A CLASSIFICATION ALGORITHM FOR COMPLEX SINGULARITIES OF CORANK AND MODALITY UP TO TWO

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Abstract. In (Arnold et al., 1985), Arnold has obtained normal forms and has developed a classifier for, in particular, all isolated hypersurface singularities over the complex numbers up to modality 2. Building on a series of 105 theorems, this classifier determines the type of the given singularity. However, for positive modality, this does not fix the right equivalence class of the singularity, since the values of the moduli parameters are not specified. In this paper, we present a simple classification algorithm for isolated hypersurface singularities of corank ≤ 2 and modality ≤ 2. For a singularity given by a polynomial over the rationals, the algorithm determines its right equivalence class by specifying a polynomial representative in Arnold's list of normal forms.

1. Introduction

In his classical paper on singularities (Arnold, 1974), Arnold has classified all isolated hypersurface singularities over the complex numbers with modality ≤ 2. He has given normal forms in the sense of polynomial families with moduli parameters such that every stable equivalence class of function germs contains at least one (but only finitely many) elements of these families. We refer to such elements as normal form equations. Two germs are stably equivalent if they are right equivalent after the direct addition of a non-degenerate quadratic form. Two function germs $f, g \in \mathfrak{m}^2 \subset \mathbb{C}[[x_1, \ldots, x_n]]$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, are right equivalent, written $f \sim g$, if there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathbb{C}[[x_1, \ldots, x_n]]$ such that $\phi(f) = g$. Using the Splitting Lemma, any germ with an isolated singularity at the origin can be written, after choosing a suitable coordinate system, as the sum of two functions of which the variables are disjoint. One function that is called the non-degenerate part, is a non-degenerate quadratic form, and the other part, called the residual part is in $\mathfrak{m}^3$. The Splitting Lemma is implemented in Singular as part of the library `classify.lib` (Krüger, 1997).

In (Arnold et al., 1985), Arnold has made this classification explicit by describing an algorithmic classifier, which is based on a series of 105 theorems. This approach determines the type of the singularity in the sense of its normal form. However, the values of the moduli parameters are not determined, that is, no normal form equation is given. Arnold's classifier is implemented in `classify.lib`.

Classification of complex singularities has a multitude of practical and theoretical applications. The classification of real singularities in (Marais and Steenpaß, 2015a, 2016; Böhm, Marais and Steenpaß, 2015b) is based on determining the complex type of the singularity.

In this paper, we develop a determinator for complex singularities of modality ≤ 2 and corank ≤ 2, which computes, for a given rational input polynomial, a normal form equation in its equivalence class. For germs with non-degenerate Newton boundary, our determinator is based on a simple and uniform approach, which does not require a case-by-case analysis (except for some trivial final steps to read off the values of the moduli parameters according to Arnold's choice of the normal form). Two series of cases with degenerate Newton boundary are handled with more specific methods. Here, we use results of (Luengo and Pfister, 1990) to compute a normal form. In this way, we obtain an approach which does not only determine the moduli

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parameters, but also allows for an elegant implementation. We have implemented our algorithm in the Singular-library `classify2.lib` (Böhm, Marais and Pfister, 2016).

It is important to note that two different normal form equations do not necessarily represent two different right equivalence classes. In (Marais and Steenpaß, 2016) the complete structure of the equivalence classes for, in particular, complex singularities of modality 1 and corank 2 is determined, in the sense that all equivalences between normal form equations are described. All normal form equations in the right equivalence class of a given unimodal corank 2 singularity can, hence, be determined by combining our classifier with the results in (Marais and Steenpaß, 2016). There is not yet a similar complete description of the structure of the equivalence classes of bimodal singularities.

This paper is structured as follows: In Section 2, we give the fundamental definitions and provide the prerequisites on singularities and their classification. In Section 3, we develop a general algorithm for the classification of complex singularities of modality ≤ 2 and corank ≤ 2. Essentially, the algorithm is structured into a subalgorithm for elimination below the Newton polygon, and a subalgorithm for elimination on and above the Newton polygon, which also determines the values of the moduli parameters. The algorithm for the two series of germs of modality 2 with degenerate Newton boundary is discussed in Section 4.

2. Definitions and Preliminary Results

In this section we give some basic definitions and results, as well as some notation that will be used throughout the paper.

Definition 1. *Let $K \subset \mathbb{C}[[x_1, \ldots, x_n]]$ be a union of equivalence classes with respect to the relation $\sim$. A* **normal form** *for K is given by a smooth map*

$$\Phi : B \longrightarrow \mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}[[x_1, \ldots, x_n]]$$

of a finite-dimensional $\mathbb{C}$-linear space B into the space of polynomials for which the following three conditions hold:

(1) $\Phi(B)$ intersects all equivalence classes of K,
(2) the inverse image in B of each equivalence class is finite,
(3) $\Phi^{-1}(\Phi(B) \setminus K)$ is contained in a proper hypersurface in B.

The elements of the image of Φ are called **normal form equations**.

Remark 2. Arnold has chosen a normal form for each of the corank 2 singularities of modality ≤ 2. He has also associated a type to each normal form, see Table 1. We denote the normal form corresponding to the type T by $\mathrm{NF}(T)$. For $b \in \mathrm{par}(\mathrm{NF}(T)) := \Phi^{-1}(K)$ with K as in Definition 1, we write $\mathrm{NF}(T)(b) := \Phi(b)$ for the corresponding normal form equation.

In the following, we give a short account on weighted jets, filtrations, and Newton polygons. See (Arnold, 1974) and (de Jong and Pfister, 2000) for more details.

Remark 3. Let $w = (c_1, \ldots, c_n) \in \mathbb{N}^n$ be a weight on the variables $(x_1, \ldots, x_n)$. The weighted degree on $\mathrm{Mon}(x_1, \ldots, x_n)$ is given by $w\text{-}\deg(\prod_{i=1}^n x_i^{s_i}) := \sum_{i=1}^n c_i s_i$. If the weight of all variables is equal to 1, we refer to the weighted degree of a monomial m as the standard degree of m and write $\deg(m)$ for $w\text{-}\deg(m)$.

Definition 4. *Let $w = (w_1, \ldots, w_s) \in (\mathbb{N}^n)^s$ be a finite family of weights on the variables $(x_1, \ldots, x_n)$. For any monomial (or term) $m \in \mathbb{C}[x_1, \ldots, x_n]$, we define the* **piecewise weight** *with respect to w as*

$$w\text{-}\deg(m) \quad := \quad \min_{i=1,\ldots,s} w_i\text{-}\deg(m).$$

A polynomial f is called **piecewise homogeneous** *of degree d with respect to w if $w\text{-}\deg(t) = d$ for any term t of f.*

TABLE 1. Normal forms of singularities of modality ≤ 2 and corank ≤ 2 as given in Arnold et al. (1985)

		Complex normal form	Restrictions			Complex normal form	Restrictions
Simple	A_k	x^{k+1}	$k \geq 1$	Bimodal	$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + cxy^7$	$4b^3 + 27 \neq 0$
	D_k	$x^2y + y^{k-1}$	$k \geq 4$		$J_{3,p}$	$x^3 + x^2y^3 + \mathbf{a}y^{9+p}$	$p > 0,\ a_0 \neq 0$
	E_6	$x^3 + y^4$	-		$Z_{1,0}$	$x^3y + dx^2y^3 + cxy^6 + y^7$	$4d^3 + 27 \neq 0$
	E_7	$x^3 + xy^3$	-		$Z_{1,p}$	$x^3y + x^2y^3 + \mathbf{a}y^{7+p}$	$p > 0,\ a_0 \neq 0$
	E_8	$x^3 + y^5$	-		$W_{1,0}$	$x^4 + \mathbf{a}x^2y^3 + y^6$	$a_0^2 \neq 4$
	X_9	$x^4 + ax^2y^2 + y^4$	$a^2 \neq 4$		$W_{1,p}$	$x^4 + x^2y^3 + \mathbf{a}y^{6+p}$	$p > 0,\ a_0 \neq 0$
	J_{10}	$x^3 + ax^2y^2 + y^6$	$4a^3 + 27 \neq 0$		$W^\sharp_{1,2q-1}$	$(x^2 + y^3)^2 + \mathbf{a}xy^{4+q}$	$q > 0,\ a_0 \neq 0$
Unimodal	J_{10+k}	$x^3 + x^2y^2 + ay^{6+k}$	$a \neq 0,\ k > 0$		$W^\sharp_{1,2q}$	$(x^2 + y^3)^2 + \mathbf{a}x^2y^{3+q}$	$q > 0,\ a_0 \neq 0$
	X_{9+k}	$x^4 + x^2y^2 + ay^{4+k}$	$a \neq 0,\ k > 0$		E_{18}	$x^3 + y^{10} + \mathbf{a}xy^7$	-
	$Y_{r,s}$	$x^r + ax^2y^2 + y^s$	$a \neq 0,\ r, s > 4$		E_{19}	$x^3 + xy^7 + \mathbf{a}y^{11}$	-
	E_{12}	$x^3 + y^7 + axy^5$	-		E_{20}	$x^3 + y^{11} + \mathbf{a}xy^8$	-
	E_{13}	$x^3 + xy^5 + ay^8$	-		Z_{17}	$x^3y + y^8 + \mathbf{a}xy^6$	-
	E_{14}	$x^3 + y^8 + axy^6$	-		Z_{18}	$x^3y + xy^6 + \mathbf{a}y^9$	-
	Z_{11}	$x^3y + y^5 + axy^4$	-		Z_{19}	$x^3y + y^9 + \mathbf{a}xy^7$	-
	Z_{12}	$x^3y + xy^4 + ax^2y^3$	-		W_{17}	$x^4 + xy^5 + \mathbf{a}y^7$	-
	Z_{13}	$x^3y + y^6 + axy^5$	-		W_{18}	$x^4 + y^7 + \mathbf{a}x^2y^4$	-
	W_{12}	$x^4 + y^5 + ax^2y^3$	-				
	W_{13}	$x^4 + xy^4 + ay^6$	-			where $\mathbf{a} = a_0 + a_1y$	

Definition 5. *Let w be a (piecewise) weight on* $\mathrm{Mon}(x_1, \ldots, x_n)$.

(1) *Let $f = \sum_{i=0}^{\infty} f_i$ be the decomposition of $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ into weighted homogeneous summands f_i of w-degree i. The* **weighted j-jet** *of f is*

$$w\text{-}\mathrm{jet}(f, j) := \sum_{i=0}^{j} f_i\,.$$

(2) *A power series in $\mathbb{C}[[x_1, \ldots, x_n]]$ has* **filtration** *$d \in \mathbb{N}$ if all its monomials are of weighted degree d or higher. The power series of filtration d form a sub-vector space*

$$E^w_d \subset \mathbb{C}[[x_1, \ldots, x_n]]\,.$$

(3) *A power series $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ is a* **germ with non-degenerate Newton boundary** *if f has filtration $d \in \mathbb{N}$ with respect to w and if the piecewise homogeneous function $w\text{-}\mathrm{jet}(f, d)$, called the* **principal part** *of f, is non-degenerate, that is, if its Milnor number is finite. If w consists out of a single weight, we call f* **semi-quasihomogeneous** *and $w\text{-}\mathrm{jet}(f, d)$ the* **quasihomogeneous part** *of f.*

(4) *A power series $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ is* **weighted k-determined** *with respect to the weight w if*

$$f \sim w\text{-}\mathrm{jet}(f, k) + g \qquad \text{for all } g \in E^w_{k+1}.$$

We define the **weighted determinacy** *of f as the minimum number k such that f is k-determined.*

Remark 6. (1) If for a given type T, $w\text{-}\mathrm{jet}(\mathrm{NF}(T)(b), j)$ is independent of $b \in \mathrm{par}(\mathrm{NF}(T))$, we denote it by $w\text{-}\mathrm{jet}(T, j)$.

(2) If the weight of each variable is 1, we write E_d and $\mathrm{jet}(f, j)$ instead of E^w_d and $w\text{-}\mathrm{jet}(f, j)$, respectively.

There is also a similar concept for jets and filtrations of coordinate transformations:

Definition 7. *Let ϕ be a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[[x_1,\dots,x_n]]$ and let w be a weight on* $\mathrm{Mon}(x_1,\dots,x_n)$*.*

(1) For $j>0$ we define $w\text{-}\mathrm{jet}(\phi,j) := \phi_j^w$ as the automorphism given by

$$\phi_j^w(x_i) := w\text{-}\mathrm{jet}(\phi(x_i), w\text{-}\deg(x_i)+j) \quad \text{for all } i=1,\dots,n\,.$$

If the weight of each variable is equal to 1, that is, $w=(1,\dots,1)$, we write ϕ_j for ϕ_j^w.

(2) ϕ has filtration d if, for all $\lambda\in\mathbb{N}$,

$$(\phi-\mathrm{id})E_\lambda^w \subset E_{\lambda+d}^w\,.$$

Remark 8. Note that $\phi_0(x_i)=\mathrm{jet}(\phi(x_i),1)$ for all $i=1,\dots,n$. Furthermore note that ϕ_0^w has filtration ≤ 0, and that, for $j>0$, ϕ_j^w has filtration j if $\phi_{j-1}^w=\mathrm{id}$.

The following definition gives an infinitesimal analogue of the above definition.

Definition 9. *A formal vector field $\mathbf{v}=\sum_i v_i\frac{\partial}{\partial x_i}$ has filtration d with respect to a weight w, if the directional derivative of $\mathbf{v}$ raises the filtration by not less than d, that is,*

$$\text{for all } g\in E_\delta^w,\quad L_{\mathbf{v}}(g) := \sum_i v_i\frac{\partial g}{\partial x_i}\in E_{\delta+d}^w.$$

In a similar way as (Marais and Steenpaß, 2015a, Proposition 8), one can prove:

Proposition 10. *Let $f,g\in\mathbb{C}[[x_1,\dots,x_n]]$ be two power series with $f\sim g$. Let $w\in\mathbb{N}^n$ and suppose that the maximal weighted filtration of f with respect to w is k. Furthermore, let ϕ be a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[[x_1,\dots,x_n]]$ such that $\phi(f)=g$. If $\mathrm{jet}(f,k)$ factorizes as*

$$w\text{-}\mathrm{jet}(f,k) = f_1^{s_1}\cdots f_t^{s_t}$$

in $\mathbb{C}[x_1,\dots,x_n]$, then $w\text{-}\mathrm{jet}(g,k)$ factorizes as

$$w\text{-}\mathrm{jet}(g,k) = \phi_0^w(f_1)^{s_1}\cdots\phi_0^w(f_t)^{s_t}\,.$$

Definition 11. *Let $f=\sum_{i,j}a_{i,j}x^iy^j\in\mathbb{C}[[x,y]]$, let T be a corank 2 singularity type. We call*

$$\begin{aligned}\mathrm{supp}(f) &:= \{x^iy^j \mid a_{i,j}\neq 0\}\\ \mathrm{supp}(T) &:= \mathrm{supp}(\mathrm{NF}(T)(b))\end{aligned}$$

where $b\in\mathrm{par}(\mathrm{NF}(T))$ is generic, the ***support*** *of f and of T, respectively. Let*

$$\begin{aligned}\Gamma_+(f) &:= \bigcup_{x^iy^j\in\mathrm{supp}(f)}((i,j)+\mathbb{R}_+^2)\\ \Gamma_+(T) &:= \bigcup_{x^iy^j\in\mathrm{supp}(T)}((i,j)+\mathbb{R}_+^2)\end{aligned}$$

and let $\Gamma(f)$ and $\Gamma(T)$ be the boundaries in $\mathbb{R}^2$ of the convex hulls of $\Gamma_+(f)$ and $\Gamma_+(T)$, respectively. Then:

(1) $\Gamma(f)$ and $\Gamma(T)$ are called the **Newton polygons** *of f and T, respectively.*

(2) The compact segments of $\Gamma(f)$ or $\Gamma(T)$ are called **faces**. *If Δ is a face, then the set of monomials of f lying on Δ is denoted by $\mathrm{supp}(f,\Delta)$ and the sum of the terms lying on Δ by $\mathrm{jet}(f,\Delta)$. Moreover, we write $\mathrm{supp}(\Delta)$ for the set of monomials corresponding to the lattice points of Δ, and set $\mathrm{supp}(T,\Delta) := \mathrm{supp}(T)\cap\mathrm{supp}(\Delta)$. We use the same notation for a set of faces, considering the monomials lying on the union of the faces.*

(3) Any face Δ induces a weight $w(\Delta)$ on $\mathrm{Mon}(x,y)$ in the following way: If Δ has slope $-\frac{w_x}{w_y}$, in lowest terms, and $w_x,w_y>0$, we set $w(\Delta)\text{-}\deg(x)=w_x$ and $w(\Delta)\text{-}\deg(y)=w_y$.

(4) If $w_1,\dots,w_s$ are the weights associated to the faces of $\Gamma(f)$, respectively $\Gamma(T)$, ordered by increasing slope, there are unique minimal integers $\lambda_1,\dots,\lambda_s\geq 1$ such that the piecewise weight associated to $w(f)=(\lambda_1w_1,\dots,\lambda_sw_s)$ by Definition 4 is constant on $\Gamma(f)$, respectively $\Gamma(T)$. We denote this constant by $d(f)$, respectively $d(T)$.

(5) Let Δ_i and Δ_j be faces with weights w_1 and w_2, respectively, and let w be the piecewise weight defined by w_1 and w_2. Let d be the w-degree of the monomials on Δ_1 and Δ_2. Then $\mathrm{span}(\Delta_1,\Delta_2)$ is the Newton polygon associated to the sum of all monomials of w-degree d.

(6) A monomial m lies strictly underneath, on or above $\Gamma(f)$, if the $w(f)$-degree of m is less than, equal to or greater than $d(f)$, respectively. We use this notation also with respect to $\Gamma(T)$, $w(T)$, and $d(T)$.

Notation 12. Given $f \in \mathbb{C}[[x,y]]$ and $m \in \operatorname{Mon}(x,y)$, we write $\operatorname{coeff}(f,m)$ for the coefficient of m in f.

Definition 13. *The* **Jacobian ideal** $\operatorname{Jac}(f) \subset \mathbb{C}[[x,y]]$ *of f is generated by the partial derivatives of f. The* **local algebra** *of f is the residue class ring of the Jacobian ideal of f.*

Definition 14. *Suppose f is a germ, $e_1,\dots,e_\mu$ are monomials representing a basis of the local algebra of f, and $e_1,\dots,e_s$ are the monomials in this basis above or on $\Gamma(f)$. We then call $e_1,\dots,e_s$ a* **system** *of the local algebra of f.*

Lemma 15 (Arnold (1974), Corollary 3.3)**.** *Let f be a semi-quasihomogeneous function with quasihomogeneous part f_0, and let $e_1,\dots,e_\mu$ be monomials representing a basis of the local algebra of f_0. Then $e_1,\dots,e_\mu$ also represent a basis of the local algebra of f.*

Theorem 16 (Arnold (1974), Theorem 7.2)**.** *Let f be a semi-quasihomogeneous function with quasihomogeneous part f_0 and let $e_1,\dots,e_s$ be a system of the local algebra of f. Then f is equivalent to a function of the form $f_0 + \sum_{k=1}^{s} c_k e_k$ with constants c_k.*

In Arnold (1974), the following results are used for the classification of singularities of corank 2.

Definition 17. *A piecewise homogeneous function f_0 of degree d satisfies* ***Condition A***, *if for every function g of filtration $d+\delta > d$ in the ideal spanned by the derivatives of f_0, there is a decomposition*

$$g = \sum \frac{\partial f_0}{\partial x_i} v_i + g',$$

where the vector field v has filtration δ and g' has filtration bigger than $d+\delta$.

Note that quasihomogeneous functions satisfy Condition A. Using (Arnold, 1974, Theorem 9.5), and taking into account that all cases under consideration in the following theorem satisfy Condition A, we obtain:

Theorem 18. *Suppose f is a function of corank 2 with non-degenerate Newton boundary such that the principal part f_0 of f coincides with the principal part of one of Arnold's normal forms of modality ≤ 2. Let $e_1,\dots,e_s$ be a system of the local algebra of f. Then f is equivalent to a function of the form $f_0 + \sum c_k e_k$ with constants c_k.*

Following Arnold's proof of Theorem 16, Theorem 18 can be proven by iteratively applying the following lemma.

Lemma 19. *Let $f_0 \in \mathbb{C}[[x_1,\dots,x_n]]$ be a piecewise homogeneous function of weighted w-degree d_w that satisfies Condition A, and let $e_1,\dots,e_r$ be the monomials of a given w-degree $d' > d_w$ in a system of the local algebra of f_0. Then, for every series of the form f_0+f_1, where the filtration of f_1 is greater than d_w, we have*

$$f_0 + f_1 \sim f_0 + f_1',$$

where the terms in f_1' of degree less than d' are the same as in f_1, and the part of degree d' can be written as $c_1e_1+\cdots+c_re_r$ with $c_i \in \mathbb{C}$.

Proof. Let $g(\mathbf{x})$ denote the sum of the terms of degree d' in f_1. There exists a decomposition of g of the form

$$g(\mathbf{x}) = \sum_i \frac{\partial f_0}{\partial x_i} v_i(\mathbf{x}) + c_1e_1 + \cdots + c_re_r, \qquad v_i \in \mathbb{C}[[x_1,\dots,x_n]],$$

since $e_1,\dots,e_r$ form a monomial vector space basis for the local algebra of f_0. Let $d(x_i)$ be the w-degree of x_i, and let $v_i' := w\text{-}\operatorname{jet}(v_i, d(x_i))$. Then

$$g(\mathbf{x}) = \sum_i \frac{\partial f_0}{\partial x_i} v_i'(\mathbf{x}) + c_1e_1 + \cdots + c_re_r - g'(\mathbf{x}),$$

where $g'(\mathbf{x})$ has filtration greater than d'. Applying the transformation defined by

$$x_i \mapsto x_i - v_i'(\mathbf{x})$$

to f, we transform f to

$$f_0(\mathbf{x}) + (f_1(\mathbf{x}) + (c_1 e_1(\mathbf{x}) + \cdots + c_r e_r(\mathbf{x}) - g(\mathbf{x})) + R(\mathbf{x}),$$

where the filtration of R is greater than d'. □

Remark 20. A system of the local algebra is in general not unique. For his lists of normal forms of hypersurface singularities with non-degenerate Newton boundary, Arnold has chosen in each case (in particular) a specific system of the local algebra. In the rest of the paper, we call these systems the **Arnold systems**.

Remark 21. Note that it follows from Lemma 19 that all hypersurface singularities of corank ≤ 2 and modality ≤ 2 with non-degenerate Newton boundary are finitely weighted determined. Moreover, we explicitely obtain the weighted determinacy for every such singularity.

3. A Classification Algorithm for Corank 2 Complex Simple, Unimodal and Bimodal Singularities

We now describe an algorithm to determine an Arnold normal form equation for a given input polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{Q}[x, y]$ of modality ≤ 2. In this section, we limit our discussion on functions with a normal form with non-degenerate Newton boundary. In the case of normal forms with degenerate Newton boundary, our algorithm will resort to special algorithms described in Section 4. Figures 1 to 4 illustrate the modality 2 types of this kind. The figures show in the gray shaded area all monomials which can possibly occur in a polynomial f of the given type T. The faces of the Newton polygon $\Gamma(T)$ are shown in blue. The dots with a thick black circle indicate the moduli monomials in the Arnold system. Red dots indicate monomials which are not in $\mathrm{Jac}(f)$. Monomials occuring in any normal form equation with non-zero coefficients are shown as blue dots.

The structure of our algorithm consists out of two basic steps, see Algorithm 1. We first determine the complex type of f by removing all the monomials underneath $\Gamma(T)$, in the semi-quasihomogeneous cases, and all the monomials underneath and on $\Gamma(T)$, not in $\mathrm{NF}(T)$, in the other cases (Algorithm 2). After that, we determine a normal form equation of f (using Algorithm 5 in the non-simple cases). More generally, we will formulate the algorithm in a way, that it is applicable to any $f \in \mathfrak{m}^2$, and will recognize if f is of modality > 2, returning an error in this case.

Algorithm 1 Algorithm to classify singularities of modality ≤ 2 corank ≤ 2

Input: A polynomial germ $f \in \mathfrak{m}^2$ over the rationals.
Output: $\mathrm{NF}(f)$ as well as the values of all moduli parameters occuring in a normal form equations that is equivalent to f, if f is of modality ≤ 2, corank ≤ 2; `false` otherwise.

Apply Algorithm 2 to f.
if T as returned by Algorithm 2 is a simple type **then**
  return $(\mathrm{NF}(T), ())$
Apply Algorithm 5 to the output of Algorithm 2 and return the result.

We first discuss Algorithm 2. If f is of corank ≤ 1, then f is of type A_k, where $k = \mu(f)$. Suppose now that f is of corank 2. Determining T in the process, we remove all monomials below $\Gamma(T)$ if $\Gamma(T)$ has only one face, and all monomials on or below $\Gamma(T)$ which are not in $\mathrm{NF}(T)$, if $\Gamma(T)$ has two faces. Let d be the maximal filtration of f. If f is of type X_9, nothing has to be done. Note that f is of type X_9 if and only if the d-jet of f has 4 different roots over the complex numbers. If f is not of type X_9, then Algorithm 3 will transform f such that $\mathrm{supp}(T, d) = \mathrm{supp}(\mathrm{jet}(f, d))$. Using (Marais and Steenpaß, 2015a, Proposition 8), we find the corresponding linear transformation by factorizing $\mathrm{jet}(f, d)$.

At this stage we know that $\operatorname{supp}(\operatorname{jet}(f,d)) \subset \operatorname{supp}(\operatorname{NF}(T))$. We store the monomials of the d-jet of f in $S_0 = \operatorname{supp}(\operatorname{jet}(f,d))$. The remainder of Algorithm 2 will proceed in an iterative way, changing f and S_0 in the process: In each step of the iteration, we can have one of the following two possibilities for $\Gamma(f)$:

(1) Note that monomials of the form $x^{n_1}y$ or xy^{n_2} cannot be intersection points of (finite) faces of $\Gamma(T)$. If any of the monomials $m_0 \in S_0$ which is not of this form lies on two faces of $\Gamma(f)$, it is clear that $\Gamma(T)$ has at least two faces with corner point m_0. The algorithm will then stay in this case. Let Δ_i and Δ_j be the two different faces of $\Gamma(f)$ on which m_0 lies. The corner point in all modality 1 and 2 cases with a Newton polygon with two faces is either x^2y^2 or x^2y^3. It follows that if $m_0 \neq x^2y^t$, $t = 2$ or $t = 3$, then f is not of modality ≤ 2. Otherwise, using the shape of $\Gamma_0 := \operatorname{span}(\Delta_i, \Delta_j)$ and the fact that $m_0 = x^2y^t$ is a corner point of Γ_0, all monomials in f on Γ_0 of the form xy^n or x^ny^{t-1} can be removed iteratively, by increasing degree, each time replacing the corresponding terms of the given degree by higher $w(f)$-degree terms using Algorithm 4. After each iteration, f, Δ_i, Δ_j and Γ_0 are recalculated. In each iteration, there will either be no terms of the considered form on Γ_0, in which case the process stops, or the number of equivalence classes in the local algebra of f represented by powers of x or y underneath Γ_0 strictly increases, except possibly in the last two steps of the process (where monomials on the final Newton polygon may be removed). Note that, if x^{m_1} and y^{m_2} are largest powers of x and y underneath Γ_0, then $1, x, \ldots, x^{m_1-1}, y, \ldots, y^{m_2-1}$ represent different equivalence classes. Since $\mu(f)$ is finite, the process must stop after finitely many iterations. No further monomials on Γ_0 can be removed without creating terms underneath Γ_0. Hence, in all cases in consideration, this algorithm will produce the Newton polygon of the normal form. In fact, if $\operatorname{supp}(f, \Gamma_0)$ does not coincide with $\operatorname{supp}(T, \Gamma_0)$ for some type T of modality ≤ 2, then the modality of f is bigger than 2. Otherwise, f is a germ of the corresponding type T, and all monomials in f underneath or on $\Gamma(T)$ not in $\operatorname{NF}(T)$ are removed.

(2) Suppose no monomials in S_0, except monomials of the form $x^{n_1}y$ or xy^{n_2}, lie on two faces of $\Gamma(f)$. Then f is not of type X_{9+k} or $Y_{r,s}$, since these cases will be recognized to have two faces in the first iteration of the above step. All the monomials in S_0 lie on only one face of $\Gamma(f)$. Let Δ be this face. If $f_1 := \operatorname{jet}(f, \Delta)$ is non-degenerate, then f is a semi-quasihomogeneous germ. Since $w\text{-}\operatorname{jet}(\phi_0^w(f), d(f)) = \phi_0^w(f_1)$ for any automorphism ϕ of filtration ≥ 0 with respect to the weight w associated to Δ, $\operatorname{span}(\Delta)$ is an invariant of the type of f. The corresponding type T can, hence, be identified. The case X_9 will already be recognized as a semi-quasihomogeneous function in the first iteration, and f will be returned by the algorithm without any change. In all other cases, the weight w associated with Δ will be such that $w\text{-}\deg(x) > w\text{-}\deg(y)$. If f_1 is degenerate, then either f has monomials underneath $\Gamma(T)$, or $\Gamma(T)$ is degenerate. For all semi-quasihomogeneous cases of modality ≤ 2, except X_9, $\operatorname{jet}(T, d)$ is divisible by a power of x and x has the highest multiplicity among all prime factors. Any weighted jet of $\operatorname{NF}(T)$ with respect to a face lying below $\Gamma(T)$ and intersecting $\Gamma(T)$ in $\operatorname{jet}(T, d)$ has the same property. Suppose Δ is such a face. Then $\operatorname{supp}(T, \Delta) = \{x^ny^m\}$ with $n > m$. Taking into account that the weighted degree of x is greater than the weighted degree of y, it follows that $f_1 = g_1^n y^m$ with $\deg_x(g_1) = 1$. The right equivalence $g_1 \mapsto x$, $y \mapsto y$ transforms f such that $\operatorname{supp}(f, \Delta) = \operatorname{supp}(T, \Delta)$. If the normal form of f has a non-degenerate Newton boundary, but is not semi-quasihomogeneous, then we can proceed in the same way: Suppose Δ lies underneath or on the face of biggest slope of $\Gamma(T)$. If g_1 is the factor of highest multiplicity of f_1 with $\deg_x(g_1) = 1$, then the right equivalence $g_1 \mapsto x$, $y \mapsto y$ transforms f such that $\operatorname{supp}(f, \Delta) = \operatorname{supp}(T, \Delta)$. We then update $S_0 := \operatorname{supp}(f, \Delta)$ and pass to the next iteration. If f_1 does not have any x-linear factor, then the normal form of f has a degenerate Newton boundary. In this case, we resort to the algorithms described in Section 4. Since $\mu(f)$ is finite, the same argument as in (1) shows that the iteration terminates after finitely many steps.

We now discuss Algorithm 5, which determines the values of the moduli parameters. Let $w = w(T)$ be the weight associated to $\Gamma(T)$. If $\Gamma(T)$ has only one face Δ, then $\operatorname{supp}(f, \Delta)$ is not

Algorithm 2 Determine the complex type of a corank ≤ 2 singularity of modality ≤ 2 with non-degenerate Newton boundary.

Input: A polynomial germ $f \in \mathfrak{m}^2$ over the rationals.
Output: If f is of modality ≤ 2 and corank ≤ 2, then the complex singularity type T of f, and a polynomial g right equivalent to f such that the span of the faces of $\Gamma(T)$ and the faces of $\Gamma(f)$ coincide; `false` otherwise.

f := residual part given by the splitting lemma applied to f, as implemented in `classify.lib`.
if corank$(f) \leq 1$ **then**
  return $(f, A_{\mu(f)})$
if corank$(f) > 2$ **then**
  return `false`
if $f \in E_5$ **then**
  return `false` (modality > 2)
f:= output of Algorithm 3 applied to $f \in \mathbb{Q}[x, y]$
$S_0 := \text{supp}(\text{jet}(f, d))$, where $d :=$ maximal filtration of f w.r.t. the standard grading.
while true do
  Let $\Delta_1, \ldots, \Delta_n$ be the faces of $\Gamma(f)$ ordered by increasing slope.
  if exist $i \neq j$ and $n_1, n_2 > 1$ with $m_0 := x^{n_1}y^{n_2} \in \text{supp}(\Delta_i) \cap \text{supp}(\Delta_j) \subset S_0$ **then**
    if $m_0 \neq x^2y^2$ and $m_0 \neq x^2y^3$ **then**
      return `false` (modality > 2)
    $\Gamma_0 := \text{span}(\Delta_i, \Delta_j)$
    $f_1 := \text{jet}(f, \Gamma_0)$
    while exist a term of the form $t = c \cdot x^{n_1-1}y^r$ or $t = c \cdot x^r y^{n_2-1}$ in f_1 **do**
      f:= output of Algorithm 4 with input f, f_1, t, and weights $w(\Delta_i), w(\Delta_j)$
      Let $\Delta_1, \ldots, \Delta_n$ be the faces of $\Gamma(f)$.
      $\Gamma_0 := \text{span}(\Delta_i, \Delta_j)$, where i and j are such that $m_0 \in \text{supp}(\Delta_i) \cap \text{supp}(\Delta_j)$
      $f_1 := \text{jet}(f, \Gamma_0)$
    if exists modal 1 or 2 type T with $\text{supp}(T, \Gamma_0) = \text{supp}(f_1)$ **then**
      return (f, T)
    else
      return `false` (modality > 2)
  else
    Let Δ be the face of $\Gamma(f)$ of smallest slope such that $S_0 \subset \text{supp}(\Delta)$.
    $f_1 := \text{jet}(\Delta, f)$
    if $\mu(f_1) = \infty$ **then**
      Let g_1 be the factor of f_1 with highest multiplicity.
      if $\deg_x(g_1) = 1$ **then**
        Replace f by $g_1 \mapsto x$, $y \mapsto y$ applied to f.
        $S_0 := \text{supp}(\text{jet}(f, \Delta))$
      else
        if $\text{supp}(f, \Delta) = \text{supp}((y^2 - x^3)^2)$ **then**
          return $(f, W^\sharp_{1,\mu(f)-15})$
        else
          return `false` (modality > 2)
    else
      if exists modal 1 or 2 type T with $\Gamma(T) = \Gamma(f_1)$ **then**
        return (f, T)
      else
        return `false` (modality > 2)

Algorithm 3 Reverse linear jet.

Input: A polynomial $f \in \mathfrak{m}^3 \subset \mathbb{Q}[x, y]$ with $\mathrm{jet}(f, 4) \neq 0$.

Output: $g \in \mathfrak{m}^3 \subset \mathbb{K}[x, y]$, where $\mathbb{K}$ is an algebraic extension field of $\mathbb{Q}$, such that $g \sim f$, and, in case f is of type $T \neq X_9$ of modality ≤ 2, then $\mathrm{supp}(\mathrm{jet}(g, d)) = \mathrm{supp}(T, d)$ where d is the maximal filtration of f w.r.t. the standard grading.

Factorize $\mathrm{jet}(f, d) = c g_1^{\alpha} g_2^{\beta} g_3^{\gamma} g_4^{\delta}$ over $\mathbb{C}$, where $0 \neq c \in \mathbb{Q}$, g_1, g_2, g_3 and g_4 are monic in x and pairwise coprime, and $4 \geq \alpha \geq \beta \geq \gamma \geq \delta \geq 0$.
if $\beta, \gamma, \delta = 0$ **then**
  if $g_1 \neq c'y$, $c' \in \mathbb{Q}$ **then**
    Replace f with $g_1 \mapsto x$, $y \mapsto y$ applied to f.
  else
    Replace f with $x \mapsto y$, $y \mapsto x$ applied to f.
if $\gamma, \delta = 0$ **then**
  Replace f with $g_1 \mapsto x$ and $g_2 \mapsto y$ applied to f.
if $\alpha = 2$ and $\beta, \gamma = 1$ and $\delta = 0$ **then**
  if $g_1 \neq c'y$, $c' \in \mathbb{Q}$ **then**
    Replace f with $g_1 \mapsto x$, $y \mapsto y$ applied to f.
  else
    Replace f with $x \mapsto y$, $y \mapsto x$ applied to f.
  Write $f = a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + R$, $a_0, a_1 \in \mathbb{Q}$, $a_2 \in \mathbb{Q}^{\times}$ and $R \in E_5$.
  Replace f with $y \mapsto y - \frac{a_1}{2a_2} x$, $x \mapsto x$ applied to f.
return f

Algorithm 4 Remove term via partials.

Input: $f, f_0 \in \mathbb{K}[x, y]$ over a field $\mathbb{K}$, with t a term of f, and weights $u_1, u_2 \in \mathbb{Z}^2$.

Output: $g \in \mathbb{K}[x, y]$ such that $f \sim g$. If called with input as in Algorithms 2 or 5, then $f = g + t +$ terms of higher (u_1, u_2)-degree than t.

$m_x :=$ the sum of the terms of $\frac{\partial f_0}{\partial x}$ of lowest u_2-degree
$m_{x,y} :=$ the term of m_x of lowest u_1-degree
$m_y :=$ the sum of the terms of $\frac{\partial f_0}{\partial y}$ of lowest u_1-degree
$m_{y,x} :=$ the term of m_y of lowest u_2-degree
if $m_{x,y} | t$ **then**

$$\begin{array}{rcl} \alpha : \mathbb{K}[x, y] & \to & \mathbb{K}[x, y] \\ x & \mapsto & x - t/m_{x,y} \\ y & \mapsto & y \end{array}$$

  return $\alpha(f)$
if $m_{y,x} | t$ **then**

$$\begin{array}{rcl} \alpha : \mathbb{K}[x, y] & \to & \mathbb{K}[x, y] \\ x & \mapsto & x \\ y & \mapsto & y - t/m_{y,x} \end{array}$$

  return $\alpha(f)$
return f

necessarily equal to supp(NF(T), Δ). We achieve equality by a weighted linear transformation. In the cases where $\Gamma(T)$ has two faces, equality has already been achieved in Algorithm 2. Above $\Gamma(T)$, we then use the method described in the proof of Lemma 19 to reduce f modulo Jac(f): We iteratively apply Algorithm 4 to each term, in the two face case only considering terms in Jac(f), proceeding weighted degree by weighted degree in increasing order (and in each weighted degree according to a total (ordinary) degree ordering). Note that a term is in Jac(f) if and only if it is in Jac(f_0) for $f_0 = \text{jet}(f, \Gamma(T))$. After handling a given weighted degree, if Arnold's system for type T contains a monomial m of this degree, we write the sum of the remaining terms in the form

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2 + cm,$$

where $v_1, v_2 \in \mathbb{C}[x, y]$ are weighted homogeneous, $c \in \mathbb{C}$, and as

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2,$$

otherwise. Applying $x \mapsto x - v_1$, $y \mapsto y - v_2$, results in replacing the sum of the remaining terms by a sum of terms which are either in Arnold's system in the w-degree under consideration, or of higher w-degree. Since f is weighted d'-determined, we stop the iteration when we reach degree $d' + 1$, where d' is the w-degree of the highest w-degree monomial in Arnold's system.

Remark 22. In the semi-quasihomogeous cases, line 11 in Algorithm 5 can be omitted, since the reduction modulo Jac(f) is also handled by lines 15 to 18.

Remark 23. In Algorithm 5, Arnold's system can be replaced by any other choice of a system of the local algebra.

Remark 24. The algebraic extension of $\mathbb{Q}$ introduced for representing the moduli parameters can arise in two steps of the overall algorithm: Reversal of the linear jet in Algorithm 3, and rescaling of the variables at the end of Algorithm 5. Note that the transformation reversing the linear jet is obtained from the factorization $\text{jet}(f, d) = c g_1^{\alpha} g_2^{\beta} g_3^{\gamma} g_4^{\delta}$. Here, a field extension can only occur if $\alpha = \beta = 2$ and $\gamma = \delta = 0$.

4. A Classification Algorithm for Corank 2, Bimodal Singularities with Degenerate Newton Boundary

In this section we give a classification algorithm for the singularities $W^{\sharp}_{1,\mu-15}$, where μ is the Milnor number, in Arnold's list. They have the property that in all coordinate systems the Newton boundary is degenerate, which is the reason that they have to be treated separately. They are of multiplicity 4 and the 4-jet is a 4-th power of a linear homogeneous polynomial. After a suitable automorphism of $\mathbb{C}[[x, y]]$, we may assume that the corresponding polynomial is of the form

$$f = (x^2 + y^3)^2 + \sum_{3i+2j \geq 12+d} w_{ij} x^i y^j \;,\; d \geq 1.$$

This automorphism was already constructed in the previous section. Singularities of this type have been studied in Luengo and Pfister (1990). It is proved that the Milnor number satisfies $\mu(f) \geq 15 + d$, and equality holds if and only if

$$\sum_{3i+2j=12+d} (-1)^{[i/2]} w_{ij} \neq 0.$$

If the Milnor number $\mu(f) = 15 + d$ is even, then the germ of the curve defined by f is irreducible with semi–group $\langle 4, 6, 12 + d \rangle$. In the odd case, the curve has two branches. Let

$$f = (x^2 + y^3)^2 + \sum_{3i+2j>12} w_{ij} x^i y^j$$

and assume $\mu := \mu(f) < \infty$. Let $>$ be the weighted degree reverse lexicographical ordering with respect to the weights $(3, 2)$ on $\mathbb{C}[[x, y]]$ with $x > y$.

Algorithm 5 Determine the moduli parameters of a normal form equation of a corank 2 uni- or bimodal singularities.

Input: $f \in \mathfrak{m}^3 \subset \mathbb{K}[x,y]$, a germ of modality 1 or 2 and corank 2 of type T over an algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, as returned by Algorithm 2. In particular, the set of faces of $\Gamma(T)$ equals the set of faces of $\Gamma(f)$.

Output: The normal form of f, as well as the values of all moduli parameters occuring in a normal form equations that is equivalent to f, specified as elements of an algebraic extension field of $\mathbb{K}$.

```
if T = W♯_{1,μ−15} for some μ then
    return result of Algorithm 6 applied to f
w := w(T) and d := d(T)
if Γ(T) has exactly one face Δ then
    Apply a weighted homogeneous transformation to f such that supp(f,Δ) = supp(T,Δ).
d' := highest w-degree of a monomial in Arnold's system of T
f_0 := w-jet(f,d)
for j = d+1,...,d' do
    for all terms t of f of w-degree j in increasing order by a total degree ordering do
        if Γ(T) has one face then
            f := result of Algorithm 4 with input f, f_0, t and (1,1), (1,1)
        else
            if t ∈ Jac(f_0) then
                f := result of Algorithm 4 with input f, f_0, t and w_2, w_1
    if exists monomial m of w-degree j in Arnold's system then
        Write w-jet(f,j) − w-jet(f,j−1) = ∂f_0/∂x v_1 + ∂f_0/∂y v_2 + cm with c ∈ K, v_1, v_2 ∈ K[x,y]
        weighted homogeneous.
    else
        Write w-jet(f,j) − w-jet(f,j−1) = ∂f_0/∂x v_1 + ∂f_0/∂y v_2 with v_1, v_2 ∈ K[x,y] weighted
        homogeneous.
    Apply x ↦ x − v_1, y ↦ y − v_2 to f.
Delete all terms in f of w-degree > d'.
Apply transformation x ↦ ax, y ↦ by over an algebraic extension of K to transform the
non-parameter terms to the terms of NF(T).
Read off the parameters a_i.
return (NF(T), (a_i))
```

In Luengo and Pfister (1990) it is proved that in case of μ being even the leading ideal of the Jacobian ideal $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ is generated by $x^3, x^2y^2, xy^{\frac{\mu-2}{2}}$. If μ is odd, then the leading ideal is generated by $x^3, x^2y^2, xy^{\frac{\mu-5}{2}}, y^{\frac{\mu+1}{2}}$. We obtain a monomial basis of $\mathbb{C}[[x,y]]/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ as $\{x^i y^i\}_{(i,j)\in B}$ with

$$B = \{(i,j) \mid i \leq 2, j \leq 1\} \cup \left\{(i,j) \,\middle|\, i \leq 1, 2 \leq j \leq \frac{\mu-4}{2}\right\}$$

in case that μ is even and

$$B = \{(i,j) \mid i \leq 2, j \leq 1\} \cup \left\{(1,j) \,\middle|\, 2 \leq j \leq \frac{\mu-7}{2}\right\} \cup \left\{(0,j) \,\middle|\, 2 \leq j \leq \frac{\mu-1}{2}\right\},$$

in case that μ is odd. Let

$$B_1 := \{(1, \frac{\mu-6}{2}), (1, \frac{\mu-4}{2})\}$$

if μ is even and

$$B_1 := \{0, \frac{\mu-3}{2}), (0, \frac{\mu-1}{2})\}$$

if μ is odd.

In Luengo and Pfister (1990), the following theorem is proved.

Theorem 25. *There exists an automorphism φ of $\mathbb{C}[[x,y]]$ such that*

$$\varphi(f) = (x^2+y^3)^2 + \sum_{(i,j)\in B_1} w_{ij}x^iy^j.$$

Note 26. In particular, it follows that these singularities are bimodal.

Remark 27. The normal form given in this way for the case that the Milnor number is odd differs from Arnold's normal form. Instead of $y^{\frac{\mu-3}{2}}$ and $y^{\frac{\mu-1}{2}}$, he used the monomials $x^2y^{\frac{\mu-9}{2}}$ and $x^2y^{\frac{\mu-7}{2}}$. From a computational point of view, our choice is better. It is easy to convert our normal form to Arnold's normal form. See Figure 5, for an illustration of the normal forms (using our choice of parameter monomials).

The construction of the automorphism in the theorem is done separately for each weighted degree: Assume we have already

$$f = (x^2+y^3)^2 + \sum_{3i+2j\geq 12+a} w_{ij}x^iy^j$$

for some a (with Milnor number $\mu = 15+d$). If $a < d$, then we have

$$\sum_{3i+2j=12+a} (-1)^{[i/2]}w_{ij} = 0.$$

This implies that

$$\sum_{3i+2j=12+a} w_{ij}x^iy^j = l\cdot(x^2+y^3).$$

We obtain

$$f = \left(x^2+y^3+\frac{1}{2}l\right)^2 + \sum_{3i+2j>12+a} \widetilde{w}_{ij}x^iy^j$$

for suitable $\widetilde{w}_{ij}\in\mathbb{C}$. Now we can choose an automorphism φ of $\mathbb{C}[[x,y]]$ such that

$$\varphi\left(x^2+y^3+\frac{1}{2}l\right) = x^2+y^3+ \text{ terms of weighted degree } \geq \mu$$

(note that we could even find an automorphism mapping $x^2+y^3+\frac{1}{2}l$ to x^2+y^3). We obtain

$$\varphi(f) = (x^2+y^3)^2 + \sum_{3i+2j>12+a} \overline{w}_{ij}x^iy^j$$

for suitable $\overline{w}_{ij}\in\mathbb{C}$.

If $a = d$ then we have

$$\sum_{3i+2j=12+d} (-1)^{[i/2]}w_{ij} \neq 0.$$

Similarly as before, we can write

$$\sum_{3i+2j=12+d} w_{ij}x^iy^j = w_{i_0j_0}x^{i_0}y^{j_0} + l\cdot(x^2+y^3)$$

with

$$(i_0,j_0) = \begin{cases} (0,\frac{\mu-3}{2}) & \text{if } \mu \text{ is odd} \\ (1,\frac{\mu-6}{2}) & \text{if } \mu \text{ is even.} \end{cases}$$

Since the Milnor number is $15+d$, we obtain $w_{i_0,j_0}\neq 0$. Using a similar automorphism as in the previous case, we may assume with $a_0 := w_{i_0,j_0}$ (the first modulus), that

$$f = (x^2+y^3)^2 + a_0\cdot x^{i_0}y^{j_0} + \sum_{3i+2j>12+d} w_{ij}x^iy^j\ .$$

Note, that $12 + d = \mu - 3$, and we have to compute the normal form of f up to degree $\mu - 1$. Now we can write

$$\sum_{3i+2j=13+d} w_{ij} x^i y^j = e \cdot x^{i_1} y^{j_1} + l \cdot (x^2 + y^3)$$

with

$$(i_1, j_1) = \begin{cases} (1, \frac{\mu-5}{2}) & \text{if } \mu \text{ is odd} \\ (0, \frac{\mu-2}{2}) & \text{if } \mu \text{ is even.} \end{cases}$$

Using an automorphism as before, we may assume that $l = 0$.

If $e = 0$, we are done with weighted degree $\mu - 2$.

If $e \neq 0$ we define an automorphism φ of $\mathbb{C}[[x, y]]$ by the exponential of the vector field

$$\delta = c \cdot (3y^2 \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y})$$

with

$$c = (-1)^{\mu-1} \frac{e}{(\mu - 3)a_0}.$$

Since by construction, $\varphi(x^2 + y^3) = x^2 + y^3$, we obtain

$$\varphi(f) = (x^2 + y^3)^2 + a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i+2j \geq 14+d} \widetilde{w}_{ij} x^i y^j$$

for suitable $\widetilde{w}_{ij} \in \mathbb{C}$.

Remark 28. Note, that for practical purposes, we have to compute φ only up to weighted degree 5, and apply it to $a_0 \cdot x^{i_0} y^{j_0} + \sum_{3i+2j=13+d} w_{ij} x^i y^j$ since we know that $\varphi((x^2 + y^3)^2) = (x^2 + y^3)^2$.

Now let

$$(i_1, j_1) = \begin{cases} (0, \frac{\mu-1}{2}) & \text{if } \mu \text{ is odd} \\ (1, \frac{\mu-4}{2}) & \text{if } \mu \text{ is even} \end{cases}$$

and write

$$\sum_{3i+2j=14+d} \widetilde{w}_{ij} x^i y^j = a_1 \cdot x^{i_1} y^{j_1} + l \cdot (x^2 + y^3).$$

Using an automorphism as in the first case, we may assume $l = 0$, and obtain as normal form

$$(x^2 + y^3)^2 + a_0 x^{i_0} y^{j_0} + a_1 \cdot x^{i_1} y^{j_1}.$$

We summarize the approach in Algorithm 6.

Remark 29. The approach described in Algorithm 5 in case of a non-degenerate Newton boundary can be adapted to also handle the cases $W^\sharp_{1,\mu-15}$. However, this strategy requires more iterations than Algorithm 6. To adapt Algorithm 5, we remove lines 1 and 2, and in line 11, we call Algorithm 7 instead of Algorithm 4 if f is of type $W^\sharp_{1,\mu-15}$.

Note that, in these cases, Algorithm 2 does not require a field extension, hence, Algorithm 7 is called with input defined over $\mathbb{Q}$. Note also, that Algorithm 7 is applicable with any choice of a system B of the local algebra.

References

Arnold, V.I., 1974. Normal forms of functions in neighbourhoods of degenerate critical points. Russ. Math. Surv. 29(2), 10-50.

Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N., 1985. Singularities of Differential Maps, Vol. I. Birkhäuser, Boston.

Böhm, J., Marais, M., Steenpaß, A., 2015b. The classification of real singularities using Singular Part III: Unimodal Singularities of Corank 2. `http://arxiv.org/pdf/1512.09028v1`.

Böhm, J., Marais, M., Pfister, G., 2016. `classify2.lib`. A Singular 4-0-2 library for classifying isolated hypersurface singularities of corank and modality up to 2, and to determine the moduli parameters. Singular distribution.

Algorithm 6 Algorithm to determine parameters for the bimodal singularities of type $W^\sharp_{1,\mu-15}$.

Input: $f = \gamma \cdot (\alpha x^2 + \beta y^3)^2 +$ terms of weighted $(3,2)$-degree $> 12 \in \mathbb{K}[x,y]$ with $\alpha, \beta, \gamma \in \mathbb{K}$ and $\mu := \mu(f) < \infty$.

Output: *A* normal form of f of the form

$$\begin{array}{ll} (x^2+y^3)^2 + a_0 \cdot xy^{\frac{\mu-6}{2}} + a_1 \cdot xy^{\frac{\mu-4}{2}} & \text{if} \quad \mu \text{ is even} \\ (x^2+y^3)^2 + a_0 \cdot y^{\frac{\mu-3}{2}} + a_1 \cdot y^{\frac{\mu-1}{2}} & \text{if} \quad \mu \text{ is odd} \end{array}$$

with $a_0 \neq 0$, as well as the corresponding moduli parameters of a normal form equation defined over an algebraic extension field of $\mathbb{K}$.

Apply transformation $x \mapsto ax$, $y \mapsto by$ over an algebraic extension field of $\mathbb{K}$ to f to transform the weighted homogeneous part of f to $(x^2+y^3)^2$.
Let $>$ be the local weighted degree reverse lexicographical ordering with weights $(3,2)$ and $x > y$.
Compute a standard basis G of $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ with respect to $>$.
Compute μ the Milnor number of f, and set $d := \mu - 15$.
$a := 13$
while $a < 12 + d$ **do**
    $g :=$ weighted homogeneous part of f of degree a
    Write $g = l \cdot (x^2+y^3)$.
    Construct automorphism φ with $\varphi(x^2+y^3+\frac{1}{2}l) = x^2+y^3$ up to degree $\mu - 1$.
    $f := \varphi(f)$, $a := a + 1$
$g :=$ weighted homogenous part of f of degree $12 + d$
if μ is odd **then**
    $m_0 := y^{\frac{\mu-3}{2}}$, $m_1 := xy^{\frac{\mu-5}{2}}$, $m_2 := y^{\frac{\mu-1}{2}}$
else
    $m_0 := xy^{\frac{\mu-6}{2}}$, $m_1 := y^{\frac{\mu-2}{2}}$, $m_2 := xy^{\frac{\mu-4}{2}}$
Write $g = a_0 \cdot m_0 + l \cdot (x^2+y^3)$.
Construct automorphism φ with $\varphi(x^2+y^3+\frac{1}{2}l) = x^2+y^3$ up to degree $\mu - 1$.
$f := \varphi(f)$
$g :=$ weighted homogeneous part of φ of degree $13 + d$
Write $g = e \cdot m_1 + l \cdot (x^2+y^3)$.
Construct automorphism φ with $\varphi(x^2+y^3+\frac{1}{2}l) = x^2+y^3$ up to degree $\mu - 1$.
$f := \varphi(f)$
if $e \neq 0$ **then**
    $c := (-1)^{\mu-1}\frac{e}{(\mu-3)a}$
    Construct automorphism φ defined by the vector field $c \cdot (3y^2\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y})$ up to degree 5.
    $f := (x^2+y^3)^2 + \varphi(f - (x^2+y^3)^2)$
    $g :=$ the weighted homogeneous part of f of degree $14 + d$
    Write $g = a_1 \cdot m_2 + l \cdot (x^2+y^3)$.
return $(\mathrm{NF}(W^\sharp_{1,\mu-15}), (a_0, a_1))$

Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H., 2015. Singular 4-0-2 – A computer algebra system for polynomial computations. http://www.singular.uni-kl.de

de Jong, T., Pfister, G., 2000. Local Analytic Geometry. Vieweg, Braunschweig.

Greuel, G.-M., Lossen, C., Shustin, E., 2007. Introduction to Singularities and Deformations. Springer, Berlin.

Greuel, G.-M., Pfister, G., 2008. A Singular Introduction to Commutative Algebra, second ed. Springer, Berlin.

Algorithm 7 Remove terms above the diagonal in cases with degenerate Newton boundary.

Input: $f, f_0 \in \mathbb{Q}[x,y]$, $t \in \mathbb{Q}[x,y]$ a term, and weights $u_1, u_2 \in \mathbb{Z}^2$.
Output: $h \in \mathbb{Q}[x,y]$ such that $f \sim h$.

```
w := w(f) and j := w-deg(t)
g := output of Algorithm 4 with input f, f_0, t and u_1, u_2
B := Arnold's system of Q[x,y]/Jac(f)
if t ∈ Jac(f_0) or g ≠ f or (g = f and B contains an element of degree j) then
    return g
m := monomial in B of minimal w-degree
Factorize f_0 = γ · g_0^2 over Q with γ ∈ Q and g_0 ∈ Q[x,y] linear.
φ := automorphism defined by (∂g_0/∂y ∂/∂x − ∂g_0/∂x ∂/∂y) up to w-degree 5
s := coeff(f,m) · m
t' := w-jet(φ(s) − s, j)
for all terms t̃ of t' in increasing order by standard degree do
    t' := −f_0+ result of Algorithm 4 with input t' + f_0, f_0, t̃ and u_1, u_2
    t' := w-jet(t', j)
c := −t/t'
φ_c := automorphism defined by c · (∂g_0/∂y ∂/∂x − ∂g_0/∂x ∂/∂y) up to w-degree 5.
h := f_0 + φ_c(f − f_0)
for all terms t̃ of h of w-degree j in increasing order by standard degree do
    h := result of Algorithm 4 with input h, f_0, t̃ and u_1, u_2
return h
```

Krüger, K., 1997. `classify.lib`. A Singular 4-0-2 library for classifying isolated hypersurface singularities w.r.t. right equivalence, based on the determinator of singularities by V.I. Arnold. Singular distribution.

Luengo, I., Pfister, G., 1990: Normal forms and moduli spaces of curve singualrities with semigroup <2p, 2q, 2pq+d>, Compositio Math. 76, 247-264.

Marais, M., Steenpaß, A., 2013. `realclassify.lib`. A Singular 4-0-2 library for classifying isolated hypersurface singularities over the reals. Singular distribution.

Marais, M., Steenpaß, A., 2015a. The classification of real singularities using Singular Part I: Splitting Lemma and Simple Singularities. J. Symb. Comput. 68, 61-71.

Marais, M., Steenpaß, A., 2016. The classification of real singularities using Singular Part II: The Structure of the Equivalence Classes of the Unimodal Singularities. J. Symb. Comput. 74, 346-366.

Tobis, E., 2012. `rootsur.lib`. A Singular 4-0-2 library for counting the number of real roots of a univariate polynomial.

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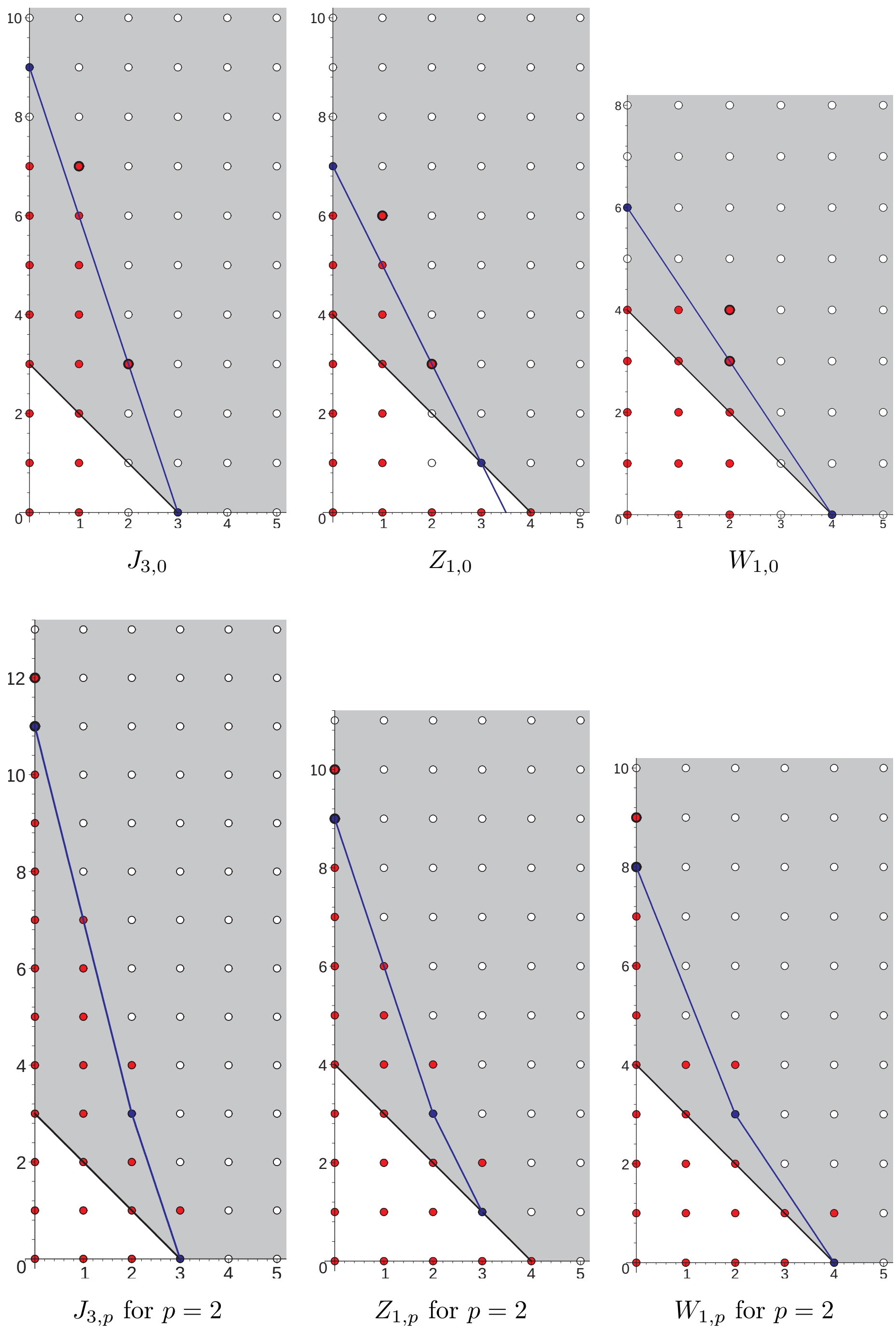
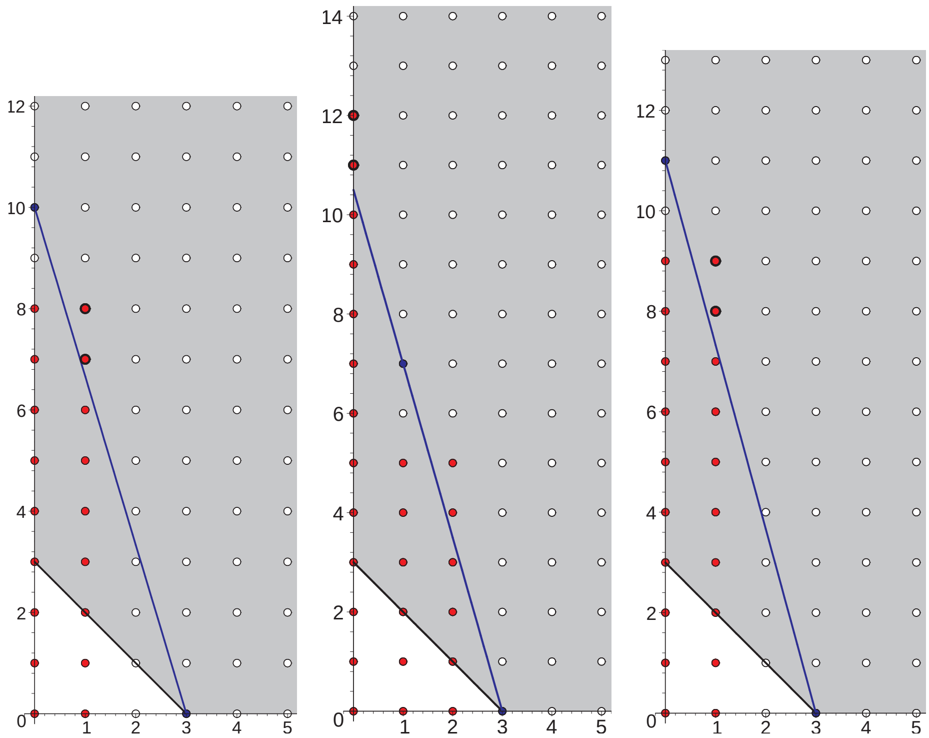
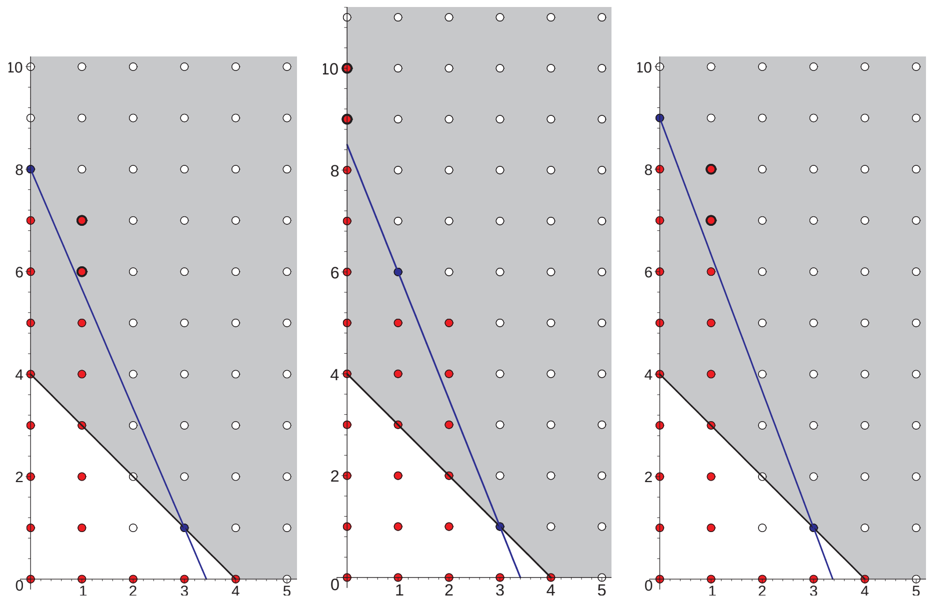


FIGURE 1. Infinite series of bimodal corank 2 singularities with non-degenerate Newton boundary.



E_{18} E_{19} E_{20}

FIGURE 2. Exceptional bimodal corank 2 singularities of type E.



Z_{17} Z_{18} Z_{19}

FIGURE 3. Exceptional bimodal corank 2 singularities of type Z.

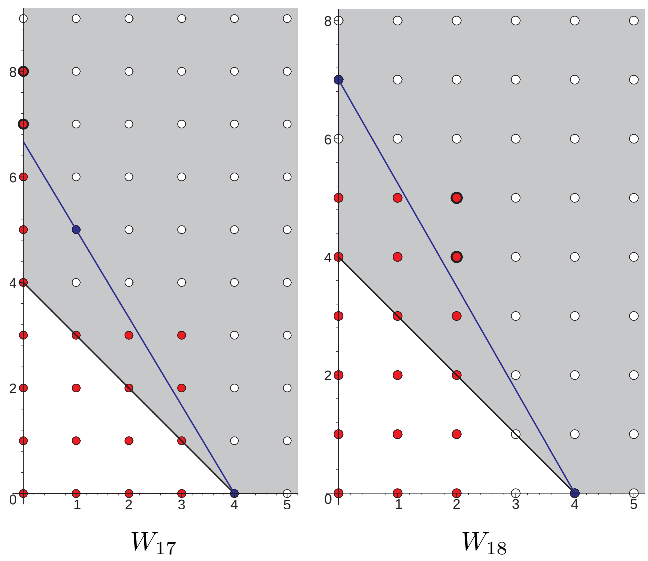


FIGURE 4. Exceptional bimodal corank 2 singularities of type W.

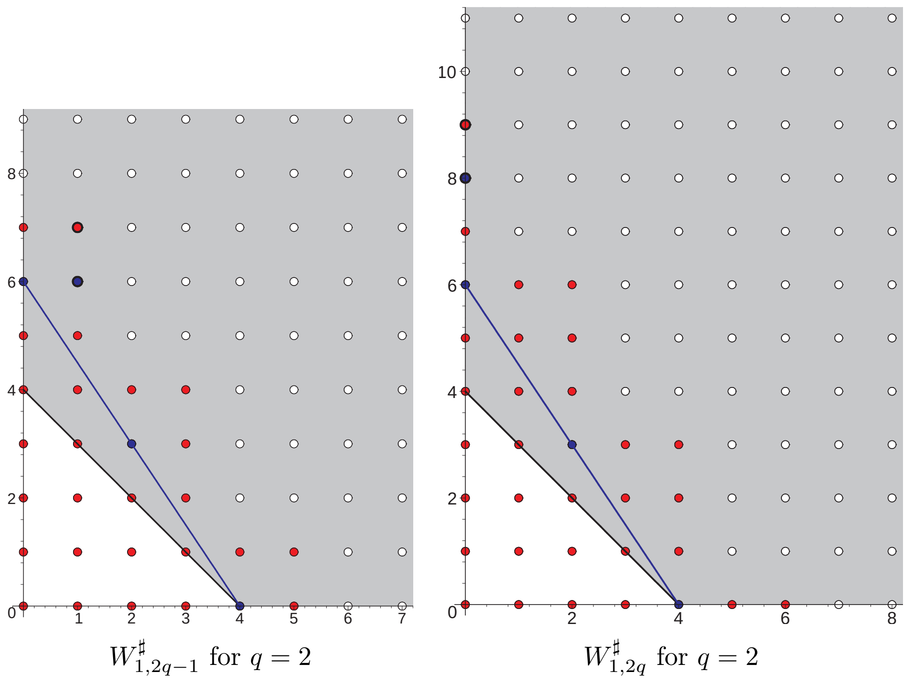


FIGURE 5. Infinite series of bimodal corank 2 singularities with degenerate Newton boundary.