{\partial x}-\left(4 x^{3} y^{13}+2 x y^{15}\right) \frac{\partial f}{\partial y}
$$

The corresponding transformation $\phi$ is given by

$$
\left.\begin{array}{rl}
\phi(x) & =x-\left(2 x^{4} y^{12}+4 x^{2} y^{14}+40 y^{50}-10 x^{20} y^{10}\right)=x-y^{11} \overline{\partial f}_{\partial y}^{y^{11}}-5 x^{16} y^{9} \overline{\partial f}^{x^{16}} y^{9}+p_{x} \\
\phi(y) & =y+\left(4 x^{3} y^{13}+2 x y^{15}\right)=y-\left(-y^{11} \overline{\overline{\partial f}}^{y^{11}}\right.
\end{array}-5 x^{16} y^{9} \frac{\overline{\partial f}}{\partial x}^{x^{16} y^{9}}\right)+p_{y}, ~ l
$$

where $p_{x} \frac{\partial f}{\partial x}$ and $p_{y} \frac{\partial f}{\partial y}$ is of $w$-degree 700 or higher. Note that the filtration of $\phi$ is $l=16$. Now let

$$
q=w-\operatorname{jet}\left(\phi(f)-f-\phi(x) \frac{\partial f}{\partial x}-\phi(y) \frac{\partial f}{\partial y}, 700\right)
$$

Then $q$ is the contribution of the orders $>1$ in the binomial expansion of $\phi$ in the 700 -jet of $\phi(f)$.

$$
\begin{aligned}
q= & -24 x^{10} y^{26}-12 x^{8} y^{28}-12 x^{6} y^{30}-24 x^{4} y^{32}-224 x^{7} y^{44}-32 x^{5} y^{46}+144 x^{6} y^{60} \\
& +12160 x^{6} y^{64}+12640 x^{4} y^{66}+2800 x^{2} y^{68}+384 x^{7} y^{74}+952320 x^{7} y^{78} \\
& +472320 x^{5} y^{80}+79680 x^{3} y^{82}+256 x^{8} y^{88}+11704320 x^{6} y^{94}+1467360 x^{4} y^{96} \\
& +9600 x^{2} y^{102}+1600 y^{104}+21065216 x^{5} y^{110}+38400 x^{5} y^{114}-12800 x^{3} y^{116} \\
& +12800 x y^{118}+245661440 x^{6} y^{124}+38400 x^{4} y^{130}+38400 x^{2} y^{132}+51200 x^{3} y^{146} \\
& -256000 x y^{152}+25600 x^{4} y^{160}+2560000 y^{202}
\end{aligned}
$$

Note that the order of $q$ is 36 .
To write $q$ as in line 13 in Algorithm 1, we are computing $\phi_{x}$ and $\phi_{y}$. We do this by forming an ideal $I$ generated by

$$
\left\{x^{i} y^{j} \frac{\partial f}{\partial x}, \left.x^{i} y^{j} \frac{\partial f}{\partial y} \right\rvert\, \quad i+j=l+1=17\right\}
$$

and all monomials of higher piecewise degree than 700 . We write $q$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, by using a Gröbner basis of $I$ with a global ordering, and then lifting the result back to the original generators using the command liftstd(). We now form $\phi$ as described in line 15 of Algorithm 1. Let $\phi_{x}$ and $\phi_{y}$ be as defined in line(14) in Algorithm 1. It turns out that

$$
\begin{aligned}
\phi_{x}= & 256 x^{21} y^{10}+512 x^{19} y^{12}+128 x^{17} y^{14}+448 x^{15} y^{16}+288 x^{13} y^{18}-144 x^{11} y^{20} \\
& +72 x^{9} y^{22}-42 x^{7} y^{24}+18 x^{5} y^{26}-12 x^{3} y^{28}+\frac{12650}{567} x^{34} y^{10}-320 x^{10} y^{36} \\
& +160 x^{8} y^{38}-80 x^{6} y^{40}-16 x^{4} y^{42}+288 x^{9} y^{52}-144 x^{7} y^{54}+72 x^{5} y^{56}-4800 x^{9} y^{56} \\
& +2400 x^{7} y^{58}-1200 x^{5} y^{60}+3640 x^{3} y^{62}+1320 x y^{64}-384 x^{8} y^{68}+192 x^{6} y^{70} \\
& -326400 x^{8} y^{72}+163200 x^{6} y^{74}+156480 x^{4} y^{76}+39840 x^{2} y^{78}+128 x^{7} y^{84} \\
& -8769600 x^{7} y^{88}+4384800 x^{5} y^{90}+733680 x^{3} y^{92}-38400 x^{7} y^{92}+19200 x^{5} y^{94} \\
& -9600 x^{3} y^{96}+4800 x y^{9} 8-21065216 x^{6} y^{104}+10532608 x^{4} y^{106}-115200 x^{6} y^{108} \\
& +57600 x^{4} y^{110}-19200 x^{2} y^{112}+6400 y^{114}+122830720 x^{5} y^{120}+38400 x^{5} y^{124} \\
& -19200 x^{3} y^{126}+19200 x y^{128}-51200 x^{4} y^{140}+25600 x^{2} y^{142}-512000 x^{4} y^{144} \\
& +256000 x^{2} y^{146}-128000 y^{148}+12800 x^{3} y^{156}+192000 x^{3} y^{160}-128000 x y^{162}
\end{aligned}
$$

and

$$
\phi_{y}=-512 x^{20} y^{11}-256 x^{18} y^{13}-256 x^{16} y^{15}-512 x^{14} y^{17}-2560 x^{36} y^{9}+40 y^{65}+64000 y^{163} .
$$

Note that order of $\phi_{x}$ and $\phi_{y}$, and hence the filtration of $\phi$, is 31, respectively. Unfortunately the higher order binomial expansions of $\phi(f)$ again contributes terms of $w$-degree $\leq 700$. Note that this time the order of these terms is $66>36$. Now let

$$
q=w-\operatorname{jet}\left(\phi(f)-f-\phi(x) \frac{\partial f}{\partial x}-\phi(y) \frac{\partial f}{\partial y}, 700\right) .
$$

Then

$$
\begin{aligned}
q= & 1296 x^{8} y^{58}-1008 x^{6} y^{60}-3072 x^{7} y^{74}-7488 x^{8} y^{88}+736800 x^{6} y^{94}+189120 x^{4} y^{96} \\
& +6049280 x^{5} y^{110}+13692880 x^{6} y^{124}-98264000 x^{4} y^{130}-12833600 x^{2} y^{132} \\
& -681446400 x^{3} y^{146}-23950380800 x^{4} y^{160}-77184000 x^{2} y^{166}-5408000 y^{168} \\
& -94208000 x y^{182}+33001344000 x^{2} y^{196}+13038080000 y^{232} .
\end{aligned}
$$

Let $f=\phi(f)$. Repeating the process, we again write $q$ in terms of

$$
\left\{x^{i} y^{j} \frac{\partial f}{\partial x}, x^{i} y^{j} \frac{\partial f}{\partial y} \left\lvert\, \begin{array}{c|c}
i+j=l+1=32
\end{array}\right.\right\},
$$

and all terms of piecewise degree higher than 700 to compute $\phi_{x}$ and $\phi_{y}$. It turns out that

$$
\begin{aligned}
\phi_{x}= & -3312 x^{9} y^{52}+1656 x^{7} y^{54}-504 x^{5} y^{56}+3072 x^{8} y^{68}-1536 x^{6} y^{70}-3744 x^{7} y^{84} \\
& -358560 x^{7} y^{88}+179280 x^{5} y^{90}+94560 x^{3} y^{92}-6049280 x^{6} y^{104}+3024440 x^{4} y^{106} \\
& +68461440 x^{5} y^{120}+73408000 x^{5} y^{124}-36704000 x^{3} y^{126}-6146400 x y^{128} \\
& +681446400 x^{4} y^{140}-340723200 x^{2} y^{142}-11975190400 x^{3} y^{156}+77184000 x^{3} y^{160} \\
& -38592000 x y^{162}+94208000 x^{2} y^{176}-47104000 y^{178}+15848768000 x y^{192}
\end{aligned}
$$

and

$$
\phi_{y}=-135200 y^{129}+325952000 y^{193} .
$$

Applying $\phi$, formed as in line 15$)$,

$$
w-\operatorname{jet}(\phi(f), 700)=x^{4} y^{2}+x^{2} y^{4}+x^{20}+y^{40}
$$

We now turn to the proof of our two main lemmata, and begin with Lemma 4.1.
Proof. We say that terms of $f_{x}:=\frac{\partial f_{\Gamma^{\prime}}}{\partial x}$ and $f_{y}:=\frac{\partial f_{\Gamma^{\prime}}}{\partial y}$ are in correspondence if they originate from the same term of $f_{\Gamma^{\prime}}$.

- If $a \neq 0$, then $x^{a}$ is the largest power of $x$ dividing $f_{x}$, and $x^{a+1}$ is the largest power of $x$ dividing $f_{y}$. To see this, we first note that, excluding a pure $x$-power term in $f_{x}$, all terms of $f_{x}$ and $f_{y}$ are in one-to one correspondence. If $f_{x}$ would be divisible by $x^{a+1}$ then, since all terms of $f_{y}$ are in correspondence to a term in $f_{x}$, all terms of $f_{y}$ would be divisible by $x^{a+2}$, contradicting the minimal choice of $a$. Thus $x^{a}$ is the largest power of $x$ dividing $f_{x}$, and $f_{y}$ is divisible by $x^{a+1}$. If $x^{a+2}$ would divide $f_{y}$, then $x^{a+1}$ would also divide every corresponding term of $f_{x}$. Since also a possible $x^{\alpha}$-term of $f_{x}$ is then divisible by $x^{a+2}$ (since it is the term with the largest $x$-exponent), this again contradicts the minimal choice of $x$. Moreover the argument above implies that $x^{a+1}$ is the highest power of $x$ dividing $f$.
- If $a=0$, then $x \nmid f_{y}$ or $x \nmid f_{x}$.
- If $x \nmid f_{y}$ then $f_{\Gamma^{\prime}}$ has a term that is a pure $y$-power. Hence $x \nmid f$.
- If $x \mid f_{y}$, then $x \nmid f_{x}$, which implies that $f_{\Gamma^{\prime}}$ has a term of the form $x y^{l}$ with $l \geq 1$. Hence, in this case, if $x^{s}$ divides $f_{y}$ (and hence $f$ ), $s=1$.
Now, let $x^{a^{\prime}} y^{b^{\prime}}$ be the product of the highest power of $x$ and the highest power of $y$ dividing $\operatorname{jet}\left(f, \Gamma^{\prime}\right)$.
- Then $a^{\prime}=a+1$, if $a \neq 0$.
- If $a=0$, we have two cases:
- If $x \nmid f_{y}$, then $a^{\prime}=0$.
- If $x \mid f_{y}$, then 1 is the highest power of $x$ dividing $f_{\Gamma^{\prime}}$. Hence $a^{\prime}=a+1$.

Similarly $b^{\prime}=b+1$, if $b \neq 0$. If $b=0$ and $y \nmid f_{x}$, then $b^{\prime}=0$. If $b=0, y \mid f_{x}$ (implying that $y \nmid f_{y}$, then 1 is the highest power of $y$ dividing $f_{\Gamma^{\prime}}$. Hence $b^{\prime}=b+1$.

We assume that

$$
g \cdot f_{x}+h \cdot f_{y}=0
$$

Then

$$
x^{a} y^{b}\left(g \cdot y^{t} \cdot x^{i} \cdot \operatorname{sat}\left(f_{x}\right)+h \cdot x^{s} \cdot y^{\nu} \cdot \operatorname{sat}\left(f_{y}\right)\right)=0
$$

where $s, t, i$ and $\nu$ are defined as above. To see this, consider first the case $a \neq 0, b \neq 0$, then

$$
x^{a} y^{b}\left(g \cdot y \cdot \operatorname{sat}\left(f_{x}\right)+h \cdot x \cdot \operatorname{sat}\left(f_{y}\right)\right)=x^{a^{\prime}} y^{b^{\prime}}\left(\frac{g}{x} \cdot \operatorname{sat}\left(f_{x}\right)+\frac{h}{y} \cdot \operatorname{sat}\left(f_{y}\right)\right)=0
$$

which can be verified by the arguments above. The other cases can be similarly verified. If $a \neq 0$, or if $a=0$, and $x \mid f_{y}$, then $s=1$ and $a^{\prime}=a+1$. In both these cases $i=0$. If $a=0$ and $x \nmid f_{y}$, then $a^{\prime}=a=0$ and $s=0$. Therfore we have in general.

$$
x^{a^{\prime}} y^{b}\left(g \cdot y^{t} \cdot x^{i} \cdot \operatorname{sat}\left(f_{x}\right)+h \cdot x^{s} \cdot y^{\nu} \cdot \operatorname{sat}\left(f_{y}\right)\right)=0
$$

Similarly, if $b \neq 0$, or if $b=0$, and $y \mid f_{x}$, then $\nu=1$ and $b^{\prime}=b+1$. In both these cases $t=0$. If $b=0$ and $y \nmid f_{x}$, then $b^{\prime}=b=0$ and $t=0$. Therfore we have in general.

$$
x^{a^{\prime}} y^{b^{\prime}}\left(\frac{g}{x^{s}} \cdot x^{i} \cdot \operatorname{sat}\left(f_{x}\right)+\frac{h}{y^{t}} \cdot y^{\nu} \cdot \operatorname{sat}\left(f_{y}\right)\right)=0
$$

Therefore

$$
\frac{g}{x^{s}} \cdot x^{i} \cdot \operatorname{sat}\left(f_{x}\right)+\frac{h}{y^{t}} \cdot y^{\nu} \cdot \operatorname{sat}\left(f_{y}\right)=0
$$

Hence $x \mid g$, if $a \neq 0$, or if $a=0$ and $x \mid f_{y}$, and $y \mid h$, if $b \neq 0$, or if $b=0$ and $y \mid f_{x}$.
We now show that $x^{i} \cdot \operatorname{sat}\left(f_{x}\right)$ and $y^{\nu} \cdot \operatorname{sat}\left(f_{y}\right)$ have no common monomial factor. To do that, first note that $\operatorname{sat}\left(f_{\Gamma^{\prime}}\right)$ is nondegenerate by definition. This implies that sat $\left(f_{\Gamma^{\prime}}\right)$ does not have a multiple monomial factor, hence $\operatorname{sat}\left(f_{\Gamma^{\prime}}\right)$ and $\frac{\partial \text { sat } f_{\Gamma^{\prime}}}{\partial x}$ have no common monomial factor, and $\operatorname{sat}\left(f_{\Gamma^{\prime}}\right)$ and $\frac{\partial \text { sat } f_{\Gamma^{\prime}}}{\partial y}$ have no common monomial factor. Thus, $x^{a^{\prime}} y^{b^{\prime}}$ accounts for the only multiple factors in $f_{\Gamma^{\prime}}$.

Consider the following equations for the partial derivatives of $f_{\Gamma^{\prime}}$ :

$$
\begin{aligned}
& f_{x}=a^{\prime} x^{a^{\prime}-1} y^{b^{\prime}}\left(\operatorname{sat}\left(f_{\Gamma^{\prime}}\right)\right)+x^{a^{\prime}} y^{b^{\prime}}\left(\frac{\partial \operatorname{sat} f_{\Gamma^{\prime}}}{\partial x}\right) \\
& f_{y}=b^{\prime} x^{a^{\prime}} y^{b^{\prime}-1}\left(\operatorname{sat}\left(f_{\Gamma^{\prime}}\right)\right)+x^{a^{\prime}} y^{b^{\prime}}\left(\frac{\partial \operatorname{sat} f_{\Gamma^{\prime}}}{\partial y}\right)
\end{aligned}
$$

If the saturations of the partial derivatives of $f_{\Gamma^{\prime}}$ with respect to $x$ and $y$ share a common factor, then this factor would also be a factor of sat $\left(f_{\Gamma^{\prime}}\right)$. However, this contradicts the previous equations. Furthermore $y \nmid \operatorname{sat}\left(f_{x}\right)$ if $\nu>0$ and $x \nmid \operatorname{sat}\left(f_{y}\right)$ if $i>0$.

Since $x^{i} \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right), y^{\nu}$ sat $\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial y}\right)$ is a regular sequence, the vector $\left(\frac{g}{x^{s}}, \frac{h}{y^{t}}\right)$ is a polynomial multiple of the Koszul syzygy of $\left(x^{i} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma^{\prime}}}{\partial y}\right)\right)$. This completes the proof.

We turn now to the proof of Lemma 4.4, where we use the following observation:
Remark 4.7. With the notation of Lemma 4.4 , let $m$ be a monomial corresponding to a lattice point of $\Gamma$. Suppose $g$ is a germ with non-degenerate Newton boundary $\Gamma(g)=\Gamma$ and with monomial $m$, such that the corresponding term cannot be removed from $g$ by a right equivalence. Then Theorem 3.20 implies that $m$ is a monomial of $f_{0}$ or $m \in B$.

We now prove Lemma 4.4.
Proof. We prove the first part of the lemma: It is sufficient to show that for any monomial $m$ in the relative interior of a face $\Delta$ of $\Gamma$, there exists a germ $g$ as in Remark 4.7. For any germ $g$ with $\Gamma(g)=\Gamma$, we can write

$$
\operatorname{jet}(g, \Delta)=x^{a} \cdot y^{b} \cdot g_{1} \cdots g_{n} \cdot \widetilde{g}
$$

where $a, b$ are integers, $g_{1}, \ldots, g_{n}$ are linear homogeneous polynomials not associated to $x$ or $y$, and $\widetilde{g}$ is a product of non-associated irreducible non-linear homogeneous polynomials. Note that $a$ and $b$ are the distances of $\Delta$ from the $y$ - and $x$-coordinate axes. Note also that smooth faces meeting the coordinate axes do not contain interior lattice points. After possibly exchanging $x$ and $y$, we may assume that for the weight $w$ of $\Delta$ we have $w(x) \geq w(y)$. First consider the case that $w(x)>w(y)$ :
(1.a) Suppose that $a \geq 2$. Let $g$ be a germ non-degenerate Newton boundary $\Gamma$. Then any right-equivalence which does not act only as a rescaling of variables on $\operatorname{jet}(g, \Delta)$ generates terms on the coordinate axes below $\Gamma$, hence, changes the Newton polygon. The claim follows directly (choosing a non-degenerate $g=f_{0}+c \cdot m$ with $\left.c \in \mathbb{C}^{*}\right)$.
(1.b) Consider now the case $a=0$. Since $\Gamma$ corresponds to a normalized germ with respect to $\Delta$, we have $n=0$ and $\widetilde{g} \neq 1$. This implies that $w(y) \nmid w(x)$, hence there does not exist a $w$-weighted homogeneous right-equivalence except rescaling of variables.
(1.c) Finally, suppose $a=1$. If $w(y) \nmid w(x)$, then, as above, $n=0$ and $\tilde{g} \neq 0$ which implies that rescalings are the only right-equivalences on the face. If $w(y) \mid w(x)$, then, writing $\tau=w(x) / w(y)$, any right-equivalence which does not create any terms of lower $w$-weight than that of $\Delta$ is of the form

$$
x \mapsto c_{1} x+c_{2} y^{\tau}, y \mapsto c_{3} y,
$$

where $c_{1}, c_{3} \in \mathbb{C}^{*}$ and $c_{2} \in \mathbb{C}$, hence acts on $y$ as a rescaling. We may therefore assume that $b=0$. The vertices of $\Delta$ correspond to monomials of the form $x^{p}$ and $x y^{(p-1) \cdot \tau}$ with $p \geq 2$. Any monomial in the interior of $\Delta$ is of the form $m=$ $x^{s} y^{\tau \cdot(p-s)}$ with $0<s<p$. For $g=f_{0}+c \cdot m$ with $c \in \mathbb{C}^{*}$, the jet with regard to $\Delta$ is then $\operatorname{jet}(g, \Delta)=x^{p}+c \cdot m+x y^{(p-1) \cdot \tau}$. We now show that there is no right-equivalence which keeps $\Gamma(g)$ and, hence, is of the above form, that removes $m$. Keeping the face $\Delta$ and removing $m$ amounts to the conditions

$$
\begin{aligned}
& \binom{p}{p-s} c_{1}^{s} c_{2}^{p-s}+c \cdot c_{1}^{s} c_{3}^{\tau \cdot(p-s)}=0 \\
& c_{2}^{p}+c \cdot c_{2}^{s} c_{3}^{\tau \cdot(p-s)}+c_{2} c_{3}^{(p-1) \cdot \tau}=0
\end{aligned}
$$

which correspond to the vanishing of the coefficients of $m$ and $y^{p \cdot \tau}$. Using $c_{1} \neq 0$ a solution of the first equation for $c_{2}$ is of the form

$$
c_{2}=\tilde{c} \cdot c_{3}^{\tau}
$$

where $\tilde{c}^{p-s}=-c /\binom{p}{s}$. Inserting this into the second equation, leads to the equation

$$
\begin{aligned}
0 & =\tilde{c}^{p} \cdot c_{3}^{\tau \cdot p}+c \cdot \tilde{c}^{s} c_{3}^{\tau \cdot p}+\tilde{c} \cdot c_{3}^{\tau \cdot p} \\
& =c_{3}^{\tau \cdot p} \cdot\left(\tilde{c}^{p}+c \cdot \tilde{c}^{s}+\tilde{c}\right) \\
& =c_{3}^{\tau \cdot p} \cdot\left(\tilde{c}^{p}-\binom{p}{s} \cdot \tilde{c}^{p}+\tilde{c}\right) \\
& =c_{3}^{\tau \cdot p} \cdot \tilde{c} \cdot\left(\left(1-\binom{p}{s}\right) \cdot \tilde{c}^{p-1}+1\right)
\end{aligned}
$$

Since $\binom{p}{s} \neq 1$, there is a Zariski open set of values of $\tilde{c}$, equivalently of $c$, such that the expression in the bracket does not vanish and $g$ is non-degenerate. For such a choice of $c$ and thus of $g$, it follows that $c_{3}=0$, a contradiction.

Now consider the case that $w(x)=w(y)$. Since jet $(g, \Delta)$ is homogeneous, hence factorizes into linears, in this case we have $a \geq 1, b \geq 1$ and $\tilde{g}=0$.
(1.d) If $a, b \geq 2$, and $g$ is a germ with non-degenerate Newton boundary and Newton polygon $\Gamma$, then any right-equivalence which does not only act as a rescaling of variables on $\operatorname{jet}(g, \Delta)$ changes the Newton polygon (with the same argument as in the case $w(x)>w(y), a \geq 2)$.
(1.e) The case $a \geq 2$ and $b=1$, can be handled in the same way as the case $w(x)>w(y)$, $a=1, w(y) \mid w(x)$ above. An analoguos argument also applies to the case $b \geq 2$ and $a=1$.
(1.f) In the case $a=b=1$, a Gröbner basis calculation shows directly that any monomial corresponding to a lattice point of $\Delta$ is either a monomimal of $f_{0}$ or an element of $B$ (note that smooth faces do not contain any interior lattice points): The germ
$x^{p} y+x y^{p}, p \geq 2$, is right-equivalent to a germ $h$ (applying for instance the rightequivalance $x \mapsto x+y, y \mapsto y+2 x$ ) with vertex monomials $x^{p+1}$ and $y^{p+1}$, hence, by Theorem has Milnor number $\mu=(p+1)^{2}-2(p+1)+1=p^{2}$. The germ $g=x^{p} y+x y^{p}+x^{p^{2}+2}+y^{p^{2}+2}$ is right-equivalent to $h$, hence also has Milnor number $\mu=p^{2}$. This implies that $\langle x, y\rangle^{p^{2}+2} \subset \operatorname{Jac}(g)$. We determine a standard basis of

$$
\operatorname{Jac}(g)=\left\langle g_{x}, g_{y}\right\rangle+\langle x, y\rangle^{p^{2}+2}
$$

where

$$
g_{x}=p \cdot x^{p-1} y+y^{p}+\left(p^{2}+2\right) \cdot x^{p^{2}+1}, g_{y}=x^{p}+p \cdot x y^{p-1}+\left(p^{2}+2\right) \cdot y^{p^{2}+1}
$$

with regard the local degree reverse lexicographic ordering. The S-polynomial of $g_{x}$ and $g_{y}$ leads to the standard basis element $x y^{p}$ after reducing the tail by $\langle x, y\rangle^{p^{2}+2}$. The S-polynomial of $x y^{p}$ with $g_{x}$ leads to the standard basis element $y^{2 p-1}$ after reducing the tail by $\langle x, y\rangle^{p^{2}+2}$. The S-polynomial of $x y^{p}$ with $g_{y}$ reduces to zero, while all remaining ones vanish. Hence, the classes of the monomials

$$
y^{p+1}, x^{2} y^{p-1}, \ldots, x^{p-2} y^{3}
$$

form a basis of the Milnor algebra in degree $p+1$. Using the relation $g_{x}$ these monomials are equivalent to

$$
x^{2} y^{p-1}, \ldots, x^{p-2} y^{3}, x^{p-1} y^{2}
$$

which form a regular basis in degree $p+1$. From the standard basis of the Jacobian it is clear that this is the only option for a regular basis, which proves the claim.

We now prove the second part of the lemma. Let $m_{1}, \ldots, m_{n}$ be the monomials of $f_{0}$ and let $g=\sum_{i} k_{i} m_{i}$. For a face $\Delta$ of $\Gamma$, we denote again its weight by $w$. In the first part of the proof we have seen that for all faces $\Delta$, except those where $a=1$, $w(x) \geq w(y), w(y) \mid w(x)$, and $b \geq 2$ if $w(x)=w(y)$, or those where $b=1, w(x) \leq w(y)$, $w(x) \mid w(y)$, and $b \geq 2$ if $w(x)=w(y)$, there does not exist a $w$-weighted homogeneous right-equivalence except rescaling of variables. ${ }^{2}$ In order to describe in these two cases the $w$-homogeneous transformations on $\operatorname{jet}(g, \Delta)$ keeping the face, by symmetry, it is sufficient to consider the first of the two. Here, writing $\tau=w(x) / w(y)$, the jet is of the from $\operatorname{jet}(g, \Delta)=y^{q} \cdot\left(k_{i} x^{p}+k_{j} x y^{(p-1) \cdot \tau}\right), k_{i}, k_{j} \neq 0, p \geq 2$. All homogeneous transformations are of the form $x \mapsto c_{1} x+c_{2} y^{\tau}, y \mapsto c_{3} y$, and keeping $\Delta$ amounts to the condition

$$
k_{i} c_{2}^{p}+k_{j} c_{2} c_{3}^{(p-1) \cdot \tau}=0
$$

which implies that either $c_{2}=0$ (which corresponds to a rescaling of variables), or that between $c_{2}$ and $c_{3}$ there is an algebraic relation $c_{2}=k \cdot c_{3}^{\tau}\left(\right.$ with $\left.k^{p-1}=-k_{j} / k_{i}\right)$.

We now show that there is a germ $g$ with the same monomials as $f_{0}$ and non-degenerate Newton boundary, such that only two terms of $g$ can be normalized to coefficient one by a right-equivalence keeping the Newton polygon. Since this $g$ is right-equivalent to a germ in the normal form described in Theorem 3.7, this then implies that at most two monomials of $f_{0}$ are not in $B$.

In case there are only two monomials of $f_{0}$ of degree $\leq d t$, the claim is trivial choosing $g=f_{0}{ }^{3}$ Otherwise, let $m_{s}, m_{t}$ and $m_{l}$ be three distinct monomials of $f_{0}$ of degree

[^2]$\leq d t$. We prove that there is a Zariski open set of germs $g$ such that not all three monomials $m_{s}, m_{t}, m_{l}$ can be normalized to coefficient one. To see this, it is enough to prove that, for any $g$ with the same monomials as $f_{0}$, after choosing two monomials out of $m_{s}, m_{t}, m_{l}$ and normalizing their coefficients to one, when restricting the action of the right-equivalence group to the Newton boundary and stabilizing the two normalized coefficients, the stabilizer acts as a finite group.
(2.a) If all three monomials lie on faces of $\Gamma$ which permit only rescaling of variables, then the claim is obvious.
(2.b) If exactly two of the three monomials, say $m_{s}, m_{t}$, lie on faces of $\Gamma$ which permit only rescaling of variables, then, without loss of generality, we may assume that $m_{l}$ lies on a face with $a=1, w(x) \geq w(y), w(y) \mid w(x)$. If $w(x)=w(y)$, then it is sufficient to consider the case $b \geq 2$. After normalization of the coefficients of $m_{s}$ and $m_{t}$ to value one via rescaling of variables, stabilizing the two normalized coefficients together with the above condition $c_{2}=0$ or $c_{2}=k \cdot c_{3}^{\tau}$ admits only finitely many solutions.
(2.c) Suppose that exactly one of the three monomials, say $m_{s}$, lies on a face of $\Gamma$ which permits only rescaling of variables.

If $m_{t}$ and $m_{l}$ lie on the same face $\Delta$, we may assume that the jet of $\Delta$ is of the form $\operatorname{jet}(g, \Delta)=y^{q} \cdot\left(k_{l} x^{p}+k_{t} x y^{(p-1) \cdot \tau}\right)$. Then coefficients of the monomials $m_{s}$ and $m_{l}$ can only be changed via rescaling of variables. Then, similar to the previous case, normalization of the coefficients of $m_{s}$ and $m_{l}$ to value one together with the condition $c_{2}=0$ or $c_{2}=k \cdot c_{3}^{\tau}$ admits only finitely many solutions.

If $m_{t}$ and $m_{l}$ lie on different faces, we may assume that $m_{t}$ lies on a face $\Delta$ with weight $w$ and $a=1, w(x) \geq w(y), w(y) \mid w(x)$. Moreover, in case $w(x)=w(y)$, we can assume that $b \geq 2$, since the case $b=1$ has already been discussed. Then $\operatorname{jet}(g, \Delta)$ is of the form

$$
\operatorname{jet}(g, \Delta)=y^{q} \cdot\left(k_{i} x^{p}+k_{j} x y^{(p-1) \cdot \tau}\right)
$$

with $\tau=w(x) / w(y)$ and $t \in\{i, j\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$
x \mapsto c_{1} x+c_{2} y^{\tau}, y \mapsto c_{3} y
$$

satisfy the condition $c_{2}=0$ or $c_{2}=k \cdot c_{3}^{\tau}$ with $k^{p-1}=-k_{j} / k_{i}$. Similarly, we may assume that $m_{l}$ lies on a face $\Pi$ with weight $v$ with $b=1, v(x) \leq v(y), v(x) \mid v(y)$. Again, in case $v(x)=v(y)$, we may assume that $a \geq 2$. Then $\operatorname{jet}(g, \Pi)$ is of the form

$$
\operatorname{jet}(g, \Pi)=x^{q^{\prime}} \cdot\left(k_{r} y^{p^{\prime}}+k_{s} y x^{\left(p^{\prime}-1\right) \cdot \tau^{\prime}}\right)
$$

with $\tau^{\prime}=v(y) / v(x)$ and $l \in\{r, s\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$
x \mapsto c_{1} x, y \mapsto c_{3} y+c_{2}^{\prime} x^{\tau^{\prime}}
$$

satisfying the condition $c_{2}^{\prime}=0$ or $c_{2}^{\prime}=k^{\prime} \cdot c_{1}^{\tau^{\prime}}$ with $\left(k^{\prime}\right)^{p-1}=-k_{s} / k_{r}$. If $m_{t}$ is the monomial of $\operatorname{jet}(g, \Delta)$ of larger $x$-degree (that is, $m_{t}=y^{q} x^{p}$ and $t=i$ ), or if $m_{l}$ is the monomial of jet $(g, \Pi)$ of larger $y$-degree (that is, $m_{l}=y^{q^{\prime}} x^{p^{\prime}}$ and $l=r$ ), then the coefficient of the respective monomial can only be changed via rescaling of variables, and we can argue as in previous case. Suppose now that $m_{t}$ and $m_{l}$ are the $x$-linear monomials of the respective jets. Normalizing the coefficients of $m_{t}$
and $m_{l}$ then amounts to the relations

$$
\begin{aligned}
c_{1} c_{3}^{q} \cdot\left(k_{t} c_{3}^{(p-1) \cdot \tau}+k_{i} c_{2}^{p-1}\right) & =1 \\
c_{3} c_{1}^{q^{\prime}} \cdot\left(k_{l} c_{1}^{\left(p^{\prime}-1\right) \cdot \tau^{\prime}}+k_{r}\left(c_{2}^{\prime}\right)^{p^{\prime}-1}\right) & =1
\end{aligned}
$$

After inserting $c_{2}=k \cdot c_{3}^{\tau}$ with $k^{p-1}=-k_{t} / k_{i}$ the first equation implies that $c_{2}=0$, a contradiction. Similarly, inserting $c_{2}^{\prime}=k^{\prime} \cdot c_{1}^{\tau^{\prime}}$ with $\left(k^{\prime}\right)^{p-1}=-k_{l} / k_{r}$ into the second equation, yields a contradiction. Hence, $c_{2}=c_{2}^{\prime}=0$, that is, rightequivalences keeping the Newton polygon act on the jets as rescaling of variables. So after normalizing the coeffcients of two monomials, the coefficient of the third one can take only finitely many values.
(2.d) If none of the three monomials $m_{s}, m_{t}$ and $m_{l}$ lies on a face of $\Gamma$ which permits only rescaling of variables, then we may assume that $m_{s}, m_{t}$ lie on a face with $a=1$, $w(x) \geq w(y), w(y) \mid w(x)$, and $b \geq 2$ if $w(x)=w(y)$, and $m_{l}$ lies on a face with $b=1$, $w(x) \leq w(y), w(x) \mid w(y)$, and $a \geq 2$ if $w(x)=w(y)$. We can thus argue as previous cases to obtain only finitely many solutions for the action of the right-equivalence group on the Newton boundary.

## 5. Next steps

We have also developed an algorithm to enumerate all normal form families up to a specified Milnor number or modality. These algorithms will be presented in an up-coming paper.

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[^1]:    ${ }^{1}$ In convex geometry, codimension 1 faces of the convex hull $\Gamma_{+}(f)$ are referred to as facets.

[^2]:    ${ }^{2}$ Note that these cases correspond to the second part of (1.c) and to (1.e), which are the only settings where there may exist (and, in fact, exist) transformations on the jet of the respective face, which are not just rescalings of variables.
    ${ }^{3}$ Note that this includes the case where there is a face with $w(x)=w(y)$ and $a=b=1$.

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