MODULI PARAMETERS OF COMPLEX SINGULARITIES WITH NON-DEGENERATE NEWTON BOUNDARY

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ABSTRACT. Our recent extension of Arnold's classification includes all singularities of corank ≤ 2 equivalent to a germ with a non-degenerate Newton boundary, thus broadening the classification's scope significantly by a class which is unbounded with respect to modality and Milnor number. This method is based on proving that all right-equivalence classes within a μ -constant stratum can be represented by a single normal form derived from a regular basis of a suitably selected special fiber. While both Arnold's and our preceding work on normal forms addresses the determination of a normal form family containing the given germ, this paper takes the next natural step: We present an algorithm for computing for a given germ the values of the moduli parameters in its normal form family, that is, a normal form equation in its stable equivalence class. This algorithm will be crucial for understanding the moduli stacks of such singularities. The implementation of this algorithm, along with the foundational classification techniques, is implemented in the library **arnold.lib** for the computer algebra system SINGULAR.

1. INTRODUCTION

Our recent extension Böhm, Marais, and Pfister (2020) of Arnold's classification of isolated hypersurface singularities (see Arnold (1976); Arnold et al. (1985)) includes all singularities of corank ≤ 2 which are equivalent to a germ with non-degenerate Newton boundary in the sense of Kouchnirenko. This broadens the scope of the classification by a class of singularities, which is unbounded both in terms of modality and Milnor number. We establish that there is a single polynomial normal form that contains representatives (at least one, but only finitely many) of all right equivalence classes within a given μ -constant stratum of a given input germ. Based on this result, and algorithmic methods developed in Böhm, Marais, and Pfister (2016a,b), an algorithm for effectively determining a normal form from a regular basis of a suitably chosen special fiber is given.

While both Arnold's and our preceding work on normal forms only addresses the determination of a normal form family containing the given germ, this continuation takes the next natural step: determining the moduli parameters in the normal form families associated with these input germs. That is, we find for a given input germ an element in the normal form associated to its μ -constant stratum such that this element is right equivalent to the input germ, hence determining exactly its stable and right equivalence class. While one could argue that, with the normal form known, such a germ could be found using an Ansatz for the right equivalence by taking finite determinacy into account, this is practically inefficient and will not yield a result except for trivial input. Taking our

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clue from some case-by-case studies in Marais and Steenpaß (2015, 2016); Böhm, Marais and Steenpaß (2019); Böhm, Marais, and Pfister (2016a) we thus develop an iterative method for eliminating the terms of the germ which are not present in the normal form family. This methods has some similarities the algorithm finding the normal form. The main challenge arises here from the fact that singularities with non-degenerate Newton boundary in general do not satisfy Condition A. In an iterative process we thus have to control higher order contributions in the binomial expansion when applying a right equivalence to the germ (since those can be of lower piecewise degree). Applications of our result could occur in the context of the study of Baikov polynomials, see Böhm et al. (2018); Lee and Pomeransky (2013).

Our paper is structured as follows:

In Section 2 we give a review of the foundational concepts and preliminary results on singularities and classification.

In Section 3, we recall the main results on the determination of normal forms for singularities of corank ≤ 2 equivalent to a germ with non-degenerate Newton boundary. We also recall the algorithmic framework determining the normal form.

In Section 4, we address the determination of the moduli parameters in the normal form corresponding to a given input germ.

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2. Definitions and Preliminary Results

In this section, fundamental definitions, theorems, and notations relevant to our discussion are presented. We use $\mathbb{C}\{x_1,\ldots,x_n\}$ to denote the ring of convergent power series, that is, power series that converge in open neighborhoods of the point $(0,\ldots,0)$. We use \mathfrak{m} for the maximal ideal of $\mathbb{C}\{x_1,\ldots,x_n\}$.

Notation 2.1. We denote by $mon(x_1, \ldots, x_n)$ the monoid of monomials in x_1, \ldots, x_n . For $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ and a monomial $m \in mon(x_1, \ldots, x_n)$, the coefficient of m in f is denoted by coeff(f, m).

Definition 2.2. If $w = (c_1, \ldots, c_n) \in \mathbb{N}^n$ is a weight for the variables (x_1, \ldots, x_n) , the *w*-weighted degree on $mon(x_1, \ldots, x_n)$ is defined by the expression

$$w - \deg\left(\prod_{i=1}^n x_i^{s_i}\right) = \sum_{i=1}^n c_i s_i.$$

In the case that the weight of all variables is one, the weighted degree of a monomial m is called its standard degree, denoted by deg(m). This notation is also used for terms in polynomials.

Definition 2.3. Consider a finite family of weights $w = (w_1, \ldots, w_s) \in (\mathbb{N}^n)^s$ for (x_1, \ldots, x_n) . For a term $m \in \mathbb{C}[x_1, \ldots, x_n]$, its **piecewise weight** with respect to w is defined as

$$w \operatorname{-deg}(m) := \min_{i=1,\dots,s} w_i \operatorname{-deg}(m).$$

Definition 2.4. Fix a (piecewise) weight w on $mon(x_1, \ldots, x_n)$.

(1) Suppose

$$f = \sum_{i=0}^{\infty} f_i$$

is the decomposition of $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ into weighted homogeneous components f_i with w-degree of *i*. In the case that $f_i = 0$ for i > d and $f_d \neq 0$ is the lowest non-zero component, we set w-deg(f) = d. The (piecewise) weighted *j*-jet of *f*, denoted by w-jet(f, j), is given by

$$w$$
-jet $(f,j) := \sum_{i=0}^{j} f_i$

The sum of terms of f with the lowest w-degree is called the **principal part** of f with respect to w. The **order** of f with respect to w is defined as the degree of its principal part, and is denoted by w-ord(f).

(2) A power series in $\mathbb{C}\{x_1, \ldots, x_n\}$ is said to have filtration $d \in \mathbb{N}$ with respect to w if all its monomials have a w-weighted degree $\geq d$. By E_d^w we denote the subvector space of $\mathbb{C}\{x_1, \ldots, x_n\}$ of power series of filtration d with respect to w. The sub-spaces E_d^w , for varying $d \in \mathbb{N}$, form a filtration on $\mathbb{C}\{x_1, \ldots, x_n\}$.

Definition 2.5. A piecewise homogeneous germ f_0 of degree d satisfies Condition A, if for every germ g of filtration $d + \delta > d$ in the ideal spanned by the derivatives of f_0 , there is a decomposition

$$g = \sum_{i} \frac{\partial f_0}{\partial x_i} v_i + g',$$

where the vector field v has filtration δ and g' has filtration bigger than $d + \delta$.

Definition 2.6. We say that $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ is k-determined if

$$f \sim \operatorname{jet}(f,k) + g$$
 for all $g \in E_{k+1}$,

with respect to right-equivalence. The **determinacy** of f, denoted by dt(f), is the smallest integer k for which f is k-determined.

Definition 2.7. For $f \in \mathbb{C}\{x_1, \ldots, x_n\}$, the Jacobian ideal

$$\operatorname{Jac}(f) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

is the ideal of $\mathbb{C}\{x_1,\ldots,x_n\}$ generated by the partial derivatives of f. The local algebra

$$Q_f = \mathbb{C}\{x_1, \dots, x_n\} / \operatorname{Jac}(f)$$

of f is the quotient of $\mathbb{C}\{x_1, \ldots, x_n\}$ by the Jacobian ideal. The **Milnor number** of f is the dimension of Q_f as a \mathbb{C} -vector space.

Remark 2.8. If the germ f defines an isolated singularity, then f is k-determined if $k \ge \mu(f) + 1$, hence f is finitely determined. So an isolated singularity can be represented by a polynomial.

Definition 2.9. The annihilator of a germ f, denoted by $\operatorname{ann}(f)$, is the ideal of all elements of $\mathbb{C}\{x_1, \ldots, x_n\}$ that yield zero when multiplied with f.

Definition 2.10. Suppose that ϕ is a \mathbb{C} -algebra automorphism of $\mathbb{C}\{x_1, \ldots, x_n\}$, and w is a single weight on $\operatorname{mon}(x_1, \ldots, x_n)$.

(1) For any positive integer j, the automorphism w-jet $(\phi, j) := \phi_j^w$, is defined by

$$\phi_j^w(x_i) := w \operatorname{-jet}(\phi(x_i), w \operatorname{-deg}(x_i) + j) \quad \text{for } i = 1, \dots, n.$$

If w = (1, ..., 1), we use the notation ϕ_j for ϕ_j^w . (2) We say that ϕ has filtration d if

$$(\phi - \mathrm{id})E^w_\lambda \subseteq E^w_{\lambda+d}$$

for all $\lambda \in \mathbb{N}$.

Remark 2.11. We note that $\phi_0(x_i) = \text{jet}(\phi(x_i), 1)$ for $i = 1, \ldots, n$. Moreover, note that ϕ_0^w has filtration ≤ 0 . For j > 0, ϕ_j^w has filtration j if $\phi_{j-1}^w = \text{id}$.

Definition 2.12. For
$$f = \sum_{i_1,...,i_n} a_{i_1,...,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}\{x_1,...,x_n\}$$
, write
 $mon(f) := \{x_1^{i_1} \cdots x_n^{i_n} \mid a_{i_1,...,i_n} \neq 0\}$

for the set of **monomials** of f, and

$$\sup(f) := \{i_1 \cdots i_n \mid a_{i_1, \dots, i_n} \neq 0\}$$

for the support of f. We set

$$\Gamma_+(f) := \bigcup_{\substack{x_1^{i_1} \cdots x_n^{i_n} \in \text{supp}(f)}} ((i_1, \dots, i_n) + \mathbb{R}^n_+)$$

and define $\Gamma(f)$ as the boundary of the convex hull of $\Gamma_+(f)$ in \mathbb{R}^n_+ . The set $\Gamma(f)$ is called the Newton boundary of f. Then:

- (1) The compact segments of Γ(f) are referred to as facets.¹ If Δ is a facet, we write supp(f, Δ) for the set of monomials of f with exponent vector on Δ. The sum of the terms lying on Δ is denoted by jet(f, Δ). Moreover, we write supp(Δ) for the set of monomials corresponding to the lattice points of Δ. Considering the monomials lying on a union of facets, we use the same terminology for a set of facets.
- (2) To a facet Δ we associate a weight $w(\Delta)$ on the monomials in $mon(x_1, \ldots, x_n)$ as follows: If $-(w_{x_1}, \ldots, w_{x_n})$ is the normal vector of Δ in lowest terms with integers $w_{x_1}, \ldots, w_{x_n} > 0$, we define

$$w(\Delta)$$
 - deg $(x_1) = w_{x_1}, \dots, w(\Delta)$ - deg $(x_n) = w_{x_n}$

(3) Now suppose that w_1, \ldots, w_s are the weight vectors of the facets of $\Gamma(f)$ ordered by increasing slope. Then there are uniquely determined minimal integers $\lambda_1, \ldots, \lambda_s \geq 1$ with the property that the piecewise weight with respect to

$$w(f) := (\lambda_1 w_1, \dots, \lambda_s w_s)$$

is constant on the Newton boundary $\Gamma(f)$. We refer to this constant by d(f).

(4) Suppose that Δ₁ and Δ₂ are adjacent facets with weights w₁ and w₂, respectively, w is the piecewise weight defined by w₁ and w₂, and d is the w-degree of the monomials on Δ₁ and Δ₂. We then write span(Δ₁, Δ₂) for the Newton polygon associated to the sum of all monomials of (w₁, w₂)-degree d.

¹In convex geometry, codimension 1 faces of the convex hull $\Gamma_+(f)$ are referred to as facets.

- (5) If $\Gamma(f)$ has at least one facet, we say that a monomial m is strictly below, on or above $\Gamma(f)$, if the w(f)-degree of m is less than, equal to or larger than d(f), respectively.
- (6) We write $jet(f, \Gamma(f))$ for the expansion of f up to w(f)-order d(f).

Definition 2.13. Assume that f has finite Milnor number. A basis $\{e_1, \ldots, e_{\mu}\}$ of the local algebra of f consisting out of homogeneous elements is **regular** with respect to the filtration given the piecewise weight w, if for each $D \in \mathbb{N}$, the basis elements of degree D with respect to w are independent modulo the sum vector space $\operatorname{Jac}(f) + E_{>D}^w$ of germs of filtration larger than D.

Remark 2.14. Arnold has proven in Arnold (1974) that for each germ $f \in \mathbb{C}\{x, \ldots, x_n\}$ there exists a regular basis consisting out of monomials.

Definition 2.15. For a union of right-equivalence classes $K \subset \mathbb{C}\{x_1, \ldots, x_n\}$ a normal form for K is a smooth map

$$\Phi: \mathcal{B} \longrightarrow \mathbb{C}[x_1, \dots, x_n] \subset \mathbb{C}\{x_1, \dots, x_n\}$$

of a finite-dimensional \mathbb{C} -linear space \mathcal{B} into the space of polynomials such that:

- (1) $\Phi(\mathcal{B})$ intersects all equivalence classes of K,
- (2) for each equivalence class the inverse image in \mathcal{B} is finite
- (3) $\Phi^{-1}(\Phi(\mathcal{B}) \setminus K)$ is contained in a proper hypersurface in \mathcal{B} .

We denote the elements of the image of Φ as normal form equations. A normal form is called a polynomial normal form if the map Φ is polynomial.

Example 2.16. For the germ $f = x^4 + y^4$ of Arnold's type X_9 , the μ -constant stratum of f is covered by the normal form $\Phi : \mathbb{C} \to \mathbb{C}[x, y]$, $\Phi(a) = x^4 + ax^2y^2 + y^4$. For instance, the function $g = x^4 + \epsilon x^3y + y^4$, with a fixed value of ϵ , lies in the same μ -constant stratum as f. Hence, there is a \mathbb{C} -algebra isomorphism ϕ_1 transforming g into $x^4 + ax^2y^2 + y^4$ with some a in \mathbb{C} . As a result, there is also a \mathbb{C} -algebra isomorphism ϕ_2 that maps g to $x^4 - ax^2y^2 + y^4$.

Definition 2.17. (Arnold et al. (1985)) Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ and let k be an upper bound for the determinacy of f. The **modality** of a germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ is the least number such that a sufficiently small neighborhood of jet(f, k) in the k-jet space can be covered by a finite number of m-parameter families of orbits under the right-equivalence action.

Definition 2.18. (Arnold, 1974) Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ be a germ with a non-degenerate Newton boundary. The **inner modality** is the number of monomials in a regular basis for Q_f lying on or above $\Gamma(f)$.

Remark 2.19. In the subsequent section, we will recall that the inner modality of a germ $f \in \mathbb{C}\{x, y\}$ is equal to the number of parameters in the normal form of the germ. Moreover it is shown in Böhm, Marais, and Pfister (2020), using results from Gabriélov (1974), that the inner modality and modality of a germ with a non-degenerate Newton boundary coincide.

3. Classification of Singularities with Non-Degenerate Newton Boundary

In this section we recall the results from Böhm, Marais, and Pfister (2020), where it is, in particular, shown that

- (1) the μ -constant stratum of a germ $f \in \mathbb{C}\{x, y\}$ with a non-degenerate Newton boundary can be covered, up to right-equivalence, by a single normal form.
- (2) there is a normalization condition for the Newton boundary of a germ with a non-degenerate Newton boundary, that ensures the following: In a μ-constant stratum which contains a germ with non-degenerate Newton boundary, all germs with normalized non-degenerate Newton boundary have the same Newton polygon. Hence, the Newton polygon can be considered as a name of the μ-constant stratum, replacing Arnolds notation of a type.
- (3) there is an effective algorithm to compute the normal form (satisfying the normalization condition) for a given input germ $f \in \mathbb{C}\{x, y\}$ which is equivalent to germ a non-degenerate Newton boundary.

Note that, in this section, we only consider germs in the bivariante convergent power series ring $\mathbb{C}\{x, y\}$.

The following result from Greuel et al. (2007), Corollary 2.71, and Böhm, Marais, and Pfister (2020), Theorem 3.9, gives a local description of the μ -constant stratum of a germ with a non-degenerate Newton boundary.

Theorem 3.1. Let $f \in \mathbb{C}\{x, y\}$ be a germ with a non-degenerate Newton boundary at the origin. A miniversal, equisingular unfolding is given by

$$F(x, y, t) = f + \sum_{i=1}^{m} t_i g_i,$$

where m is the modality of f, and g_1, \ldots, g_m represent a regular basis for Q_f on and above $\Gamma(f)$.

A first step to find a normal form for the entire μ -constant stratum of a germ with a non-degenerate Newton boundary is to investigate how a regular basis of the germs in the stratum change, while moving through the stratum.

Proposition 3.2. (Böhm, Marais, and Pfister (2020), Proposition 3.12) Let f_0 be a germ with a non-degenerate Newton boundary $\Gamma(f_0)$ and let f be a germ with the same Newton polygon as f_0 and non-degenerate Newton boundary. Then for f sufficiently close to f_0 with respect to the Euclidean distance in the $(\mu + 1)$ -jet space, the monomials in mon(x, y) representing a regular basis for f_0 with respect to the filtration defined by $\Gamma(f_0) = \Gamma(f)$ also represent a regular basis for f with respect to the same filtration.

Next, it is important to observe that all the germs in the μ -constant stratum of a germ with a non-degenerate Newton boundary have the same topological type (see Böhm, Marais, and Pfister (2020), Remark 3.16). Since germs with a non-degenerate Newton boundary has the same topological type if and only if their characteristic exponents and intersection numbers coincide (see Brieskorn, Knörrer (1986), Theorem 15), and the characteristic exponents and intersection numbers of a germ in $\mathbb{C}\{x, y\}$ determines the non-degenerate Newton boundaries of a germ that is equivalent to a germ with a nondegenerate Newton boundary (see Böhm, Marais, and Pfister (2020), Proposition 4.17 and Corollary 4.18), the next result follows:

Theorem 3.3. (Böhm, Marais, and Pfister (2020), Theorem 3.18) Suppose $f \in \mathbb{C}\{x, y\}$ is a convenient germ with non-degenerate Newton boundary Γ . Then all the germs in the μ -constant stratum of f are equivalent to a germ with the same Newton polygon Γ and non-degenerate Newton boundary.

Let f be a germ with a non-degenerate Newton boundary Γ . Taking the previous result into account, the next result shows that there exists a set of monomials that is a regular basis for at least one germ, a germ with a non-degenerate Newton boundary and a Newton polygon that coincide with that of Γ , in each right-equivalence class of the germs in the μ -constant strantum of f.

Lemma 3.4. (Böhm, Marais, and Pfister (2020), Lemma 3.20) Let f be a convenient germ with non-degenerate Newton boundary. Define f_0 as the sum of the monomials of f lying on the vertex points of $\Gamma(f)$. Then any regular basis of f_0 is also a regular basis for every germ with a non-degenerate Newton boundary in the μ -constant stratum of f.

Corollary 3.5. (Böhm, Marais, and Pfister (2020), Corollary 3.21) Suppose f is a convenient germ with non-degenerate Newton boundary. Define f_0 as the sum of the monomials of f lying on the vertex points of $\Gamma(f)$, and f'_0 as the sum of the terms of f on the vertex points of $\Gamma(f)$. Then any regular basis for f_0 is also a regular basis for f'_0 and for f.

By the next theorem, every germ in the μ -constant with non-degenerate Newton boundary can be written in terms of its Newton boundary and a regular basis of the germ.

Proposition 3.6. (Boubakri et al. (2011), Corollary 4.6) Let $f \in \mathbb{C}\{x, y\}$ be a convenient germ with a non-degenerate Newton boundary. Let f_0 be the principal part of f and let $\{e_1, \ldots, e_n\}$ be the set of all monomials in a regular basis for f_0 lying above $\Gamma(f_0)$. Then there are α_i such that

$$f \sim f_0 + \sum_{i=1}^n \alpha_i e_i.$$

Using Theorem 3.1, Proposition 3.2, Theorem 3.3, Lemma 3.4 and Proposition 3.6 the following theorem can be proved (see Böhm, Marais, and Pfister (2020), Theorem 3.22).

Theorem 3.7. Suppose f is a convenient germ with non-degenerate Newton boundary. Define f_0 as the sum of the monomials of f lying on the vertex points of $\Gamma(f)$, and let $\{e_1, \ldots, e_n\}$ be the set of all monomials in a regular basis for f_0 lying on or above $\Gamma(f)$. Then the family

$$f_0 + \sum_{i=1}^n \alpha_i e_i$$

defines a normal form of the μ -constant stratum containing f. Restricting the parameters $\alpha_1, \ldots, \alpha_n$ to values such that every germ $f_0 + \sum_{i=1}^n \alpha_i e_i$ has a non-degenerate Newton boundary and the same Newton polygon as that of f, we obtain all germs in the μ -constant stratum of f.

Remark 3.8. Using Theorem 3.7 a normal form can be constructed for any germ $f \in \mathbb{C}\{c, y\}$ with a non-degenerate Newton boundary. Note that the Newton polygon of f fixes the Newton polygon of all the germs in the constructed normal form. Since the normal form constructed in Theorem 3.7 is a normal form for the full μ -constant stratum, it follows that if f' is a germ with a non-degenerate Newton boundary and a different Newton polygon than f, then the normal form constructed for f' describes the same μ -constant stratum as that for f. In fact, by Theorem 3.3, f is equivalent to a germ with a non-degenerate Newton polygon as f'. Hence, the normal form for the μ -constant stratum of f as constructed using Theorem 3.7, depends

on the choice of non-degenerate Newton boundary of germs in the equivalence class of f and the choice of regular basis for the chosen f_0 . In his lists of normal forms, Arnold associate a **type** T to each μ -constant stratum. He then fixes a Newton polygon and a choice of moduli monomials (which boils down to the choice of regular basis for f_0 in Theorem 3.7). For distinguishing between different types, it is sufficient to know the Newton polygon of the normal form.

To achieve uniqueness of the Newton polygon associated to a fixed type (in order to label types by Newton polygons), a normalization condition on the Newton boundary of a germ with a non-degenerate Newton boundary is needed. Such a condition ensures that the same Newton polygon for any germ in the μ -constant stratum is consistently chosen in order to construct a normal form by using Theorem 3.7.

It is important to distinguish between smooth and non-smooth facets:

Definition 3.9. A facet of the Newton polygon of a germ is called a **smooth** if the saturation of its jet is smooth.

Definition 3.10. Suppose $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ is a convenient germ with non-degenerate Newton boundary. Let Δ be a facet of $\Gamma(f)$, and write $w = w(\Delta)$. Then $jet(f, \Delta)$ factorizes in $\mathbb{C}[x, y]$ as

$$\operatorname{jet}(f,\Delta) = x^a \cdot y^b \cdot g_1 \cdots g_n \cdot \widetilde{g}_s$$

where a,b are integers, g_1, \ldots, g_n are linear homogeneous polynomials not associated to x or y, and \tilde{g} is a product of non-associated irreducible non-linear homogeneous polynomials. We say that f is **normalized** with respect to the facet Δ , if

$$\begin{array}{lll} w(x) = w(y) & \implies & a, b \neq 0 \\ w(x) > w(y) & and & a = 0 & \implies & n = 0 \\ w(x) < w(y) & and & b = 0 & \implies & n = 0 \end{array}$$

A germ for which all the facets satisfy the above normalization condition is not necessarily convenient. We can transform such a germ to a convenient germ in the same right-equivalence class by adding the terms x^d or y^d with $d = \mu(f) + 2$, if needed. This will create non-normalized smooth facets that cut the coordinate axes. We address this in the following definition:

Definition 3.11. A germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ satisfies the normalization condition if all of its facets, except smooth facets cutting the coordinate axes, are normalized, and each of its smooth facets that cut a coordinate axis, cut the axis in standard degree d, where $d = \mu(f) + 1$.

It directly follows from Theorem 3.7 that two germs with the same normalized Newton boundary have the same normal form and hence lie in the same μ -constant stratum. The following theorem states that the normalization condition is reasonable:

Theorem 3.12. In a μ -constant stratum which contains a germ with a non-degenerate Newton boundary, every right-equivalence class contains a normalized germ, and all germs in the μ -constant stratum satisfying the normalization condition have the same Newton polygon (up to permutation of the variables).

In Böhm, Marais, and Pfister (2020), algorithms are given to determine a normal form for the μ -constant stratum of a germ which is equivalent to a germ with a non-degenerate Newton boundary. This algorithm also detects if the given input germ is not equivalent to one with a non-degenerate Newton boundary. As a first step, this algorithm uses Algorithm 4 of Böhm, Marais, and Pfister (2020) to transform an input germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ to a germ which is right-equivalent and has a normalized non-degenerate Newton boundary (this step will detect non-degeneracy).

Knowing the normalized non-degenerate Newton boundary, Algorithm 6 of Böhm, Marais, and Pfister (2020) is used to find a regular basis for the sum of the vertex monomials of the non-degenerate Newton polygon.

Finally, Algorithm 7 applies Theorem 3.7 to construct a normal form for the μ -constant stratum of the input germ f.

Building on this construction and its implementation in Böhm, Marais, and Pfister (2024), the subsequent section will address finding the moduli parameters corresponding to the given input germ in the normal form.

4. Determining the Values of the Moduli Parameters in the Normal From of a germ with a Non-Degenerate Newton Boundary

After finding the normal form as discussed in Section 3, the values of the moduli parameters can, in theory, be computed via an Ansatz for the right equivalence mapping the given germ under consideration to an element of the normal form (making use of finite determinacy). However, this is not practicable except for very small examples. In this section, we discuss an efficient algorithm for this problem. For brevity of presentation, we introduce the following shorthand notation: if Δ is a set of facets of the Newton polygon of f, we write $f_{\Delta} = \text{jet}(f, \Delta)$ as introduced in Definition 2.12.

Now, let $f \in \mathfrak{m}^2$ be a polynomial with a normalized non-degenerate Newton boundary, with $w(f) = (w_1, \ldots, w_n)$ the induced weight on the Newton polygon. Let f_0 be the sum of the vertex monomials of $\Gamma(f)$ and write $f'_0 = w(f) - \operatorname{jet}(f, d(f))$ for the sum of the terms of f on $\Gamma(f)$.

Recall from Lemma 3.4 and Corollary 3.5 that a regular basis B for f_0 is also a regular basis for f'_0 and for f. Assume that f has a term above $\Gamma(f)$ not in B, and let d' be the lowest w(f)-degree occurring among these terms. Let t be a term of piecewise degree d'in f. Note that by the properties of a regular basis we can write

$$t = g\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y} + \text{terms of } w(f)\text{-degree } d' \text{ in } B + \text{terms of } w(f)\text{-degree } > d', \qquad (1)$$

where $g, h \in \mathbb{C}[x, y]$. Define the right equivalence $\phi : \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ by

$$\phi(x) = x - g, \ \phi(y) = y - h,$$
(2)

where g, h are as in equation (1). Note that

$$\phi(f) = f - \underbrace{\left(\frac{\partial f}{\partial x}g + \frac{\partial f}{\partial y}h\right)}_{0} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 f}{\partial^2 y}h^2 + \frac{\partial^2 f}{\partial x \partial y}gh + \frac{\partial^2 f}{\partial^2 x}g^2\right) + \cdots}_{0} (3)$$

first order of the binomial expansion of $\phi(f)$

higher order of the binomial expansion of $\phi(f)$

A germ with a non-degenerate Newton boundary does not necessarily satisfy Condition A. Hence we cannot be certain that terms in the higher-order terms in the binomial expansion of $\phi(f)$ are of w(f)-degree larger than d'. Thus the method introduced in Böhm, Marais, and Pfister (2020) for finding a normal form equation in general cannot be applied. Algorithm 1 provides a method applicable to any germ with a non-degenerate Newton boundary.

We rely on the following lemma to formulate the algorithm. Fix a set Γ' of connected facets of the Newton polygon of f.

Lemma 4.1. Suppose that a and b are the maximal exponents such that the monomial $x^a y^b$ divides both $\frac{\partial f_{\Gamma'}}{\partial x}$ and $\frac{\partial f_{\Gamma'}}{\partial y}$, and define the exponents a' and b' by $f_{\Gamma'} = x^{a'} y^{b'} \operatorname{sat}(f_{\Gamma'})$. If $0 \neq g, h \in \mathbb{C}[x, y]$ satisfy

$$g \cdot \frac{\partial f_{\Gamma'}}{\partial x} + h \cdot \frac{\partial f_{\Gamma'}}{\partial y} = 0,$$

then

$$\left(\frac{g}{x^s}, \frac{h}{y^t}\right)$$

is a syzygy of

$$\left(x^i \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right)\right),$$

where s, i, t, ν are given by

$$s = \begin{cases} \max\{\alpha \mid x^{\alpha} \text{ divides } \frac{\partial f_{\Gamma'}}{\partial y}\} & \text{if } a = 0, \\ 1 & \text{if } a \neq 0, \end{cases} \quad i = \begin{cases} \max\{\alpha \mid x^{\alpha} \text{ divides } \frac{\partial f_{\Gamma'}}{\partial x}\} & \text{if } a = 0, \\ 0 & \text{if } a \neq 0, \end{cases}$$
$$t = \begin{cases} \max\{\beta \mid y^{\beta} \text{ divides } \frac{\partial f_{\Gamma'}}{\partial x}\} & \text{if } b = 0, \\ 1 & \text{if } b \neq 0, \end{cases} \quad \nu = \begin{cases} \max\{\beta \mid y^{\beta} \text{ divides } \frac{\partial f_{\Gamma'}}{\partial y}\} & \text{if } b = 0, \\ 0 & \text{if } b \neq 0, \end{cases}$$

Moreover, the vector $\left(\frac{g}{x^s}, \frac{h}{y^t}\right)$ is a polynomial multiple of the Koszul syzygy of

$$\left(x^i \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right)\right)$$

We postpone the proof of the lemma to the end of this section. We only remark at the current point that $x^i \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right)$, $y^{\nu} \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right)$ is a regular sequence, hence the vector $\left(\frac{g}{x^s}, \frac{h}{y^t}\right)$ is a polynomial multiple of the mentioned Koszul syzygy.

Definition 4.2. Let $f \in \mathbb{C}[x, y]$ and suppose $m = \frac{m_1}{m_2} \in \text{Quot}(\mathbb{C}[x, y])$ with monomial $m_1, m_2 \in \mathbb{C}[x, y]$, that is, m is a **Laurent monomial** in x, y. Then \overline{f}^m is defined as the sum of all terms t of f such that $m \cdot t \in \mathbb{C}[x, y]$.

Using the next result, we will be able to show in the proof of Algorithm 1 that higher order terms t' of the binomial expansion of $\phi(f)$, where ϕ is defined in (2), can be written as

$$t' = g' \frac{\partial f}{\partial x} + h' \frac{\partial f}{\partial y} + \text{terms of } w(f) \text{-degree } d' \text{ in } B + \text{terms of } w(f) \text{-degree} > d',$$

where $g', h' \in \mathbb{C}[x, y]$. Moreover we will show that the transformation $\phi_{\text{new}} : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ defined by $\phi_{\text{new}}(x) = x - g', \ \phi_{\text{new}}(y) = y - h'$ has a higher filtration than ϕ .

Theorem 4.3. Let I be the ideal generated by all the monomials of w-degree d' with $d' \ge d(f)$ fixed. If (a,b) is a syzygy of $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ over $\mathbb{C}[x,y]/I$, then there exist Laurent monomials z_j in x, y such that

$$a - \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \in \operatorname{Ann}\left(\frac{\partial f}{\partial x}\right) \quad and \quad b + \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \in \operatorname{Ann}\left(\frac{\partial f}{\partial y}\right)$$

over $\mathbb{C}[x,y]/I$.

Furthermore if no y^{α} , $\alpha > 0$, is a monomial of $\overline{\frac{\partial f}{\partial y}}^{z_j}$, then x divides all terms of $z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$ that are not in $\operatorname{Ann}(\frac{\partial f}{\partial x})$ and do not get cancelled in the sum $\sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$. Similarly, if no x^{β} , $\beta > 0$, is a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$, then y divides all terms of $z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$ that are not in $\operatorname{Ann}(\frac{\partial f}{\partial y})$ and do not get cancelled in the sum $\sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$.

Proof. Let (a, b) be a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y]/I$, where I is the ideal generated by all the monomials of w-degree d'. If the syzygy equation $g_1 = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = 0$ has no cancellation below w-degree d' then we are finished. Now suppose g_1 has cancellation below degree d'. Suppose in particular g_1 has cancellation below w_1 -degree d'. Let

$$l_1 = \min\left(w_1 \operatorname{-ord}(a\frac{\partial f}{\partial x}), w_1 \operatorname{-ord}(b\frac{\partial f}{\partial y})\right).$$

For a face Δ of the Newton polygon, we write

$$f_{\Delta,x} = \frac{\partial \operatorname{jet}(f,\Delta)}{\partial x}$$
 and $f_{\Delta,y} = \frac{\partial \operatorname{jet}(f,\Delta)}{\partial y}$.

Then

$$w_{1} - \operatorname{jet}(g_{1}, l_{1}) = m^{(1)} \left(\underbrace{x^{s_{1}} y^{\nu_{1}} \operatorname{sat}(f_{\Delta_{1}, y})}_{w_{1} - \operatorname{jet}(a, l_{1} - d(f))} f_{\Delta_{1}, x} \underbrace{-y^{t_{1}} x^{i_{1}} \operatorname{sat}(f_{\Delta_{1}, x})}_{w_{1} - \operatorname{jet}(b, l_{1} - d(f))} f_{\Delta_{1}, y} \right),$$

where $m^{(1)}$ is a monomial, Δ_1 is the face with the smallest slope of $\Gamma(f)$, and i_1, ν_1, s_1 and t_1 is as in Lemma 4.1.

Now consider the syzygy $\left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$ of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y]$, with syzygy equation $g_0 = \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = 0$. Let $l_0 = w_1 - \operatorname{ord}\left(\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}\right)$, then

$$w_1 - jet(g_0, l_0) = m_{w_1} \left(x^{s_1} y^{\nu_1} \operatorname{sat} \left(f_{\Delta_1, y} \right) f_{\Delta_1, x} - y^{t_1} x^{i_1} \operatorname{sat} \left(f_{\Delta_1, x} \right) f_{\Delta_1, y} \right),$$

where m_{w_1} is the product of the maximal power of x and the maximal power of y dividing both $f_{\Delta_1,x}$ and $f_{\Delta_1,y}$. Let $n^{(1)} = \operatorname{lcm}(m_{w_1}, m^{(1)})$. Then the lowest nonzero w_1 -jet of $\frac{n^{(1)}}{m^{(1)}}(a,b)$ and $\frac{n^{(1)}}{m_{w_1}}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)$ coincide.

Then

$$a^{(1)}, b^{(1)}) = \left(\frac{n^{(1)}}{m^{(1)}}a - \frac{n^{(1)}}{m_{w_1}}\frac{\partial f}{\partial y}, \frac{n^{(1)}}{m^{(1)}}b + \frac{n^{(1)}}{m_{w_1}}\frac{\partial f}{\partial x}\right)$$

is a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ in $\mathbb{C}[x, y]/I^{(1)}$, where $I^{(1)} = \frac{n^{(1)}}{m^{(1)}}I$.

(

Let $d'_2 = d'\left(w_1 - \deg\left(\frac{n^{(1)}}{m^{(1)}}\right)\right)$. Now, if the equation $g_2 = a^{(1)}\frac{\partial f}{\partial x} + b^{(1)}\frac{\partial f}{\partial y}$ has any terms of w_1 -degree less than d'_2 , then the lowest non-zero w_1 -jet of $(a^{(1)}, b^{(1)})$ is

$$m^{(2)}\left(x^{s_2}y^{\nu_2}\operatorname{sat}\left(f_{\Delta_1,y}\right), -y^{t_2}x^{i_2}\operatorname{sat}\left(f_{\Delta_1,x}\right)\right)$$

Let $n^{(2)} = \text{lcm}\left(m^{(2)}, m_{w_1}\right)$, then the lowest non-zero w_1 -jet of the syzygies $\frac{n^{(2)}}{m^{(2)}}\left(a^{(1)}, b^{(1)}\right)$ and $\frac{n^{(2)}}{m_{w_1}}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)$ coincide. Similarly, as before, we can create the syzygy

$$\left(a^{(2)}, b^{(2)}\right) = \left(\frac{n^{(2)}}{m^{(2)}}a^{(1)} - \frac{n^{(2)}}{m_{w_1}}\frac{\partial f}{\partial y}, \frac{n^{(2)}}{m^{(2)}}b^{(1)} + \frac{n^{(2)}}{m_{w_1}}\frac{\partial f}{\partial x}\right)$$

in $\mathbb{C}[x, y]/I^{(2)}$, where $I^{(2)} = \frac{n^{(2)}}{m^{(2)}}I^{(1)}$.

We can go on with this process until the syzygy equation $g_k = a^{(k-1)} \frac{\partial f}{\partial x} - b^{(k-1)} \frac{\partial f}{\partial y}$ has no terms of w_1 -degree less than w_1 -degree $d'_k = d'_{(k-1)} \left(w_1 - \deg\left(\frac{n^{(k-1)}}{m^{(k-1)}}\right) \right)$. We show now that this will eventually happen. Now, $n^{(j)} = \gcd(c, d, e, h)$ and $m^{(j+1)}$ be the the monomial with the maximal x- and y-power dividing c - d and e - h, where $c = \frac{n^{(j+1)}}{m^{(j+1)}} a^{(j)}, d = \frac{n^{(j+1)}}{m_{w_1}} \frac{\partial f}{\partial y}, e = \frac{n^{(j+1)}}{m^{(j+1)}} b^{(j)}$ and $h = \frac{n^{(j+1)}}{m_{w_1}} \frac{\partial f}{\partial x}$. Since the lowest non-zero w_1 -jet of c and d, and e and h, cancel, respectively, in c - d and e - h, it follows that w_1 -deg $(m^{(j+1)}) > w_1$ -deg $(n^{(j)})$. Now,

$$w_{1} - \deg\left(\frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1} - \deg\left(\frac{\partial f}{\partial y}\right) - d'_{j+1}$$

$$= w_{1} - \deg\left(\frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1} - \deg\left(\frac{\partial f}{\partial y}\right) - w_{1} - \deg\left(\frac{n^{(j+1)}}{m^{(j+1)}}\right) d'_{j}$$

$$= w_{1} - \deg\left(\frac{m^{(j+1)}}{n^{(j+1)}}\frac{n^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1} - \deg\left(\frac{\partial f}{\partial y}\right) - d'_{j}$$

$$= w_{1} - \deg\left(\frac{m^{(j+1)}}{m_{w_{1}}}\right) \cdot w_{1} - \deg\left(\frac{\partial f}{\partial y}\right) - d'_{j}$$

$$< w_{1} - \deg\left(\frac{n^{(j)}}{m_{w_{1}}}\right) \cdot w_{1} - \deg\left(\frac{\partial f}{\partial y}\right) - d'_{j}.$$

This implies that the difference in the degree of the lowest non-zero w_1 -jet of $(a^{(j)}, b^{(j)})$ and the degree of the lowest order elements in d'_k becomes smaller. Hence eventually the w_1 -degree of $(a^{(j)}, b^{(j)})$ will be $\geq d'_j$.

Suppose now that $(a^{(j-1)}, b^{(j-1)})$ is such that g_j has no terms below w_1 -degree d'_{j-1} . We now consider the w_2 -degree of $(a^{(j-1)}, b^{(j-1)})$. We follow the same strategy. Suppose g_j has terms of w_2 -degree less than $d^{(2)}_{j-1}$, where

$$d_k^{(r)} = w_r \cdot \deg\left(\frac{n^{(1)}n^{(2)}\cdots n^{(k)}}{m^{(1)}m^{(2)}\cdots m^{(k)}}\right)d',$$

then let

$$l_j = \min\left(w_2 \operatorname{-ord}(a^{(j-1)}\frac{\partial f}{\partial x}), w_2 \operatorname{-ord}(b^{(j-1)}\frac{\partial f}{\partial y})\right).$$

Then

$$w_2 - jet(g_j, l_j) = m^{(j)} \left(x^{s_j} y^{\nu_j} \operatorname{sat} \left(f_{\Delta_2, y} \right) f_{\Delta_2, x} - y^{t_j} x^{i_j} \operatorname{sat} \left(f_{\Delta_2, x} \right) f_{\Delta_2, y} \right),$$

where $m^{(j)}$ is a monomial, Δ_2 is the face with the second smallest slope of $\Gamma(f)$, and s_j , ν_j , i_j and t_j is as in Lemma 4.1. Furthermore

$$w_2 - jet(g_0, l_0) = m_{w_2} \left(x^{s_j} y^{\nu_j} \operatorname{sat} \left(f_{\Delta_2, y} \right) f_{\Delta_2, x} - y^{t_j} x^{i_j} \operatorname{sat} \left(f_{\Delta_2, x} \right) f_{\Delta_2, y} \right),$$

where $m_{w_2} = \gcd(f_{\Delta_2,x}, f_{\Delta_2,y})$. With

$$n^{(j)} := \operatorname{lcm}(m_{w_2}, m^{(j)})$$

the lowest nonzero w_2 -jets of

$$\frac{n^{(j)}}{m^{(j)}} \left(a^{(j-1)}, b^{(j-1)} \right) \text{ and } \frac{n^{(j)}}{m^{w_2}} \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$$

coincide. Similarly, as before, we can create the syzygy

$$\left(a^{(j)}, b^{(j)}\right) = \left(\frac{n^{(j)}}{m^{(j)}}a^{(j-1)} - \frac{n^{(j)}}{m_{w_j}}\frac{\partial f}{\partial y}, \frac{n^{(j)}}{m^{(j)}}b^{(j-1)} + \frac{n^{(j)}}{m_{w_j}}\frac{\partial f}{\partial x}\right)$$

in $\mathbb{C}[x, y]/I^{(j)}$, where

$$I^{(j)} = \frac{n^{(j)}}{m^{(j)}} I^{(j-1)}$$

Note that the syzygy equation g_j has no terms below w_2 -degree

$$d'_j := w_2 \operatorname{-deg}\left(rac{n^{(j)}}{m^{(j)}}
ight) d^{(2)}_{j-1}$$

and w_1 -degree

$$d''_j := \left(w_1 \cdot \deg(\frac{n^{(j)}}{m^{(j)}}) \right) \cdot d'_{j-1}.$$

If this would not be the case, this would imply that

$$-\frac{n^{(j)}}{m_{w_2}}\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}+\frac{n^{(j)}}{m_{w_2}}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$$

has terms below w_1 -degree d''_i .

Let $l'_{(x,w_1)}$ and $l'_{(y,w_1)}$ be the lowest non-zero w_1 -orders of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Let $l'_{(x,w_2)}$ and $l'_{(y,w_2)}$ be the lowest w_2 -orders of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. We observe that

$$w_1$$
-jet $\left(\frac{\partial f}{\partial x}, l'_{(x,w_1)}\right)$ and w_2 -jet $\left(\frac{\partial f}{\partial x}, l'_{(x,w_2)}\right)$

have coinciding terms at a vertex monomial. The same holds true for

$$w_1$$
-jet $\left(\frac{\partial f}{\partial y}, l'_{(y,w_1)}\right)$ and w_2 -jet $\left(\frac{\partial f}{\partial y}, l'_{(y,w_2)}\right)$.

All the terms of

$$w_1$$
-jet $\left(\frac{\partial f}{\partial x}, l'_{(x,w_1)}\right)$

have the same w_1 -degree. The same holds true for

$$w_1$$
-jet $\left(\frac{\partial f}{\partial y}, l'_{(y,w_1)}\right)$.

Similarly, all terms of

$$w_2$$
-jet $\left(\frac{\partial f}{\partial x}, l'_{(x,w_2)}\right)$

have the same w_2 -degree, and the same holds true for the terms of w_2 -jet $\left(\frac{\partial f}{\partial y}, l'_{(y,w_2)}\right)$. Hence, we conclude that

$$w_2$$
-jet $\left(-\frac{n^{(j)}}{m_{w_2}}\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} + \frac{n^{(j)}}{m_{w_2}}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}, l_j\right)$

also has terms below w_1 -degree d''_i , which means that

$$w_2$$
-jet $\left(\frac{n^{(j)}}{m^{(j)}}a^{(j-1)}\frac{\partial f}{\partial x} + \frac{n^{(j)}}{m^{(j)}}b^{(j-1)}\frac{\partial f}{\partial y}, l_j\right)$

has terms below w_1 -degree d''_j . This again implies that $w_2 - jet(a^{(j-1)}\frac{\partial f}{\partial x} - b^{(j-1)}\frac{\partial f}{\partial y}, l_{j-1})$ has terms below w_1 -degree d'_{j-1} . We conclude that $a^{(j-1)}\frac{\partial f}{\partial x} - b^{(j-1)}\frac{\partial f}{\partial y}$ has terms below w_1 -degree d'_{j-1} , a contradiction.

Continuing as above, we can construct a syzygy $(a^{(k)}, b^{(k)})$ of $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\mathbb{C}[x, y]/I^{(k)}$, where the syzygy equation g_k has no terms below w_i -degree $d_k^{(i)}$, for $i = 1, 2, \ldots, n$. Let $r_0 = \frac{n^{(1)} \dots n^{(k)}}{m^{(1)} \dots m^{(k)}}$ and $r_j = \frac{n^{(j)} \dots n^{(k)}}{m^{(j+1)} \dots m^{(k)} m_{w_j}}$, for $j = \{1, \ldots, k-1\}$, and $r_k = \frac{n^{(k)}}{m_{w_{i_k}}}$. By construction

$$w_i \operatorname{-jet}\left(r_0 a, d_k^{(i)} - (w_i \operatorname{-ord}(\frac{\partial f}{\partial x}))\right) = w_i \operatorname{-jet}\left(\sum_{j=1}^k r_j \frac{\partial f}{\partial y}, d_k^{(i)} - (w_i \operatorname{-ord}(\frac{\partial f}{\partial x}))\right).$$
(4)

Since r_0 divides all the terms on the right side of the equation, it follows that if $\frac{r_j}{r_0}t \notin \mathbb{C}[x, y]$, for some term t of $\frac{\partial f}{\partial y}$, then t either gets cancelled on the righthand side of the equation or $r_j t$ is of w_i -degree higher than $d_k^{(i)} - (w_i - \operatorname{ord}(\frac{\partial f}{\partial x}))$ for all i, that is $r_j t \frac{\partial f}{\partial x}$ is of w_i -order higher than d' for all i. Hence $r_j t \frac{\partial f}{\partial x}$ is contained in $I^{(k)}$. Therefore

$$w_i \operatorname{-jet}\left(r_0 a, d_k^{(i)} - (w_i \operatorname{-ord}(\frac{\partial f}{\partial x}))\right) = w_i \operatorname{-jet}\left(\sum_{j=1}^k r_j \overline{\frac{\partial f}{\partial y}}^{z_j}, d_k^{(i)} - (w_i \operatorname{-ord}(\frac{\partial f}{\partial x}))\right), \quad (5)$$

where

$$z_j = \frac{m^{(1)} \cdots m^{(j)}}{n^{(1)} \cdots n^{(j-1)} m_{w_j}},$$

Therefore

$$a - \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \in \operatorname{Ann}(\frac{\partial f}{\partial y}) \quad \text{and} \quad b + \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \in \operatorname{Ann}(\frac{\partial f}{\partial y})$$

over $\mathbb{C}[x, y]/I$.

We will now show that if y^{α} is not a term of $\overline{\frac{\partial f}{\partial y}}^{z_j}$, then x divides all terms of $z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$ that are not in $\operatorname{Ann}(\frac{\partial f}{\partial x})$ and do not get cancelled in the sum $\sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$.

So suppose that $\overline{\frac{\partial f}{\partial y}}^{z_j}$, has only mixed terms. Let $cx^{\alpha}y^{\beta}$, $c \in \mathbb{C}$ be such a term of $\overline{\frac{\partial f}{\partial y}}^{z_j}$ such that $z_j cx^{\alpha}y^{\beta} \notin \operatorname{Ann}(\frac{\partial f}{\partial x})$ and $z_j cx^{\alpha}y^{\beta}$ is not cancelled in the sum

$$\sum_{j=1}^{k} z_j \frac{\overline{\partial f}}{\partial y}^{z_j}$$

Clearly $x^{\alpha-1}y^{\beta+1}$ is a monomial of $\frac{\partial f}{\partial x}$, and there exists a monomial $lx^{\eta}y^{\gamma}$, $l \in \mathbb{C}$ of $\frac{\partial f}{\partial x}$ such that

$$z_j c x^{\alpha} y^{\beta} x^{\eta} y^{\gamma} \notin I.$$

But then $lx^{\eta+1}y^{\gamma-1}$ is a monomial of $\frac{\partial f}{\partial y}$. Furthermore, notice that $r_j \frac{c\alpha}{\beta+1} x^{\alpha-1} y^{\beta+1}$ is cancelled in $\sum_{j=1}^k r_j \frac{\partial f}{\partial x}$ if and only if $r_j cx^{\alpha} y^{\beta}$ is cancelled in $\sum_{j=1}^k r_j \frac{\partial f}{\partial y}$. Since, in

addition, $z_j x^{\alpha-1} y^{\beta+1} x^{\eta+1} y^{\gamma-1} \notin I$, it follows that $x^{\alpha-1} y^{\beta+1}$ is a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$. On the other hand, since f is convenient, it follows that y^{α} , $\alpha > 0$, is a monomial of $\frac{\partial f}{\partial y}$, for some $\alpha > 0$. Furthermore, since f has a non-degenerate Newton boundary, it follows that if y^{β} is a monomial of $\frac{\partial f}{\partial y}$, $\beta < \alpha$. Hence, since y^{α} is not a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$, $z_j y^{\alpha} \notin \mathbb{C}[x, y]$. This implies that $z_j y^{\beta} \notin \mathbb{C}[x, y]$. Therefore y^{β} , for any $\beta > 0$ is not a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$. This again implies that $\alpha > 1$. Suppose $z_j = \frac{a_j}{b_j}$, then $b_j \mid x^{\alpha} y^{\beta}$ and $b_j \mid x^{\alpha-1} y^{\beta}$. Hence $x \mid z_j x^{\alpha} y^{\beta}$. In a similar way, we can prove that if $x^{\beta} \notin \overline{\frac{\partial f}{\partial x}}^{z_j}$, then $y \mid z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$, $\beta > 0$.

Analogously we can show that, if x^{β} is not a term of $\overline{\frac{\partial f}{\partial x}}^{z_j}$ for all j, then y divides all terms of $z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$ that are not in $\operatorname{Ann}(\frac{\partial f}{\partial y})$ and do not get cancelled.

In line 20 of Algorithm 1, we rely on Lemma 4.4 below. Since its statement is quite obvious, we postpone the proof of the lemma to the end of the chapter.

Lemma 4.4. Let Γ be the Newton polygon of a germ with normalized, non-degenerate Newton boundary. Let f_0 be the sum of the monomials corresponding to the vertices of Γ , and let B be a regular basis for f_0 . Then

- (1) any monomial corresponding to a lattice point of Γ is a monomial of f_0 or an element of B, and
- (2) at most two monomials of f_0 of degree $\leq dt(f_0)$ are not in B.

The above lemma shows that for a germ with a normalized non-degenerate Newton boundary every monomial corresponding to a lattice point of its Newton polygon is a monomial occurring in any normal form of the germ with the same Newton polygon. Furthermore it shows that only two vertex monomials under the determinacy and on the Newton boundary are not parameter monomials in any such normal form. This implies that the Newton boundary of the given germ can be transformed to the Newton boundary of any of its normal form equations by scaling x and y, except for terms above the determinacy.

We now prove correctness and termination of Algorithm 1.

Proof of Algorithm 1. In the transformation ϕ in line 8 we know that terms coming from first partial derivatives with degree $\leq d'$, not in B, cancel, as described in (3). We now discuss the effect of higher order terms in the binomial expansion after applying ϕ . Now, it follows from Theorem 4.3 that $\phi(x) = x + \sum_i z_i \frac{\partial f}{\partial y}^{z_i}$ and $\phi(y) = y + \sum_i z_i \frac{\partial f}{\partial x}^{z_i}$. Note that, since applying any transformation of filtration < 2 to f will change its nondegenerate Newton boundary to a degenerate boundary, ϕ has filtration ≥ 2 . That is $\operatorname{ord}\left(z_i \frac{\partial f}{\partial y}^{z_i}\right) \geq 2$ and $\operatorname{ord}\left(z_i \frac{\partial f}{\partial x}^{z_i}\right) \geq 2$. We systematically consider the terms coming from the higher order binomial expansions of $\phi(f)$. In higher order binomial expansions the terms are of the following form:

$$\frac{\partial^n f}{\partial x^s \partial y^{n-s}} \cdot \left(z_{i_1} \overline{\frac{\partial f}{\partial y}}^{z_{i_1}} \right) \cdots \left(z_{i_s} \overline{\frac{\partial f}{\partial y}}^{z_{i_s}} \right) \cdot \left(z_{i_{s+1}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{s+1}}} \right) \cdots \left(z_{i_n} \overline{\frac{\partial f}{\partial x}}^{z_{i_n}} \right), \quad (6)$$

Algorithm 1 Determining the moduli parameters in the normal Form of a germ with a normalized non-degenerate Newton boundary

- **Input:** A polynomial germ $f \in \mathbb{Q}[x, y], f \in \mathfrak{m}^3$ of corank 2 with a normalized nondegenerate Newton boundary; f_0 , the sum of the vertex monomials of $\Gamma(f)$; a set of monomials $B = \{b_1, \ldots, b_m\}$ that is the set of all monomials of w(f)-degree $\geq d(f)$ in a regular basis for f_0
- **Output:** A normal form of f and a normal form equation equivalent to f such that the normal form equation is a member of the normal form.
- 1: Let d'' be a determinacy bound for f.
- 2: Let $w := (w_1, \ldots, w_n) := w(f)$.

3: $F := f_0 + \sum_{i=1}^m \alpha_i \cdot b_i.$ 4: $S := \min(f - w(f) - \operatorname{jet}(f, d(f))).$

- 5: while $S \not\subset B$ do
- Let d' be the lowest w(f)-degree above d(f) with non-zero terms in f. 6:
- Write the sum q of the terms of w(f)-degree d' as 7:

$$q = g \frac{\partial f}{\partial x} + h \frac{\partial f}{\partial y}$$
 + terms in B of $w(f)$ -degree d' + terms of higher $w(f)$ -degree than d'.

- Define $\phi : \mathbb{C}[x, y] \to \mathbb{C}[x, y], \ \phi(x) = x + g, \ \phi(y) = y + h.$ 8:
- Let l be the filtration of ϕ . 9:
- $q := w(j) \operatorname{-jet}(\phi(f) (f + g \frac{\partial f}{\partial x} + h \frac{\partial f}{\partial u}), d').$ 10:
- $f := jet(\phi(f), d'').$ 11:
- while $q \neq 0$ do 12:
- Write 13:

$$q = \phi_x \cdot \frac{\partial f}{\partial y} + \phi_y \cdot \frac{\partial f}{\partial x}$$
, where $\phi_x, \phi_y \in \langle x, y \rangle^{l+1}$.

 $\phi_x = w \operatorname{-jet}(\phi_x, d'), \quad \phi_y = w \operatorname{-jet}(\phi_y, d')$ 14:

Define $\phi : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ by 15:

$$\phi(x):=x-w\operatorname{-jet}(\phi_x,d'), \quad \phi(y):=y-w\operatorname{-jet}(\phi_y,d').$$

 $q := \phi(f) - (f+q).$ 16:

- $f := \operatorname{jet}(\phi(f), d'').$ 17:
- $l = \min(\operatorname{ord}(\phi_x), \operatorname{ord}(\phi_u)).$ 18:
- $S := \min(f w(f) jet(f, d(f))).$ 19:
- 20: Normalize two terms of f of w(f)-degree d(f), of degree $\leq d''$, and not in B to coefficient 1.
- 21: if $\Gamma(f)$ does not intersect the x-axis then $f := f + x^a$ with $a = \mu(f) + 2$.
- 22: if $\Gamma(f)$ does not intersect the y-axis then $f := f + y^a$ with $a := \mu(f) + 2$.
- 23: return F, f

where n > 1, i_j 's $\in \mathbb{Z}$, as well as terms in J, where J is the ideal generated by all monomials of degree d' + 1. Let $z_{i_j} = \frac{a_{i_j}}{b_{i_j}}$. We distinguish between the following types of b_{i_i} 's:

(i)
$$b_{i_i} = cx^{\alpha}y^{\beta}, \ \alpha, \beta > 0;$$

(ii) $b_{i_j} = cy^{\beta}, \, \beta > 0;$ (iii) $b_{i_j} = cx^{\alpha}, \, \alpha > 0;$ (iv) $b_{i_i} = 1$.

We consider the following cases:

- (1) $b_{i_j} = 1$ for some $j \leq s$;
- (2) $b_{i_j} = 1$ for some j > s; (3) all the b_{i_j} 's are of type (i), (ii) and (iii), and the number of b_{i_j} 's of type (i) and (iii) is < s.
- (4) all the b_{i_j} 's are of type (i), (ii) and (iii), the number of b_{i_j} 's of type (ii) < n s, and the number of b_{i_j} 's of type (iii) is < s.
- (5) all the b_{i_j} 's are of type (i), (ii) and (iii), and the number of b_{i_j} 's of type (i) and (ii) is < n - s.

Note that if $z_k = \frac{a_k}{b_k}$ and $b_k = 1$, then

$$\frac{\frac{a_k}{b_k}\left(\overline{x^s \cdot y^{n-(s+1)} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}}^{z_k}\right)}{x^s \cdot y^{n-(s+1)}} = \frac{a_k\left(x^s \cdot y^{n-(s+1)} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}\right)}{x^s \cdot y^{n-(s+1)}} = z_k\left(\frac{\partial^n f}{\partial x^s \partial y^{n-s}}^{z_k}\right).$$
(7)

Similarly

$$\frac{\frac{a_k}{b_k}\left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}}^{z_k}\right)}{x^{s-1} \cdot y^{n-s}} = \frac{a_k\left(x^{s-1} \cdot y^{n-s} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}\right)}{x^{s-1} \cdot y^{n-s}} = z_k\left(\frac{\partial^n f}{\partial x^s \partial y^{n-s}}^{z_k}\right).$$

Furthermore, note that the monomials of $(\overline{x^{s-1} \cdot y^{n-s}} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}) \in \mathbb{C}[x, y]$ is a subset of the monomials of $\overline{\frac{\partial f}{\partial x}}^{z_k}$. Noticing that for all terms t in $\frac{\partial f}{\partial y} - \overline{\frac{\partial f}{\partial y}}^{z_k}$, it follows from the proof of Theorem 4.3 that $a_k \cdot t \in \operatorname{Ann}_{R/J'}(\frac{\partial f}{\partial x})$, where J' is the ideal generated by all the terms of wedgeres $(d' + w) \operatorname{deg}(h_1) + 1$ and $x = \frac{a_k}{dt} \operatorname{it} f_{t-1}^{u_k} = t^{1-t}$ the terms of w-degree $(d' + w - \deg(b_k) + 1)$, and $z_k = \frac{a_k}{b_k}$, it follows that

$$\left(x^{s-1} \cdot y^{n-s} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}\right) \cdot z_k \frac{\partial f}{\partial y}^{z_k} = z_k \left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}}^{z_k}\right) \cdot \frac{\partial f}{\partial y} + J.$$
(8)

In case (1), it hence follows from (7) and (8) that

$$\begin{split} & \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial y}}^{z_{i_{k}}} \right) \cdot \sum_{k=s+1}^{n} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{k}}} \right) \\ &= \frac{1}{x^{s-1} \cdot y^{n-s}} \cdot \left(x^{s-1} \cdot y^{n-s} \frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \right) \sum_{k=1}^{s} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial y}}^{z_{i_{k}}} \right) \cdot \sum_{k=s+1}^{n} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{k}}} \right) \\ &= \underbrace{\left(z_{i_{j}} \overline{\frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}^{z_{i_{j}}} \right) \cdot \sum_{k=1 \dots s, \ k \neq j} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial y}}^{z_{i_{k}}} \right) \cdot \sum_{k=s+1}^{n} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{k}}} \right) \\ & \in \langle x, y \rangle^{l+1} \subset \mathbb{C}[x, y] \end{split}$$

Note that $\left(z_{i_k}\frac{\overline{\partial f}}{\partial x}^{z_{i_k}}\right), \left(z_{i_k}\frac{\overline{\partial f}}{\partial y}^{z_{i_k}}\right) \in \langle x, y \rangle^l \subset \mathbb{C}[x, y]$, for all $k = s+1, \ldots, n$, where $l \ge 2$.

In case (2) it similarly follows that

$$=\underbrace{\frac{\partial^n f}{\partial x^s \partial y^{n-s}} \cdot \sum_{k=1}^s \left(z_{i_k} \overline{\frac{\partial f}{\partial y}}^{z_{i_k}} \right) \cdot \sum_{k=s+1}^n \left(z_{i_k} \overline{\frac{\partial f}{\partial x}}^{z_{i_k}} \right)}_{\substack{k=s+1,\ldots,n,\ k \neq j}} \left(z_{i_k} \overline{\frac{\partial f}{\partial x}}^{z_{i_k}} \right) \cdot \sum_{k=1}^s \left(z_{i_k} \overline{\frac{\partial f}{\partial y}}^{z_{i_k}} \right) \cdot \sum_{k=s+1,\ldots,n,\ k \neq j} \left(z_{i_k} \overline{\frac{\partial f}{\partial x}}^{z_{i_k}} \right) \cdot \frac{\partial f}{\partial x}.$$

Next we consider case (3). Note that

$$\begin{pmatrix} x^{s-1}y^{n-s}\frac{\partial^n f}{\partial x^s \partial y^{n-s}} \end{pmatrix} \cdot z_k \frac{\overline{\partial f}}{\partial y}^{z_k} \cdot z_l \frac{\overline{\partial f}}{\partial x}^{z_l} = z_k \begin{pmatrix} \overline{x^{s-1}y^{n-s}}\frac{\partial^n f}{\partial x^s \partial y^{n-s}} \end{pmatrix} \cdot \frac{\partial f}{\partial y} \cdot z_l \frac{\overline{\partial f}}{\partial x}^{z_l} \\ = z_k \begin{pmatrix} \overline{x^{s-1}y^{n-s}}\frac{\partial^n f}{\partial x^s \partial y^{n-s}} \end{pmatrix} \cdot z_l \frac{\overline{\partial f}}{\partial y}^{z_l} \cdot \frac{\partial f}{\partial x}^{z_l}$$

Using the above method, it follows that

$$\frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}} \cdot \sum_{k=1}^{s} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial y}}^{z_{i_{k}}} \right) \cdot \sum_{k=s+1}^{n} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{k}}} \right)$$

$$= \frac{z_{i_{j}}}{x^{s-1}y^{n-s}} \left(\overline{x^{s-1}y^{n-s}} \overline{\frac{\partial^{n} f}{\partial x^{s} \partial y^{n-s}}}^{z_{i_{j}}} \right) \sum_{k=1,\dots,s,\ k\neq j} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial y}}^{z_{i_{k}}} \right) \cdot \sum_{k=s+1}^{n} \left(z_{i_{k}} \overline{\frac{\partial f}{\partial x}}^{z_{i_{k}}} \right) \overline{\frac{\partial f}{\partial y}},$$

where all the b_{i_j} 's are ordered that first, b_{i_j} 's of type (i) and (iii) occur and then those of type (ii). In other words the last n - s + 1 b'_{i_j} s are of type (ii). This means by Theorem 4.3 that

$$\begin{split} & \frac{\partial^n f}{\partial x^s \partial y^{n-s}} \cdot \sum_{k=1}^s \left(z_{i_k} \overline{\frac{\partial f}{\partial y}}^{z_{i_k}} \right) \cdot \sum_{k=s+1}^n \left(z_{i_k} \overline{\frac{\partial f}{\partial x}}^{z_{i_k}} \right) \\ &= \frac{1}{x^{s-1}} \cdot z_{i_s} \left(\overline{x^{s-1} \cdot y^{n-s} \frac{\partial^n f}{\partial x^s \partial y^{n-s}}}^{z_{i_s}} \right) \sum_{k=1,\dots,s-1} \left(z_{i_k} \overline{\frac{\partial f}{\partial y}}^{z_{i_k}} \right) \cdot \sum_{k=s+1}^n \frac{\left(z_{i_k} \overline{\frac{\partial f}{\partial x}}^{z_{i_k}} \right)}{\underbrace{y}_{\in \mathbb{C}[x,y]}} \frac{\partial f}{\partial y} \end{split}$$

Since the number of $b'_{i_j}s$ that are of type (ii) is > n - s, b_{i_s} is of type (ii). But then, arguing similar as in (7), $x^{s-1}|z_{i_s}\left(\overline{x^{s-1}\cdot y^{n-s}\frac{\partial^n f}{\partial x^s\partial y^{n-s}}}^{z_{i_s}}\right)$. Hence (6) can be expressed as

$$\underbrace{q'\frac{z_k\frac{\partial f}{\partial x}}{y}}_{\in\langle x,y\rangle^{l+1}}\frac{\partial f}{\partial y}, \quad \text{with } q' \in \langle x,y\rangle^2, \tag{9}$$

and b_k is of type (ii). In case (4) it follows similarly that (6) can be expressed as in (9), where b_k is of type (i) or (ii).

In case (5), exchanging x and y in (9), we express (6) as

$$\underbrace{q'\frac{z_k\frac{\overline{\partial f}}{\partial y}^{z_k}}{x}}_{\in \langle x,y\rangle^{l+1}}\frac{\partial f}{\partial x}, \quad \text{with } q'\in \langle x,y\rangle^2,$$

where b_k is of type (iii).

We conclude that q can be expressed as

$$q = \phi_x \cdot \frac{\partial f}{\partial x} + \phi_y \frac{\partial f}{\partial y},\tag{10}$$

where $\phi_x, \phi_y \in \langle x, y \rangle^{l+1}$. Then the higher orders of the binomial expansion of $\phi(f)$ of degree $\leq d'$ can be removed by the first order terms of the transformation $\phi : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ defined by

$$\phi_{\text{new}}(x) := x - \phi_x, \ \phi_{\text{new}}(y) := y - \phi_y.$$

In an analogous way one can show that the sum, q_{new} , of higher order terms of the binomial expansion of degree $\leq d'$ of $\phi_{\text{new}}(\phi(f))$ can be written in terms of the same formulas, now in terms of $\phi_{\text{new}}(x) = x - \sum_i z'_i \frac{\partial f}{\partial y}^{z'_i}$ and $\phi_{\text{new}}(y) = y - \sum_i z'_i \frac{\partial f}{\partial x}^{z'_i}$, as above. Note that the filtration of ϕ_{new} is higher than that of ϕ . Hence eventually there will be no terms, not in B, of w-degree $\leq d'$.

For many examples the transformations arising from line 7 can be chosen in such a way that the higher orders of the binomial expansion of $\phi(f)$ are of degree larger than d'. Choosing g and h in such a way, the while-loop from line 12 to 18 is redundant. The next example proves that this is unfortunately not in general the case.

Example 4.5. Let $f = y^{28} + xy^7 + x^2y^3 + 11x^2y^4 + x^{22}$. Then $\frac{\partial f}{\partial x} = 2xy^3 + 22xy^4 + y^7 + 22x^{21}$ and $\frac{\partial f}{\partial y} = 3x^2y^2 + 44x^2y^3 + 7xy^6 + 28y^{27}$, and a regular basis for f is $x^{22}y, x^{21}y, x^{20}y, x^{19}y, x^{18}y, x^{17}y, x^{16}y, x^2y^3$. Note that x^2y^4 is the only monomial above the Newton Boundary, with w(f)-degree 1008. We can express $-11x^2y^4$ as follows in terms of the first partial derivatives.

$$-11x^2y^4 = -(7xy + 28y^{22})\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}$$

The corresponding transformation ϕ is given by

$$\begin{split} \phi(x) &= x - (7xy + 28y^{22}) = x - \sum_{i} z_i \frac{\partial f}{\partial y} = x - \frac{1}{y^5} \frac{\overline{\partial f}}{\partial y}^{\frac{1}{y^5}} + p_x \\ \phi(y) &= y + y^2 = y - \sum_{i} z_i \frac{\partial f}{\partial x} = y - (-\frac{1}{y^5} \frac{\overline{\partial f}}{\partial x}^{\frac{1}{y^5}}) + p_y, \end{split}$$

where $p_x \frac{\partial f}{\partial x}$ and $p_y \frac{\partial f}{\partial y}$ are of w(f)-degree 1008 or higher. Note that the filtration of ϕ is l = 2. Now let

$$q = w \operatorname{-jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 1008).$$

Then q is the contribution of the orders > 1 in the binomial expansion of ϕ in the 1008-jet of $\phi(f)$.

$$q = -28xy^9 + 182y^{30}$$

To write q as in line 13 in Algorithm 1, we are computing ϕ_x and ϕ_y . We do this by forming an ideal I generated by the set

$$\left\{ x^i y^j \frac{\partial f}{\partial x}, x^i y^j \frac{\partial f}{\partial y} \quad \left| \quad i+j=l+1=3 \quad \right\},\right.$$

and all monomials of higher piecewise degree than 1008. We then write q in terms of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, where the coefficients are of order 3 or higher, using a Gröbner basis of I with

a global ordering, and then lifting the result back to the original generators using the command liftstd() in SINGULAR. It turns out that

$$q = \left((-28 - 364y^{17} - 4004y^{18})xy^2 + 182y^{23} \right) \cdot \frac{\partial f}{\partial x}$$

By abuse of notation, let $f = \phi(f)$. We now form a new ϕ as described in line 15 of Algorithm 1. We set

$$\phi_x = 28xy^2 - 182y^{23}$$
 and $\phi_y = 0$.

We define $\phi : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ by $\phi(x) = x - \phi_x$ and $\phi(y) = y - \phi_y = y$. Hence the first order terms of the binomial expansion of $\phi(f)$ up to degree 1008 cancel q. In this case higher order terms of the binomial expansion of $\phi(f)$ does not have terms of w(f)-degree 1008 or lower. In fact w-jet $(\phi(f), 1008) = y^{28} + xy^7 + x^2y^3 + x^{22}$.

In the next example more than one iteration of the while-loop in Algorithm 1 is needed to transform the germ f to its normalform equation up to w(f)-degree 700.

Example 4.6. For $f = x^2y^4 + x^4y^2 + x^{20} + y^{40} + 60x^{21}y^{14}$, we have $\frac{\partial f}{\partial x} = 4x^3y^2 + 2xy^4 + 20x^{19} + 1260x^{20}y^{14}$ and $\frac{\partial f}{\partial y} = 2x^4y + 4x^2y^3 + 840x^{21}y^{13} + 40y^{39}$, and a regular basis is x^4y^4 , x^4y^3 , x^3y^4 , x^4y^2 , x^3y^3 , x^2y^4 . Note that $x^{21}y^{14}$ is the only monomial above the Newton boundary with w(f)-degree 700. Now $60x^{21}y^{14}$ can be written as

$$60x^{21}y^{14} = (2x^4y^{12} + 4x^2y^{14} + 40y^{50} - 10x^{20}y^{10})\frac{\partial f}{\partial x} - (4x^3y^{13} + 2xy^{15})\frac{\partial f}{\partial y}$$

The corresponding transformation ϕ is given by

$$\begin{split} \phi(x) &= x - (2x^4y^{12} + 4x^2y^{14} + 40y^{50} - 10x^{20}y^{10}) = x - y^{11}\overline{\frac{\partial f}{\partial y}}^{y^{11}} - 5x^{16}y^9\overline{\frac{\partial f}{\partial y}}^{x^{10}y^9} + p_x \\ \phi(y) &= y + (4x^3y^{13} + 2xy^{15}) = y - (-y^{11}\overline{\frac{\partial f}{\partial x}}^{y^{11}} - 5x^{16}y^9\overline{\frac{\partial f}{\partial x}}^{x^{16}y^9}) + p_y, \end{split}$$

where $p_x \frac{\partial f}{\partial x}$ and $p_y \frac{\partial f}{\partial y}$ is of *w*-degree 700 or higher. Note that the filtration of ϕ is l = 16. Now let

$$q = w \operatorname{-jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 700).$$

Then q is the contribution of the orders > 1 in the binomial expansion of ϕ in the 700-jet of $\phi(f)$.

$$\begin{array}{ll} q &=& -24x^{10}y^{26} - 12x^8y^{28} - 12x^6y^{30} - 24x^4y^{32} - 224x^7y^{44} - 32x^5y^{46} + 144x^6y^{60} \\ &\quad +12160x^6y^{64} + 12640x^4y^{66} + 2800x^2y^{68} + 384x^7y^{74} + 952320x^7y^{78} \\ &\quad +472320x^5y^{80} + 79680x^3y^{82} + 256x^8y^{88} + 11704320x^6y^{94} + 1467360x^4y^{96} \\ &\quad +9600x^2y^{102} + 1600y^{104} + 21065216x^5y^{110} + 38400x^5y^{114} - 12800x^3y^{116} \\ &\quad +12800xy^{118} + 245661440x^6y^{124} + 38400x^4y^{130} + 38400x^2y^{132} + 51200x^3y^{146} \\ &\quad -256000xy^{152} + 25600x^4y^{160} + 2560000y^{202}. \end{array}$$

Note that the order of q is 36.

To write q as in line (13) in Algorithm 1, we are computing ϕ_x and ϕ_y . We do this by forming an ideal I generated by

$$\left\{ x^i y^j \frac{\partial f}{\partial x}, x^i y^j \frac{\partial f}{\partial y} \quad \middle| \quad i+j=l+1=17 \quad \right\},$$

and all monomials of higher piecewise degree than 700. We write q in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, by using a Gröbner basis of I with a global ordering, and then lifting the result back to the original generators using the command liftstd(). We now form ϕ as described in line 15 of Algorithm 1. Let ϕ_x and ϕ_y be as defined in line(14) in Algorithm 1. It turns out that

$$\begin{split} \phi_x &= 256x^{21}y^{10} + 512x^{19}y^{12} + 128x^{17}y^{14} + 448x^{15}y^{16} + 288x^{13}y^{18} - 144x^{11}y^{20} \\ &+ 72x^9y^{22} - 42x^7y^{24} + 18x^5y^{26} - 12x^3y^{28} + \frac{12650}{567}x^{34}y^{10} - 320x^{10}y^{36} \\ &+ 160x^8y^{38} - 80x^6y^{40} - 16x^4y^{42} + 288x^9y^{52} - 144x^7y^{54} + 72x^5y^{56} - 4800x^9y^{56} \\ &+ 2400x^7y^{58} - 1200x^5y^{60} + 3640x^3y^{62} + 1320xy^{64} - 384x^8y^{68} + 192x^6y^{70} \\ &- 326400x^8y^{72} + 163200x^6y^{74} + 156480x^4y^{76} + 39840x^2y^{78} + 128x^7y^{84} \\ &- 8769600x^7y^{88} + 4384800x^5y^{90} + 733680x^3y^{92} - 38400x^7y^{92} + 19200x^5y^{94} \\ &- 9600x^3y^{96} + 4800xy^98 - 21065216x^6y^{104} + 10532608x^4y^{106} - 115200x^6y^{108} \\ &+ 57600x^4y^{110} - 19200x^2y^{112} + 6400y^{114} + 122830720x^5y^{120} + 38400x^5y^{124} \\ &- 19200x^3y^{126} + 19200xy^{128} - 51200x^4y^{140} + 25600x^2y^{142} - 512000x^4y^{144} \\ &+ 256000x^2y^{146} - 128000y^{148} + 12800x^3y^{156} + 192000x^3y^{160} - 128000xy^{162} \end{split}$$

and

$$\phi_y = -512x^{20}y^{11} - 256x^{18}y^{13} - 256x^{16}y^{15} - 512x^{14}y^{17} - 2560x^{36}y^9 + 40y^{65} + 64000y^{163}$$

Note that order of ϕ_x and ϕ_y , and hence the filtration of ϕ , is 31, respectively. Unfortunately the higher order binomial expansions of $\phi(f)$ again contributes terms of w-degree ≤ 700 . Note that this time the order of these terms is 66 > 36. Now let

$$q = w \operatorname{-jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 700).$$

Then

$$\begin{array}{ll} q &=& 1296x^8y^{58} - 1008x^6y^{60} - 3072x^7y^{74} - 7488x^8y^{88} + 736800x^6y^{94} + 189120x^4y^{96} \\ &\quad + 6049280x^5y^{110} + 136922880x^6y^{124} - 98264000x^4y^{130} - 12833600x^2y^{132} \\ &\quad - 681446400x^3y^{146} - 23950380800x^4y^{160} - 77184000x^2y^{166} - 5408000y^{168} \\ &\quad - 94208000xy^{182} + 33001344000x^2y^{196} + 13038080000y^{232}. \end{array}$$

Let $f = \phi(f)$. Repeating the process, we again write q in terms of

$$\left\{ x^{i}y^{j}\frac{\partial f}{\partial x}, x^{i}y^{j}\frac{\partial f}{\partial y} \mid i+j=l+1=32 \right\},$$

and all terms of piecewise degree higher than 700 to compute ϕ_x and ϕ_y . It turns out that

$$\begin{split} \phi_x &= -3312x^9y^{52} + 1656x^7y^{54} - 504x^5y^{56} + 3072x^8y^{68} - 1536x^6y^{70} - 3744x^7y^{84} \\ &- 358560x^7y^{88} + 179280x^5y^{90} + 94560x^3y^{92} - 6049280x^6y^{104} + 3024640x^4y^{106} \\ &+ 68461440x^5y^{120} + 73408000x^5y^{124} - 36704000x^3y^{126} - 6146400xy^{128} \\ &+ 681446400x^4y^{140} - 340723200x^2y^{142} - 11975190400x^3y^{156} + 77184000x^3y^{160} \\ &- 38592000xy^{162} + 94208000x^2y^{176} - 47104000y^{178} + 15848768000xy^{192} \end{split}$$

and

$$\phi_y = -135200y^{129} + 325952000y^{193}.$$

Applying ϕ , formed as in line (15),

$$w$$
-jet $(\phi(f), 700) = x^4y^2 + x^2y^4 + x^{20} + y^{40}.$

We now turn to the proof of our two main lemmata, and begin with Lemma 4.1,

Proof. We say that terms of $f_x := \frac{\partial f_{\Gamma'}}{\partial x}$ and $f_y := \frac{\partial f_{\Gamma'}}{\partial y}$ are in correspondence if they originate from the same term of $f_{\Gamma'}$.

- If $a \neq 0$, then x^a is the largest power of x dividing f_x , and x^{a+1} is the largest power of x dividing f_y . To see this, we first note that, excluding a pure x-power term in f_x , all terms of f_x and f_y are in one-to one correspondence. If f_x would be divisible by x^{a+1} then, since all terms of f_y are in correspondence to a term in f_x , all terms of f_y would be divisible by x^{a+2} , contradicting the minimal choice of a. Thus x^a is the largest power of x dividing f_x , and f_y is divisible by x^{a+1} . If x^{a+2} would divide f_y , then x^{a+1} would also divide every corresponding term of f_x . Since also a possible x^{α} -term of f_x is then divisible by x^{a+2} (since it is the term with the largest x-exponent), this again contradicts the minimal choice of x. Moreover the argument above implies that x^{a+1} is the highest power of x dividing f_z .
- If a = 0, then $x \nmid f_y$ or $x \nmid f_x$.
 - If $x \nmid f_y$ then $f_{\Gamma'}$ has a term that is a pure y-power. Hence $x \nmid f$.
 - If $x|f_y$, then $x \nmid f_x$, which implies that $f_{\Gamma'}$ has a term of the form xy^l with $l \ge 1$. Hence, in this case, if x^s divides f_y (and hence f), s = 1.

Now, let $x^{a'}y^{b'}$ be the product of the highest power of x and the highest power of y dividing jet (f, Γ') .

- Then a' = a + 1, if $a \neq 0$.
- If a = 0, we have two cases:
 - If $x \nmid f_y$, then a' = 0.
 - If $x|f_y$, then 1 is the highest power of x dividing $f_{\Gamma'}$. Hence a' = a + 1.

Similarly b' = b + 1, if $b \neq 0$. If b = 0 and $y \nmid f_x$, then b' = 0. If b = 0, $y \mid f_x$ (implying that $y \nmid f_y$, then 1 is the highest power of y dividing $f_{\Gamma'}$. Hence b' = b + 1.

We assume that

$$g \cdot f_x + h \cdot f_y = 0.$$

Then

$$x^{a}y^{b}(g \cdot y^{t} \cdot x^{i} \cdot \operatorname{sat}(f_{x}) + h \cdot x^{s} \cdot y^{\nu} \cdot \operatorname{sat}(f_{y})) = 0$$

where s, t, i and ν are defined as above. To see this, consider first the case $a \neq 0, b \neq 0$, then

$$x^{a}y^{b}(g \cdot y \cdot \operatorname{sat}(f_{x}) + h \cdot x \cdot \operatorname{sat}(f_{y})) = x^{a'}y^{b'}\left(\frac{g}{x} \cdot \operatorname{sat}(f_{x}) + \frac{h}{y} \cdot \operatorname{sat}(f_{y})\right) = 0,$$

which can be verified by the arguments above. The other cases can be similarly verified. If $a \neq 0$, or if a = 0, and $x | f_y$, then s = 1 and a' = a + 1. In both these cases i = 0. If a = 0 and $x \nmid f_y$, then a' = a = 0 and s = 0. Therefore we have in general.

$$x^{a'}y^{b}(g \cdot y^{t} \cdot x^{i} \cdot \operatorname{sat}(f_{x}) + h \cdot x^{s} \cdot y^{\nu} \cdot \operatorname{sat}(f_{y})) = 0.$$

Similarly, if $b \neq 0$, or if b = 0, and $y | f_x$, then $\nu = 1$ and b' = b + 1. In both these cases t = 0. If b = 0 and $y \nmid f_x$, then b' = b = 0 and t = 0. Therefore we have in general.

$$x^{a'}y^{b'}(\frac{g}{x^s} \cdot x^i \cdot \operatorname{sat}(f_x) + \frac{h}{y^t} \cdot y^{\nu} \cdot \operatorname{sat}(f_y)) = 0.$$

Therefore

$$\frac{g}{x^s} \cdot x^i \cdot \operatorname{sat}(f_x) + \frac{h}{y^t} \cdot y^{\nu} \cdot \operatorname{sat}(f_y) = 0.$$

Hence x|g, if $a \neq 0$, or if a = 0 and $x|f_y$, and y|h, if $b \neq 0$, or if b = 0 and $y|f_x$.

We now show that $x^i \cdot \operatorname{sat}(f_x)$ and $y^{\nu} \cdot \operatorname{sat}(f_y)$ have no common monomial factor. To do that, first note that $\operatorname{sat}(f_{\Gamma'})$ is nondegenerate by definition. This implies that $\operatorname{sat}(f_{\Gamma'})$ does not have a multiple monomial factor, hence $\operatorname{sat}(f_{\Gamma'})$ and $\frac{\partial \operatorname{sat} f_{\Gamma'}}{\partial x}$ have no common monomial factor, and $\operatorname{sat}(f_{\Gamma'})$ and $\frac{\partial \operatorname{sat} f_{\Gamma'}}{\partial y}$ have no common monomial factor. Thus, $x^{a'}y^{b'}$ accounts for the only multiple factors in $f_{\Gamma'}$.

Consider the following equations for the partial derivatives of $f_{\Gamma'}$:

$$f_x = a' x^{a'-1} y^{b'} (\operatorname{sat}(f_{\Gamma'})) + x^{a'} y^{b'} (\frac{\partial \operatorname{sat} f_{\Gamma'}}{\partial x})$$
$$f_y = b' x^{a'} y^{b'-1} (\operatorname{sat}(f_{\Gamma'})) + x^{a'} y^{b'} (\frac{\partial \operatorname{sat} f_{\Gamma'}}{\partial y}).$$

If the saturations of the partial derivatives of $f_{\Gamma'}$ with respect to x and y share a common factor, then this factor would also be a factor of $\operatorname{sat}(f_{\Gamma'})$. However, this contradicts the previous equations. Furthermore $y \nmid \operatorname{sat}(f_x)$ if $\nu > 0$ and $x \nmid \operatorname{sat}(f_y)$ if i > 0.

Since $x^i \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right)$, $y^{\nu} \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right)$ is a regular sequence, the vector $\left(\frac{g}{x^s}, \frac{h}{y^t}\right)$ is a polynomial multiple of the Koszul syzygy of $(x^i \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right))$. This completes the proof.

We turn now to the proof of Lemma 4.4, where we use the following observation:

Remark 4.7. With the notation of Lemma 4.4, let m be a monomial corresponding to a lattice point of Γ . Suppose g is a germ with non-degenerate Newton boundary $\Gamma(g) = \Gamma$ and with monomial m, such that the corresponding term cannot be removed from g by a right equivalence. Then Theorem 3.20 implies that m is a monomial of f_0 or $m \in B$.

We now prove Lemma 4.4:

Proof. We prove the first part of the lemma: It is sufficient to show that for any monomial m in the relative interior of a face Δ of Γ , there exists a germ g as in Remark 4.7. For any germ g with $\Gamma(g) = \Gamma$, we can write

$$jet(g,\Delta) = x^a \cdot y^b \cdot g_1 \cdots g_n \cdot \widetilde{g}$$

where a, b are integers, g_1, \ldots, g_n are linear homogeneous polynomials not associated to x or y, and \tilde{g} is a product of non-associated irreducible non-linear homogeneous polynomials. Note that a and b are the distances of Δ from the y- and x-coordinate axes. Note also that smooth faces meeting the coordinate axes do not contain interior lattice points. After possibly exchanging x and y, we may assume that for the weight w of Δ we have $w(x) \ge w(y)$. First consider the case that w(x) > w(y):

(1.a) Suppose that $a \geq 2$. Let g be a germ non-degenerate Newton boundary Γ . Then any right-equivalence which does not act only as a rescaling of variables on jet (g, Δ) generates terms on the coordinate axes below Γ , hence, changes the Newton polygon. The claim follows directly (choosing a non-degenerate $g = f_0 + c \cdot m$ with $c \in \mathbb{C}^*$).

- (1.b) Consider now the case a = 0. Since Γ corresponds to a normalized germ with respect to Δ , we have n = 0 and $\tilde{g} \neq 1$. This implies that $w(y) \nmid w(x)$, hence there does not exist a *w*-weighted homogeneous right-equivalence except rescaling of variables.
- (1.c) Finally, suppose a = 1. If $w(y) \nmid w(x)$, then, as above, n = 0 and $\tilde{g} \neq 0$ which implies that rescalings are the only right-equivalences on the face. If $w(y) \mid w(x)$, then, writing $\tau = w(x)/w(y)$, any right-equivalence which does not create any terms of lower w-weight than that of Δ is of the form

$$x \mapsto c_1 x + c_2 y^{\tau}, \ y \mapsto c_3 y,$$

where $c_1, c_3 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$, hence acts on y as a rescaling. We may therefore assume that b = 0. The vertices of Δ correspond to monomials of the form x^p and $xy^{(p-1)\cdot\tau}$ with $p \geq 2$. Any monomial in the interior of Δ is of the form $m = x^s y^{\tau \cdot (p-s)}$ with 0 < s < p. For $g = f_0 + c \cdot m$ with $c \in \mathbb{C}^*$, the jet with regard to Δ is then $\text{jet}(g, \Delta) = x^p + c \cdot m + xy^{(p-1)\cdot\tau}$. We now show that there is no right-equivalence which keeps $\Gamma(g)$ and, hence, is of the above form, that removes m. Keeping the face Δ and removing m amounts to the conditions

$$\binom{p}{p-s}c_1^s c_2^{p-s} + c \cdot c_1^s c_3^{\tau \cdot (p-s)} = 0$$
$$c_2^p + c \cdot c_2^s c_3^{\tau \cdot (p-s)} + c_2 c_3^{(p-1) \cdot \tau} = 0$$

which correspond to the vanishing of the coefficients of m and $y^{p \cdot \tau}$. Using $c_1 \neq 0$ a solution of the first equation for c_2 is of the form

$$c_2 = \tilde{c} \cdot c_3^{\tau}$$

where $\tilde{c}^{p-s} = -c/{p \choose s}$. Inserting this into the second equation, leads to the equation

$$0 = \tilde{c}^{p} \cdot c_{3}^{\tau,p} + c \cdot \tilde{c}^{s} c_{3}^{\tau,p} + \tilde{c} \cdot c_{3}^{\tau,p}$$
$$= c_{3}^{\tau,p} \cdot (\tilde{c}^{p} + c \cdot \tilde{c}^{s} + \tilde{c})$$
$$= c_{3}^{\tau,p} \cdot \left(\tilde{c}^{p} - {p \choose s} \cdot \tilde{c}^{p} + \tilde{c}\right)$$
$$= c_{3}^{\tau,p} \cdot \tilde{c} \cdot \left(\left(1 - {p \choose s}\right) \cdot \tilde{c}^{p-1} + 1\right)$$

Since $\binom{p}{s} \neq 1$, there is a Zariski open set of values of \tilde{c} , equivalently of c, such that the expression in the bracket does not vanish and g is non-degenerate. For such a choice of c and thus of g, it follows that $c_3 = 0$, a contradiction.

Now consider the case that w(x) = w(y). Since $jet(g, \Delta)$ is homogeneous, hence factorizes into linears, in this case we have $a \ge 1$, $b \ge 1$ and $\tilde{g} = 0$.

- (1.d) If $a, b \ge 2$, and g is a germ with non-degenerate Newton boundary and Newton polygon Γ , then any right-equivalence which does not only act as a rescaling of variables on jet (g, Δ) changes the Newton polygon (with the same argument as in the case $w(x) > w(y), a \ge 2$).
- (1.e) The case $a \ge 2$ and b = 1, can be handled in the same way as the case w(x) > w(y), $a = 1, w(y) \mid w(x)$ above. An analoguos argument also applies to the case $b \ge 2$ and a = 1.
- (1.f) In the case a = b = 1, a Gröbner basis calculation shows directly that any monomial corresponding to a lattice point of Δ is either a monomimal of f_0 or an element of B (note that smooth faces do not contain any interior lattice points): The germ

 $x^p y + xy^p$, $p \ge 2$, is right-equivalent to a germ h (applying for instance the right-equivalance $x \mapsto x + y$, $y \mapsto y + 2x$) with vertex monomials x^{p+1} and y^{p+1} , hence, by Theorem has Milnor number $\mu = (p+1)^2 - 2(p+1) + 1 = p^2$. The germ $g = x^p y + xy^p + x^{p^2+2} + y^{p^2+2}$ is right-equivalent to h, hence also has Milnor number $\mu = p^2$. This implies that $\langle x, y \rangle^{p^2+2} \subset \text{Jac}(g)$. We determine a standard basis of

$$\operatorname{Jac}(g) = \langle g_x, g_y \rangle + \langle x, y \rangle^{p^2 + 2}$$

where

$$g_x = p \cdot x^{p-1}y + y^p + (p^2 + 2) \cdot x^{p^2 + 1}, g_y = x^p + p \cdot xy^{p-1} + (p^2 + 2) \cdot y^{p^2 + 1}$$

with regard the local degree reverse lexicographic ordering. The S-polynomial of g_x and g_y leads to the standard basis element xy^p after reducing the tail by $\langle x, y \rangle^{p^2+2}$. The S-polynomial of xy^p with g_x leads to the standard basis element y^{2p-1} after reducing the tail by $\langle x, y \rangle^{p^2+2}$. The S-polynomial of xy^p with g_y reduces to zero, while all remaining ones vanish. Hence, the classes of the monomials

$$y^{p+1}, x^2 y^{p-1}, \dots, x^{p-2} y^3,$$

form a basis of the Milnor algebra in degree p + 1. Using the relation g_x these monomials are equivalent to

$$x^2y^{p-1}, \dots, x^{p-2}y^3, x^{p-1}y^2$$

which form a regular basis in degree p+1. From the standard basis of the Jacobian it is clear that this is the only option for a regular basis, which proves the claim.

We now prove the second part of the lemma. Let m_1, \ldots, m_n be the monomials of f_0 and let $g = \sum_i k_i m_i$. For a face Δ of Γ , we denote again its weight by w. In the first part of the proof we have seen that for all faces Δ , except those where a = 1, $w(x) \ge w(y), w(y)|w(x)$, and $b \ge 2$ if w(x) = w(y), or those where $b = 1, w(x) \le w(y)$, w(x)|w(y), and $b \ge 2$ if w(x) = w(y), there does not exist a w-weighted homogeneous right-equivalence except rescaling of variables.² In order to describe in these two cases the w-homogeneous transformations on $jet(g, \Delta)$ keeping the face, by symmetry, it is sufficient to consider the first of the two. Here, writing $\tau = w(x)/w(y)$, the jet is of the from $jet(g, \Delta) = y^q \cdot (k_i x^p + k_j x y^{(p-1)\cdot \tau}), k_i, k_j \neq 0, p \ge 2$. All homogeneous transformations are of the form $x \mapsto c_1 x + c_2 y^{\tau}, y \mapsto c_3 y$, and keeping Δ amounts to the condition

$$k_i c_2^p + k_j c_2 c_3^{(p-1)\cdot\tau} = 0,$$

which implies that either $c_2 = 0$ (which corresponds to a rescaling of variables), or that between c_2 and c_3 there is an algebraic relation $c_2 = k \cdot c_3^{\tau}$ (with $k^{p-1} = -k_j/k_i$).

We now show that there is a germ g with the same monomials as f_0 and non-degenerate Newton boundary, such that only two terms of g can be normalized to coefficient one by a right-equivalence keeping the Newton polygon. Since this g is right-equivalent to a germ in the normal form described in Theorem 3.7, this then implies that at most two monomials of f_0 are not in B.

In case there are only two monomials of f_0 of degree $\leq dt$, the claim is trivial choosing $g = f_0$.³ Otherwise, let m_s, m_t and m_l be three distinct monomials of f_0 of degree

²Note that these cases correspond to the second part of (1.c) and to (1.e), which are the only settings where there may exist (and, in fact, exist) transformations on the jet of the respective face, which are not just rescalings of variables.

³Note that this includes the case where there is a face with w(x) = w(y) and a = b = 1.

 $\leq dt$. We prove that there is a Zariski open set of germs g such that not all three monomials m_s, m_t, m_l can be normalized to coefficient one. To see this, it is enough to prove that, for any g with the same monomials as f_0 , after choosing two monomials out of m_s, m_t, m_l and normalizing their coefficients to one, when restricting the action of the right-equivalence group to the Newton boundary and stabilizing the two normalized coefficients, the stabilizer acts as a finite group.

- (2.a) If all three monomials lie on faces of Γ which permit only rescaling of variables, then the claim is obvious.
- (2.b) If exactly two of the three monomials, say m_s, m_t , lie on faces of Γ which permit only rescaling of variables, then, without loss of generality, we may assume that m_l lies on a face with a = 1, $w(x) \ge w(y)$, w(y)|w(x). If w(x) = w(y), then it is sufficient to consider the case $b \ge 2$. After normalization of the coefficients of m_s and m_t to value one via rescaling of variables, stabilizing the two normalized coefficients together with the above condition $c_2 = 0$ or $c_2 = k \cdot c_3^{\tau}$ admits only finitely many solutions.
- (2.c) Suppose that exactly one of the three monomials, say m_s , lies on a face of Γ which permits only rescaling of variables.

If m_t and m_l lie on the same face Δ , we may assume that the jet of Δ is of the form $jet(g, \Delta) = y^q \cdot (k_l x^p + k_t x y^{(p-1)\cdot\tau})$. Then coefficients of the monomials m_s and m_l can only be changed via rescaling of variables. Then, similar to the previous case, normalization of the coefficients of m_s and m_l to value one together with the condition $c_2 = 0$ or $c_2 = k \cdot c_3^{\tau}$ admits only finitely many solutions.

If m_t and m_l lie on different faces, we may assume that m_t lies on a face Δ with weight w and a = 1, $w(x) \ge w(y)$, w(y)|w(x). Moreover, in case w(x) = w(y), we can assume that $b \ge 2$, since the case b = 1 has already been discussed. Then $jet(g, \Delta)$ is of the form

$$\operatorname{jet}(g,\Delta) = y^q \cdot \left(k_i x^p + k_j x y^{(p-1)\cdot\tau}\right),$$

with $\tau = w(x)/w(y)$ and $t \in \{i, j\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$x \mapsto c_1 x + c_2 y^{\tau}, y \mapsto c_3 y$$

satisfy the condition $c_2 = 0$ or $c_2 = k \cdot c_3^{\tau}$ with $k^{p-1} = -k_j/k_i$. Similarly, we may assume that m_l lies on a face Π with weight v with b = 1, $v(x) \leq v(y)$, v(x)|v(y). Again, in case v(x) = v(y), we may assume that $a \geq 2$. Then $jet(g, \Pi)$ is of the form

$$\operatorname{jet}(g,\Pi) = x^{q'} \cdot \left(k_r y^{p'} + k_s y x^{(p'-1)\cdot\tau'}\right),$$

with $\tau' = v(y)/v(x)$ and $l \in \{r, s\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$x \mapsto c_1 x, \ y \mapsto c_3 y + c_2' x^{\tau'}$$

satisfying the condition $c'_2 = 0$ or $c'_2 = k' \cdot c_1^{\tau'}$ with $(k')^{p-1} = -k_s/k_r$. If m_t is the monomial of $jet(g, \Delta)$ of larger x-degree (that is, $m_t = y^q x^p$ and t = i), or if m_l is the monomial of $jet(g, \Pi)$ of larger y-degree (that is, $m_l = y^q x^{p'}$ and l = r), then the coefficient of the respective monomial can only be changed via rescaling of variables, and we can argue as in previous case. Suppose now that m_t and m_l are the x-linear monomials of the respective jets. Normalizing the coefficients of m_t and m_l then amounts to the relations

$$c_1 c_3^q \cdot (k_t c_3^{(p-1)\cdot\tau} + k_i c_2^{p-1}) = 1$$

$$c_3 c_1^{q'} \cdot (k_l c_1^{(p'-1)\cdot\tau'} + k_r (c_2')^{p'-1}) = 1,$$

After inserting $c_2 = k \cdot c_3^{\tau}$ with $k^{p-1} = -k_t/k_i$ the first equation implies that $c_2 = 0$, a contradiction. Similarly, inserting $c_2' = k' \cdot c_1^{\tau'}$ with $(k')^{p-1} = -k_l/k_r$ into the second equation, yields a contradiction. Hence, $c_2 = c_2' = 0$, that is, right-equivalences keeping the Newton polygon act on the jets as rescaling of variables. So after normalizing the coefficients of two monomials, the coefficient of the third one can take only finitely many values.

(2.d) If none of the three monomials m_s, m_t and m_l lies on a face of Γ which permits only rescaling of variables, then we may assume that m_s, m_t lie on a face with a = 1, $w(x) \ge w(y), w(y)|w(x)$, and $b \ge 2$ if w(x) = w(y), and m_l lies on a face with b = 1, $w(x) \le w(y), w(x)|w(y)$, and $a \ge 2$ if w(x) = w(y). We can thus argue as previous cases to obtain only finitely many solutions for the action of the right-equivalence group on the Newton boundary.

5. Next steps

We have also developed an algorithm to enumerate all normal form families up to a specified Milnor number or modality. These algorithms will be presented in an up-coming paper.

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