UNIMODULAR ICIS, A CLASSIFIER

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Abstract. We present the algorithms for computing the normal form of unimodular complete intersection surface singularities classified by C. T. C. Wall. He indicated in the list only the μ-constant strata and not the complete classification in each case. We give a complete list of surface unimodular singularities. We also give the description of a classifier which is implemented in computer algebra system Singular .

1. Introduction

In this article we report about a classifier for unimodular isolated complete intersection surface singularities in the computer algebra system Singular (cf. [DGPS13],[GP07]). Marc Giusti gave the complete list of simple isolated complete intersection singularities which are not hypersurfaces (cf. [GM83]). Wall achieved the classification of unimodular singularities which are not hypersurfaces (cf. [Wal83]). Two of the authors described Giusti's classification in terms of certain invariants. Based on this description it is not necessary to compute the normal form for finding the type of the singularity. This is usually more complicated and may be space and time consuming (cf. [ADPG1], [ADPG2]). Similarly the type of singularities in terms of certain invariants (Milnor number, Tjurina number and semi groups) for unimodular isolated complete intersection space curve singularities given by C. T. C. Wall is characterized [ADPG3].

A basis for a classifier is a complete list of these singularities together with a list of invariants characterizing them. Since Wall gave only representatives of the μ-constant strata in his classification (cf. [Wal83]), we complete his list by computing the versal μ-constant deformation of the singularities. The new list obtained in this way contains all unimodular complete intersection surface singularities.

In section 2 we characterize the 2-jet of the unimodular complete intersection singularities using primary decomposition, Krull dimension and Hilbert polynomials. In section 3 we give the complete list of unimodular complete intersection surface singularities by fixing the 2-jet of the singularities and develop algorithms for each case. In section 4 we give the main algorithm for computing the type of unimodular complete intersection surface singularity.

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2. Characterization of normal form of 2-jet of singularities

Consider $I = \langle f_1, f_2 \rangle \subseteq \langle x, y, z, w\rangle^2 \mathbb{C}[[x, y, z, w]]$ defining a complete intersection singularity and let I_2 be the 2-jet of I. According to C.T.C. Wall's classification the 2-jet of $\langle f_1, f_2 \rangle$ is a homogenous ideal generated by 2 polynomials of degree 2. We want to give a description of the type of a singularity without producing the normal form. C.T.C. Wall's classification is based on the classification of the 2-jet I_2 of $\langle f, g\rangle$. Let $\bigcap_{i=1}^{s} Q_i$ be the irredundant primary decomposition of I_2 in $\mathbb{C}[[x, y, z, w]]$. Let t be the number of prime ideals appearing in primary decomposition of I_2 and j_i be the number of conjugates corresponding to each prime ideal. Let $d_i = dim_{\mathbb{C}} \mathbb{C}[[x, y, z, w]]/Q_i, i = 1, ..., s$ and h_i be the Hilbert polynomial of $\mathbb{C}[[x, y, z, w]]/Q_i$. According to C.T.C. Wall's classification we obtain unimodular singularities only in the following cases.

Name	*Characterization*	*Normalform*
$T_{2,2,2,2}$	$s=1,$ $d_1=2, h_1=4t$ $t=1, j_1=1, j_2=1$	$\langle x^2+y^2+z^2, y^2+\lambda z^2+w^2\rangle \lambda\neq 0,1$
$T_{p,2,2,2}$ $p>2$	$s=1$ $d_1=2, h_1=4t$ $t=1, j_1=1$ A_1 *after blowing up*[1]	$\langle xy+z^2+w^2, zw+y^2\rangle$
$T_{p,q,2,2}$ $p,q>2$	$s=2,$ $d_1=d_2=2$ $h_1=h_2=1+2t$	$\langle xy+z^2+w^2, zw\rangle$
$T_{p,q,r,2}$ $p,q,r>2$	$s=3,$ $d_1=d_2=d_3=2$ $h_1=h_2=1+t, h_3=1+2t$ $t=3, j_1=j_2=j_3=1$ $Q_3\nsubseteq Q_1+Q_2$ *and* Q_3 *has a generator of order* 2	$\langle xy+w^2, zw\rangle$
$T_{p,q,r,s}$ $p,q,r,s>2$	$s=4, d_1=d_2=d_3=d_4=2$ $h_1=h_2=h_3=h_4=1+t$	$\langle xy, zw\rangle$
I	$s=4, d_1=d_2=d_3=d_4=1$ $h_1=h_2=h_3=h_4=1+t$	$\langle xy-xz, yz-yx\rangle$
$J^{'}$	$s=1$ $d_1=2, h_1=4t$ $t=1, j_1=1$ A_2 *after blowing up*	$\langle xy+z^2, w^2+xz\rangle$
$K^{'}$	$s=1$ $d_1=2, h_1=4t$ $t=1, j_1=2$	$\langle xy+z^2, w^2+x^2\rangle$
L	$s=2, d_1=d_2=2$ $h_1=1+t, h_2=1+3t$	$\langle xy+z^2, w^2+xz\rangle$
M	$s=3, d_1=d_2=d_3=2$ $h_1=h_2=1+t, h_3=1+2t$ $t=3, j_1=j_2=j_3=1$ $Q_3\subseteq Q_1+Q_2$ *and* Q_3 *has a generator of order* 2	$\langle wy+x^2-z^2, wx\rangle$

TABLE 1. Charecterization table of normal forms of $2-jet$ of I

[1]Here after blowing up means the singularity type of the strict transform of I_2 in the blowing up of $\langle x,y,z,w\rangle$.

3. Unimodular complete intersection surface singularities

We set

$$l_i(x,y) = \begin{cases} xy^q, & \text{if } i = 2q \\ y^{q+2}, & \text{if } i = 2q+1 \end{cases}$$

for brevity.

3.1. **I singularities.** Assume the $2-jet$ of $< \langle f, g\rangle$ has normal form $\langle xy-xz, yz-xy\rangle$. In this case according to C.T.C.Wall's classification the unimodular surface singularities with their Milnor number say μ and Tjurina number τ are given in the table below.

$Name$	$Normal\ \ form$	μ	τ
$I_{1,0}$	$\langle x(y-z)+w^3, y(z-x)+\lambda w^3\rangle \quad \lambda \neq 0, 1$	13	13
$I_{1,i}$	$\langle x(y-z)+w^3, y(z-x)+w^2 l_{i-1}(x,w)\rangle$	$13+i$	$13+i-2$

Table 2

Proposition 3.1. *The unimodular complete intersection surface singularity with Milnor number $\mu = 13$ are $I_{1,0}$ with Tjurina number $\tau = 13$ defined by the ideal*

$$\langle xy - xz + w^3, yz - xy + \lambda w^3\rangle$$

and $I_{1,0,1}$ with Tjurina number $\tau = 12$ defined by the ideal

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 + w^4\rangle.$$

Proof. In the list of C.T.C. Wall $I_{1,0}$ is the singularity defined by the ideal

$$\langle xy - xz + w^3, yz - xy + \lambda w^3\rangle$$

with Milnor number $\mu = 13$ and Tjurina number $\tau = 13$. The versal deformation of $I_{1,0}$ is given by

$$\langle xy - xz + w^3 + t_1 zw + t_2 w + t_3 z + t_4, yz - xy + \lambda w^3 + \lambda_1 w^4 \\ + \lambda_2 w^3 + \lambda_3 w^2 + \lambda_4 zw + \lambda_5 yw + \lambda_6 w + \lambda_7 z + \lambda_8 y + \lambda_9\rangle.$$

$I_{1,0}$ defines a weighted homogenous singularity with weights, $(w_1, w_2, w_3, w_4) = (6, 6, 6, 4)$ and the degrees $(d_1, d_2) = (12, 12)$. The versal μ-constant deformation of $I_{1,0}$ is given by

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 + \lambda_1 w^4\rangle.$$

Using the coordinate change $x \to \xi^6 x$, $y \to \xi^6 y$, $z \to \xi^6 z$, $w \to \xi^4 w$. We obtain

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 + \lambda_1 \xi^4 w^4\rangle.$$

Choosing ξ such that $\lambda_1 \xi^4 = 1$. So we obtain

$$\langle xy - xz + w^3, yz - xy + \lambda w^3 + w^4\rangle$$

with Tjurina number $\tau = 12$ different from $I_{1,0}$. □

Summarizing the results of the above preposition we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal forms $\langle xy - xz, yz - xy\rangle$.

Type	*Normalform*	μ	τ
$I_{1,0,1}$	$\langle xy - xz + w^3, yz - xy + \lambda w^3 + w^4\rangle$	13	12

Table 3

Proposition 3.2. *Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^4, 0)$ be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy - xz, yz - xy\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Tables 2 and 3.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.1. □

We summarize our approach in this case in Algorithm 1

Algorithm 1 `Isingularity(I)`

Input: $I = \langle f, g\rangle \subseteq \langle x, y, z, w\rangle^2\mathbb{C}[[x, y, z, w]]$ such that 2-jet of I has normal form $\langle xy - xz, yz - xy\rangle$
Output: the type of the singularity

compute μ =Milnor number of I;
compute τ =Tjurina number of I;
compute B =the singularity type of the strict transform of I in the blowing up of $\langle x, y, z, w\rangle$ [2]
if $\mu = 13$ and $B = A[1]$ **then**
  if $\mu - \tau = 0$ **then**
    return $(I_{1,0})$;
  if $\mu - \tau = 1$ **then**
    return $(I_{1,0,1})$;
if $\mu = 13 + i$, $i > 0$ and $B = A[i + 1]$ **then**
  if $\mu - \tau = 2$ **then**
    return $(I_{1,i})$;
return (*not unimodular*);

Name	*Normal form*	μ	τ
$T_{2,2,2,2}$	$\langle x^2 + y^2 + z^2, y^2 + z^2 + w^2\rangle$	7	7
$T_{p,q,r,s}$ $p > 2, q, r, s \geq 2$	$\langle xy + z^2 + w^2, zw + x^2 + y^2\rangle$	$p + q + r + s - 1$	$p + q + r + s - 2$

Table 4

[2]List B gives the singularity type of the strict transform of I in the blowing up of $\langle x, y, z, w\rangle$ and it returns 1 if it is smooth.

3.2. **T singularities.** Assume the $2-jet$ of $\langle f,g\rangle$ has normal form $\langle x^2+y^2+z^2, y^2+z^2+w^2\rangle$, $\langle xy+z^2+w^2, zw+y^2\rangle$, $\langle xy+z^2+w^2, zw\rangle$ or $\langle xy, zw\rangle$. In this case according to C.T.C.Wall's classification the unimodular surface singularities with Milnor number μ and Tjurina number τ are given in the table 4 above.

Algorithm 2 `Tsingularity(I)`

Input: $I=\langle f,g\rangle \subseteq \langle x,y,z,w\rangle^2\mathbb{C}[[x,y,z,w]]$ such that the2-jet of I has normal form $\langle xy,zw\rangle$,$\langle xy+z^2+w^2, zw+x^2+y^2\rangle$, $\langle xy+z^2+w^2, zw+y^2\rangle$, $\langle xy+z^2+w^2, zw\rangle$
Output: the type of the singularity

compute μ =Milnor number of I; τ =Tjurina number of I and B =the singularity type of the strict transform of I in the blowing up of $\langle x,y,z,w\rangle$
if $\mu=\tau$ and 2-jet has normal form $\langle xy+z^2+w^2, zw+x^2+y^2\rangle$ and B=1 **then**
  return $(T_{2,2,2,2})$;
if $\mu-\tau=1$ **then**
  if 2-jet has normal form $< \langle xy+z^2+w^2, zw+y^2\rangle$ **then**
    if $\mu=8$ and $B=1$ **then**
      return $(T_{3,2,2,2})$;
    if $\mu>8$ and $B=A[\mu-8]$ **then**
      return $(T_{\mu-5,2,2,2})$;
  if 2-jet has normal form $\langle xy+z^2+w^2, zw\rangle$ **then**
    if $B=1$ **then**
      return $(T_{3,2,3,2})$;
    if $B=A[r-3]$ and $r>3$ **then**
      return $(T_{3,2,r,2})$;
    if $B=A[p-3], A[r-3]$ and $p,r>3$ **then**
      return $(T_{p,2,r,2})$;
  if 2-jet has normal form $< \langle xy+w^2, zw\rangle$ **then**
    if $B=1$ **then**
      return $(T_{3,3,3,2})$;
    if $B=A[r-3]$ and $r>3$ **then**
      return $(T_{3,3,r,2})$;
    if $B=A[q-3], A[r-3]$ and $q,r>3$ **then**
      return $(T_{3,q,r,2})$;
    if $B=A[p-3], A[q-3], A[r-3]$ and $p,q,r>3$ **then**
      return $(T_{p,q,r,2})$;
  if 2-jet has normal form $\langle xy, zw\rangle$ **then**
    if $B=1$ **then**
      return $(T_{3,3,3,3})$;
    if $B=A[s-3]$ and $s>3$ **then**
      return $(T_{3,3,3,s})$;
    if $B=A[r-3], A[s-3]$ and $r,s>3$ **then**
      return $(T_{3,3,r,s})$;
    if $B=A[q-3]A[r-3], A[s-3]$ and $q,r,s>3$ **then**
      return $(T_{3,q,r,s})$;
    if $B=A[p-3], A[q-3], A[r-3], A[s-3]$ and $p,q,r,s>3$ **then**
      return $(T_{p,q,r,s})$;
return $(not\quad unimodular)$;

3.3. **J′ singularities.** Assume the $2-jet$ of $\langle f,g\rangle$ has normal form $\langle xy+z^2, w^2+xz\rangle$. According to C.T.C. Wall's classification the unimodular surface singularities are given in the following table

$Name$	$Normal\ Form$	μ	τ
J'_{6m+9}	$\langle xy+z^2, xz+w^2+y^{3m+3}\rangle$	$6m+9$	$6m+9$
J'_{6m+10}	$\langle xy+z^2, w^2+xz+zy^{2m+2}\rangle$	$6m+10$	$6m+10$
J'_{6m+11}	$\langle xy+z^2, w^2+xz+y^{3m+4}\rangle$	$6m+11$	$6m+11$
$J'_{m+1,0}$	$\langle xy+z^2, w^2+xz+z^2y^m+\lambda y^{3m+2}\rangle\lambda\neq 0,-4/27$	$6m+7$	$6m+7$
$J'_{m+1,i}$	$\langle xy+z^2, w^2+xz+z^2y^m+y^{3m+2-i}\rangle$	$6m+7+i$	$6m+5+i$

TABLE 5

Proposition 3.3. *The unimodular complete intersection surface singularities having Milnor number of the form $\mu=6m+9$ where m is a positive integer are J'_{6m+9} with Tjurina number $\tau=6m+9$ defined by the ideal*

$$\langle xy+z^2, xz+w^2+y^{3m+3}\rangle$$

and $J'_{6m+9,i+1}$ with Tjurina number $\tau=6m+8-i$ are defined by the ideal

$$\langle xy+z^2, xz+w^2+y^{3m+3}+zy^{3m+2-i}\rangle\ for\ i=0,1,\ldots,m.$$

Proof. In C.T.C. Wall's list the only unimodular complete intersection surface singularities are the singularities J'_{6m+9} defined by the ideal

$$\langle xy+z^2, xz+w^2+y^{3m+3}\rangle.$$

The versal deformation of J'_{6m+9} is given by

$$\langle xy+z^2+Aw+Bz+C, w^2+xz+y^{3m+3}+\sum_{i=0}^{3m+2}\alpha_i y^{3m+2-i}z+\sum_{i=0}^{3m+2}\beta_i y^{3m+2-i}\rangle.$$

J'_{6m+9} defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1,w_2,w_3,w_4)=(12m+10,6,6m+8,9m+9)$$

and degrees

$$(d_1,d_2)=(12m+16,18m+18).$$

The versal $\mu-$constant deformation of J'_{6m+9} is

$$\langle xy+z^2, w^2+xz+y^{3m+3}+\sum_{i=0}^{m}\alpha_i y^{3m+2-i}z\rangle.$$

Consider $\phi\in Aut_{\mathbb{C}}(\mathbb{C}[[x,y,z,w]])$ defined by $\phi(x)=\epsilon^{12m+10}x$, $\phi(y)=\epsilon^6 y$, $\phi(z)=\epsilon^{6m+8}z$ and $\phi(w)=\epsilon^{9m+9}w$. If $\alpha_m\neq 0$, then we can write I as

$$I=\langle xy+z^2, w^2+xz+y^{3m+3}+y^{2m+2}z(\sum_{i=0}^{m-1}\alpha_i y^{m-i}+\alpha_m)\rangle.$$

Let $\mu_{m-1}=\sum_{i=0}^{m-1}\alpha_i y^{m-i}+\alpha_m$. Then I can be written

$$I=\langle xy+z^2, w^2+xz+y^{3m+3}+y^{2m+2}z\mu_m\rangle.$$

By applying the transformation ϕ we get

$$\langle xy + z^2, w^2 + xz + y^{3m+3} + \epsilon^2 y^{2m+2} z\mu'_{m-1}\rangle.$$

Choosing ε such that $\varepsilon^2\mu'_{m-1} = 1$, we obtain

$$\langle xy + z^2, xz + w^2 + y^{3m+3} + y^{2m+2}z\rangle.$$

Now we assume that $\alpha_m = 0$. This implies that

$$I = \langle xy + z^2, w^2 + xz + y^{3m+3} + y^{2m+3}z(\sum_{i=0}^{m-2} \alpha_i y^{m-1-i} + \alpha_{m-1})\rangle.$$

Let $\mu_{m-2} = \sum_{i=0}^{m-2} \alpha_i y^{m-1-i} + \alpha_{m-1}$ then we can have

$$I = \langle xy + z^2, w^2 + xz + y^{3m+3} + y^{2m+3}z\mu_{m-2}\rangle.$$

Now applying the transformation ϕ we get I as

$$\langle xy + z^2, w^2 + xz + y^{3m+3} + \epsilon^6 y^{2m+3} z\mu'_{m-2}\rangle.$$

Choosing ε such that $\varepsilon^6\mu'_{m-2} = 1$, so we obtain

$$\langle xy + z^2, xz + w^2 + y^{3m+3} + y^{2m+3}z\rangle.$$

Now we assume $\alpha_{m-1} = 0$. Iterating in the same way we get $m + 1$ different singularities defined by

$$\langle xy + z^2, w^2 + xz + y^{3m+3} + y^{3m+2-i}z\rangle$$

for $i = 0, 1, \ldots, m$ and $m \geq 1$. □

Proposition 3.4. *The unimodular complete intersection surface singularities having Milnor number of the form $\mu = 6m+10$ where m is a positive integer are J'_{6m+10} with Tjurina number $\tau = 6m + 10$ defined by the ideal*

$$\langle xy + z^2, xz + w^2 + zy^{2m+2}\rangle$$

and $J'_{6m+10,i+1}$ with Tjurina numbers $6m + 10 - i$ is defined by the ideal

$$\langle xy + z^2, xz + w^2 + zy^{2m+2} + y^{4m+4-i}\rangle \ for\ i = 0, 1, \ldots, m.$$

Proposition 3.5. *The unimodular complete intersection surface singularities having Milnor number of the form $\mu = 6m+11$ where m is a positive integer are J'_{6m+11} with Tjurina number $\tau = 6m + 11$ defined by the ideal*

$$\langle xy + z^2, w^2 + xz + y^{3m+4}\rangle$$

and $J'_{6m+11,i+1}$ with Tjurina number $6m + 10 - i$ is defined by the ideal

$$\langle xy + z^2, w^2 + xz + y^{3m+4} + y^{3m+3-i}\rangle \ for\ i = 0, 1, \ldots, m.$$

Proof. The proofs of Propositions 3.4 and 3.5 can be done similarly to the proof of Proposition 3.3. □

Summarizing the above results we have the following table

$Type$	$NormalForm$	μ	τ
$J'_{6m+9,i+1}$	$\langle xy+z^2, xy+z^2+y^{3m+3}+zy^{3m+2-i}\rangle$	$6m+9$	$6m+8-i$
$J'_{6m+10,i+1}$	$\langle xy+z^2, w^2+xz+zy^{2m+2}+y^{4m+4-i}\rangle$	$6m+10$	$6m+9-i$
$J'_{6m+11,i+1}$	$\langle xy+z^2, w^2+xz+y^{3m+4}+y^{3m+3-i}z\rangle$	$6m+11$	$6m+10-i$
	$for \quad i=0,1,...,m.$		

Table 6

As a consequence of C.T.C. Wall's classification and Propositions 3.3 - 3.5 we obtain:

Proposition 3.6. *Let* $(V(\langle f,g\rangle),0)\subseteq(\mathbb{C}^4,0)$ *be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the* 2*-jet of* $\langle f,g\rangle$ *has normal form* $\langle xy+z2, w^2+xz\rangle$. $(V(\langle f,g\rangle),0)$ *is unimodular if and only if it is isomorphic to a complete intersection in Tables 5 and 6.*

We summarize our approach in this case in Algorithm 3.

Algorithm 3 $\mathtt{J}^{'}$`singularity(I)`

Input: $I=\langle f,g\rangle\subseteq\langle x,y,z,w\rangle^2\mathbb{C}[[x,y,z,w]]$ such that 2-jet of I has normal form $\langle xy+z^2, w^2+xz\rangle$

Output: the type of the singularity

compute μ =Milnor number of I, τ =Tjurina number of I and B =the singularity type of the strict transform of I in the blowing up of $\langle x,y,z,w\rangle$
if $\mu\equiv 3$ mod 6 and $B=E[(\mu-15)+6]$ **then**
  if $\mu=\tau$ **then**
    return $(J^{'}_{\mu})$;
  else
    return $(J^{'}_{\mu,\mu-\tau})$;
if $\mu\equiv 4$ mod 6 and $B=E[(\mu-16)+7]$ **then**
  if $\mu=\tau$ **then**
    return $(J^{'}_{\mu})$;
  else
    return $(J^{'}_{\mu,\mu-\tau})$;
if $\mu\equiv 5$ mod 6 and $B=E[(\mu-17)+6]$ **then**
  if $\mu=\tau$ **then**
    return $(J^{'}_{\mu})$;
  else
    return $(J^{'}_{\mu,\mu-\tau})$;
if $\mu\equiv 1$ mod 6 and $B=J[(\mu-13)/6,0]$ and $\mu=\tau$ **then**
  return $(J^{'}_{(\mu-7)/6+1,0})$;
if $\mu\neq\tau$ and $B=J[(\mu-13)/6,\mu-\tau-1]$ **then**
  return $(J^{'}_{(\mu-7)/6+1,\mu-\tau-1})$;
return $not \quad unimodular$;

3.4. **K′ singularities.** Assume the $2-jet$ of $\langle f, g\rangle$ has normal form $\langle xy+z^2, w^2+x^2\rangle$. In this case according to C.T.C.Wall's classification the unimodular surface singularities with Milnor number μ are given in the table below.

$Name$	$Normalform$	μ	τ
$K^{'}_{10}$	$xy+z^2, w^2+x^2+y^3$	10	10
$K^{'}_{11}$	$xy+z^2, x^2+w^2+zy^2$	11	11
$K^{'}_{1,0}$	$xy+z^2, x^2+w^2+z^2y+\lambda y^4 \quad (\lambda \neq 0, \frac{1}{4})$	13	13
$K^{'}_{1,i}$	$xy+z^2, x^2+w^2+z^2y+y^{4+i}$	$13+i$	$13+i-2$
$K^{b}_{1,i}$	$xy+z^2, x^2+w^2+2z^2y+y^4+zyI_i(z,y)$	$13+i$	$13+i-2$
$K^{'}_{15}$	$xy+z^2, x^2+w^2+zy^3$	15	15
$K^{'}_{16}$	$xy+z^2, x^2+w^2+y^5$	16	16

TABLE 7

Proposition 3.7. *The unimodular complete intersection surface singularities with Milnor number $\mu = 10$ are $K^{'}_{10}$ with Tjurina number $\tau = 10$ defined by the ideal*

$$\langle xy+z^2, w^2+x^2+y^3\rangle$$

and $K^{'}_{10,1}$ with Tjurina number $\tau = 9$ defined by the ideal

$$\langle xy+z^2, w^2+x^2+y^3+yz^2\rangle.$$

Proof. In the list of C.T.C Wall $K^{'}_{10}$ is the singularity defined by the ideal

$$\langle xy+z^2, w^2+x^2+y^3\rangle$$

with Milnor number $\mu = 10$ and Tjurina number $\tau = 10$. The versal deformation of $K^{'}_{10}$ is

$$\langle xy+z^2+t_1w+t_2y+t_3, x^2+w^2+y^3+\lambda_1yz^2+\lambda_2z^2+\lambda_3yz+\lambda_4z+\lambda_5y^2+\lambda_6y+\lambda_7\rangle.$$

$K^{'}_{10}$ defines a weighted homogenous isolated complete intersection singularity with

$$(w_1, w_2, w_3, w_4) = (6, 4, 5, 6) \textit{ and the degrees } (d_1, d_2) = (10, 12).$$

The versal μ-constant deformation of $K^{'}_{10,1}$ is given by $\langle xy+z^2, w^2+x^2+y^3+\lambda_1yz^2\rangle$. Using the coordinate change $x \to \xi^6x$, $y \to \xi^4y$, $z \to \xi^5z$ and $w \to \xi^6w$, we obtain

$$\langle xy+z^2, w^2+x^2+y^3+\lambda_1\xi^2yz^2\rangle.$$

Choosing ξ such that $\lambda_1\xi^2 = 1$, we obtain

$$\langle xy+z^2, w^2+x^2+y^3+yz^2\rangle$$

with Tjurina number $\tau = 9$. □

Proposition 3.8. *The unimodular complete intersection surface singularities with Milnor number $\mu = 11$ are $K^{'}_{11}$ with Tjurina number $\tau = 11$ defined by the ideal*

$$\langle xy+z^2, x^2+w^2+zy^2\rangle$$

and $K^{'}_{11,1}$ with Tjurina number $\tau = 10$ defined by the ideal

$$\langle xy+z^2, x^2+w^2+zy^2+y^5\rangle.$$

Proof. The proof of Proposition 3.8 can be done similarly to the proof of Proposition 3.7. □

Proposition 3.9. *The unimodular complete intersection surface singularities with Milnor number* $\mu = 13$ *are* $K^{'}_{1,0}$ *with Tjurina number* $\tau = 13$ *defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4\rangle, \qquad \lambda \neq 0, \frac{1}{4}.$$

$K^{'}_{1,0,1}$ *with Tjurina number* $\tau = 12$ *defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4 + y^5\rangle.$$

Proof. In C.T.C's Wall list $K^{'}_{1,0}$ is the singularity defined by the ideal

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4\rangle \quad \lambda \neq 0, \frac{1}{4}$$

with Milnor number $\mu = 13$ and Tjurina number $\tau = 13$. The versal deformation of $K^{'}_{1,0}$ is given by

$$\langle xy + z^2 + t_1w + t_2y + t_3, x^2 + w^2 + z^2y + \lambda y^4 + \lambda_1 z^2 + \lambda_2 y^2 z + \lambda_3 yz + \lambda_4 z + \lambda_5 y^5 + \lambda_6 y^4 + \lambda_7 y^3 + \lambda_8 y^2 + \lambda_9 y + \lambda_{10}\rangle$$

$K^{'}_{1,0}$ defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (4, 2, 3, 4)$$

and the degrees

$$(d_1, d_2) = (6, 8).$$

The versal μ-constant deformation of $K^{'}_{1,0}$ is given by

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4 + \lambda_5 y^5\rangle.$$

Using the coordinate change $x \to \xi^4 x,\ y \to \xi^2 y,\ z \to \xi^3 z, w \to \xi^4 w$, we obtain

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4 + \lambda_5 \xi^2 y^5\rangle.$$

Choosing ξ such that $\lambda_5 \xi^2 = 1$ we obtain

$$\langle xy + z^2, x^2 + w^2 + z^2y + \lambda y^4 + y^5\rangle$$

with Tjurina number $\tau = 12$. □

Proposition 3.10. *The unimodular complete intersection surface singularities with Milnor number* $\mu = 15$ *are* $K^{'}_{15}$ *with Tjurina number* $\tau = 15$ *defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + zy^3\rangle$$

and $K^{'}_{15,1}$ *with Tjurina number* $\tau = 14$ *defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + zy^2 + y^6\rangle.$$

and $K^{'}_{15,2}$ *with Tjurina number* $\tau = 13$ *defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + zy^3 + y^5\rangle.$$

Proof. In C.T.C. Wall's list $K^{'}_{15}$ is the singularity defined by the ideal

$$\langle xy + z^2, x^2 + w^2 + zy^3 \rangle.$$

with Milnor number $\mu = 15$ and Tjurina number $\tau = 15$. The versal deformation of $K^{'}_{15}$ is given by

$$\langle xy + z^2 + t_1 w + t_2 y + t_3, x^2 + w^2 + zy^2 + \lambda_1 yz^2 + \lambda_2 z^2 + \lambda_3 yz \\ + \lambda_4 z + \lambda_5 y^6 + \lambda_6 y^5 + \lambda_7 y^4 + \lambda_8 y^3 + \lambda_9 y^2 + \lambda_{10} y + \lambda_{11} \rangle.$$

$K^{'}_{15}$ defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (7, 3, 5, 7)$$

and the degrees

$$(d_1, d_2) = (10, 14).$$

The versal μ-constant deformation of $K^{'}_{15}$ is given by

$$\langle xy + z^2, x^2 + w^2 + zy^3 + \lambda_5 y^6 + \lambda_6 y^5 \rangle.$$

If $\lambda_6 \neq 0$ then

$$I = \langle xy + z^2, x^2 + w^2 + zy^3 + uy^5 \rangle, \ \textit{where}\ u = \lambda_5 y + \lambda_6.$$

Using coordinate change such that $x \to \xi^7 x$, $y \to \xi^3 y$, $z \to \xi^5 z$ and $w \to \xi^7 w$ we may assume

$$I = \langle xy + z^2, x^2 + w^2 + zy^3 + \xi \bar{u} y^5 \rangle.$$

Choosing ξ such that $\xi \bar{u} = 1$ we obtain

$$I = \langle xy + z^2, x^2 + w^2 + zy^3 + y^5 \rangle$$

with Tjurina number $\tau = 13$. If $\lambda_6 = 0$ then again applying the same coordinate change we obtain

$$I = \langle xy + z^2, x^2 + w^2 + zy^3 + y^6 \rangle$$

by choosing $\lambda_5 \xi^4 = 1$ with Tjurina number $\tau = 14$. □

Proposition 3.11. *The unimodular complete intersection surface singularities with Milnor number $\mu = 16$ are $K^{'}_{16}$ with Tjurina number $\tau = 16$ defined by the ideal*

$$\langle xy + z^2, x^2 + w^2 + y^5 \rangle.$$

$K^{'}_{16,1}$ with Tjurina number $\tau = 15$ defined by the ideal

$$\langle xy + z^2, x^2 + w^2 + y^5 + z^2 y^3 \rangle.$$

and $K^{'}_{16,2}$ with Tjurina number $\tau = 14$ defined by the ideal

$$\langle xy + z^2, x^2 + w^2 + y^5 + z^2 y^2 \rangle.$$

Proof. The proof of Proposition 3.11 can be done similarly to the proof of Proposition 3.10. □

Summarizing the results of the above propositions we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal forms $\langle xy + z^2, w^2 + x^2\rangle$.

$Type$	$Normal form$	μ	τ
$K^{'}_{10,1}$	$(xy + z^2, w^2 + x^2 + y^3 + yz^2)$	10	9
$K^{'}_{11,1}$	$(xy + z^2, w^2 + x^2 + zy^2 + y^5)$	11	10
$K^{'}_{1,0,1}$	$(xy + z^2, w^2 + x^2 + yz^2 + \lambda y^4 + y^5)$	13	12
$K^{'}_{15,1}$	$(xy + z^2, w^2 + x^2 + zy^3 + y^6)$	15	14
$K^{'}_{15,2}$	$(xy + z^2, w^2 + x^2 + zy^3 + y^5)$	15	13
$K^{'}_{16,1}$	$(xy + z^2, w^2 + x^2 + y^5 + y^3z^2)$	16	15
$K^{'}_{16,2}$	$(xy + z^2, w^2 + x^2 + y^5 + y^2z^2)$	16	14

Table 8

Proposition 3.12. *Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^4, 0)$ be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy + z^2, w^2 + x^2\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Tables 7 and 8.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.7 - 3.11 □

Proposition 3.13. *Let $(\mathbf{V}(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^4, 0)$ be the germ of a complete intersection surface singularity. Assume it is not hypersurface singularity and the two jet of $\langle f, g\rangle$ has normal form $\langle xy + z^2, w^2 + xz\rangle$. $\mathbf{V}(\langle f, g\rangle, 0)$ is unimodular if and only if it is isomorphic to a complete intersection in table 3.*

We summarize our approach in this case in Algorithm 4.

Algorithm 4 `K′singularity(I)`

Input: $I = \langle f, g\rangle \subseteq \langle x,y,z,w\rangle^2\mathbb{C}[[x,y,z,w]]$ such that 2-jet of I has normal form $(xy+z^2, w^2+x^2)$
Output: the type of the singularity

compute μ =Milnor number of I;
compute τ =Tjurina number of I;
compute B =the singularity type of the strict transform of I in the blowing up of $\langle x,y,z,w\rangle$
if $\mu = 10$ and $B = 1$ **then**
  if $\mu = \tau$ **then**
    return (K'_{10});
  if $\mu - \tau = 1$ **then**
    return $(K'_{10,1})$;
if $\mu = 11$ and $B = A[1]$ **then**
  if $\mu = \tau$ **then**
    return (K'_{11});
  if $\mu - \tau = 1$ **then**
    return $(K'_{11,1})$;
if $\mu = 13$ and $B = A[3]$ **then**
  if $\mu = \tau$ **then**
    return $(K'_{1,0})$;
  if $\mu - \tau = 1$ **then**
    return $(K'_{1,0,1})$;
if $\mu = 15$ and $B = D[5]$ **then**
  if $\mu = \tau$ **then**
    return (K'_{15});
  if $\mu - \tau = 1$ **then**
    return $(K'_{15,1})$;
  if $\mu - \tau = 2$ **then**
    return $(K'_{15,2})$;
if $\mu = 16$ and $B = E[6]$ **then**
  if $\mu = \tau$ **then**
    return (K'_{16});
  if $\mu - \tau = 1$ **then**
    return $(K'_{16,1})$;
  if $\mu - \tau = 2$ **then**
    return $(K'_{16,2})$;
if $\mu \neq \tau$ **then**
  if $\mu - \tau = 2$ and $\mu > 13$ **then**
    if $B = D[\mu - 10]$ **then**
      return $(K'_{1,\mu-13})$;
    if $B = A[\mu - 10]$ **then**
      return $(K^b_{1,\mu-13})$;
return (not unimodular);

3.5. **L singularities.** Assume the $2-jet$ of $\langle f,g\rangle$ has normal form $\langle wz+xy, y^2+xz\rangle$. According to C.T.C. Wall's classification the unimodular singularities are given in the table.

$Type$	$Normal\ \ form$	μ	τ
L_{10}	$\langle wz+xy, y^2+xz+w^3\rangle$	10	10
L_{11}	$\langle wz+xy, y^2+xz+xw^2\rangle$	11	11
$L_{1,0}$	$\langle wz+xy, y^2+xz+x^2w+\lambda w^4\rangle$	11	11
L_{15}	$\langle wz+xy, y^2+xz+xw^3\rangle$	15	16
L_{16}	$\langle wz+xy, y^2+xz+w^5\rangle$	16	16

Table 9

Proposition 3.14. *The unimodular complete intersection surface singularity with Milnor number $\mu=10$ and Tjurina number $\tau=10$ is L_{10} defined by the ideal*

$$\langle wz+xy, y^2+xz+w^3\rangle$$

and $L_{10,1}$ with Tjurina number $\tau=9$ is defined by the ideal

$$\langle wz+xy, y^2+xz+w^3+yw^2\rangle.$$

Proof. In C.T.C. Wall's list the only unimodular complete intersection singularity with Milnor number $\mu=10$ is the singularity L_{10} defined by the ideal $(wz+xy, y^2+xz+w^3)$. The versal deformation of the above singularity is given by

$$\langle xy+zw+Hw^2+Iw+J, y^2+xz+w^3+Ayw^2+Bw^2+Cyw+Dw+Ez+Fy+G\rangle.$$

L_{10} defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1,w_2,w_3,w_4)=(5,6,7,4)$$

and degrees

$$(d_1,d_2)=(11,12).$$

The versal $\mu-$constant deformation of L_{10} is

$$\langle wz+xy, y^2+xz+w^3+Ayw^2\rangle.$$

Using the coordinate change

$$x\to\varepsilon^5x,\ \ y\to\varepsilon^6y,\ \ z\to\varepsilon^7z \text{ and } w\to\varepsilon^4w$$

we obtain

$$\langle wz+xy, y^2+xz+w^3+\varepsilon^2A^{'}yw^2\rangle.$$

Take ε such that $\varepsilon^2A^{'}=1$ then we obtain

$$\langle wz+xy, y^2+xz+w^3+yw^2\rangle.$$

□

Proposition 3.15. *The unimodular complete intersection surface singularity with Milnor number $\mu=11$ and Tjurina number $\tau=11$ is L_{11} defined by the ideal*

$$\langle wz+xy, y^2+xz+xw^2\rangle$$

and $L_{11,1}$ with Tjurina number $\tau=10$ is defined by the ideal

$$\langle wz+xy, y^2+xz+xw^2+w^4\rangle.$$

Proof. The proof of Proposition 3.15 can be done similarly to the proof of Proposition 3.14. □

Proposition 3.16. *The unimodular complete intersection surface singularity with Milnor number $\mu = 13$ and with Tjurina number $\tau = 13$ is $L_{1,0}$ defined by the ideal*

$$\langle wz + xy, y^2 + xz + x^2w + \lambda y^4 \rangle, \ \lambda \neq 0, -1.$$

and $L_{1,0,1}$ with Tjurina number $\tau = 12$ is defined by the ideal

$$\langle wz + xy, y^2 + xz + x^2w + \lambda w^4 + w^5 \rangle.$$

Proof. In C.T.C. Wall's list the only unimodular complete intersection singularity with Milnor number 13 is the singularity $L_{1,0}$ defined by the ideal

$$\langle wz + xy, y^2 + xz + x^2w + \lambda w^4 \rangle, \ \lambda \neq 0, -1.$$

The versal deformation of the above singularity is given by

$$\langle xy + zw + Jw^3 + Kw^2 + Lw + M, y^2 + xz + x^2w + \lambda w^4 + Aw^5 \\ + Bw^4 + Cw^3 + Dw^2 + Eyw + Fw + +Gz + Hy + I \rangle.$$

$L_{1,0}$ defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (3, 4, 5, 2)$$

and degrees

$$(d_1, d_2) = (7, 8).$$

The versal $\mu-$constant deformation of $L_{1,0}$ is

$$\langle wz + xy, y^2 + xz + x^2w + \lambda w^4 + Aw^5 \rangle.$$

Now using the coordinate change $x \to \varepsilon^3 x$, $y \to \varepsilon^4 y$, $z \to \varepsilon^5 z$ and $w \to \varepsilon^2 w$. We obtain

$$\langle wz + xy, y^2 + xz + x^2w + \lambda w^4 + \varepsilon^2 A' w^5 \rangle.$$

Choosing ε such that $\varepsilon^2 A' = 1$. We obtain

$$\langle wz + xy, y^2 + xz + x^2w + \lambda w^4 + w^5 \rangle$$

with Tjurina number $\tau = 9$. □

Proposition 3.17. *The unimodular complete intersection surface singularity with Milnor number $\mu = 15$ and with Tjurina number $\tau = 15$ is L_{15} defined by the ideal*

$$\langle wz + xy, y^2 + xz + xw^3 \rangle,$$

$L_{15,1}$ with Tjurina number $\tau = 14$ is defined by the ideal

$$\langle wz + xy, y^2 + xz + xw^3 + w^6 \rangle$$

and $L_{15,2}$ with Tjurina number $\tau = 13$ is defined by the ideal

$$\langle wz + xy, y^2 + xz + xw^3 + w^5 \rangle.$$

Proof. In C.T.C. Wall's list the only unimodular complete intersection singularity with Milnor number $\mu = 15$ is the singularity L_{15} defined by the ideal

$$\langle wz + xy, y^2 + xz + xw^3 \rangle.$$

The versal deformation of the above singularity is given by

$$\langle xy + zw + Lw^3 + Mw^2 + Nw + O, y^2 + xz + xw^3 + Aw^6 + Bw^5 \\ + Cw^4 + Dw^3 + Ew^2y + Fw^2 + Gyw + Hw + Iz + Jy + K \rangle.$$

L_{15} defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (5, 7, 9, 3)$$

and degrees

$$(d_1, d_2) = (12, 14).$$

The versal $\mu-$constant deformation of L_{15} is

$$\langle wz + xy, y^2 + xz + xw^3 + Aw^6 + Bw^5 \rangle.$$

Let $B \neq 0$,

$$\langle xy + wz, y^2 + xz + xw^3 + w^5(Aw + B) \rangle.$$

Let $\mu_B = Aw + B$, Now using the coordinate change $x \to \varepsilon^5 x$, $y \to \varepsilon^7 y$, $z \to \varepsilon^9 z$ and $w \to \varepsilon^3 w$, we obtain

$$\langle wz + xy, y^2 + xz + xw^3 + \varepsilon \acute{\mu_B} w^5 \rangle.$$

Choosing ε such that $\varepsilon \acute{\mu_B} = 1$. So we obtain

$$\langle wz + xy, y^2 + xz + xw^3 + w^5 \rangle$$

with Tjurina number $\tau = 13$. Now we if $B = 0$ then it becomes

$$\langle wz + xy, y^2 + xz + xw^3 + Aw^6 \rangle.$$

Now using again the same coordinate change we obtain

$$\langle wz + xy, y^2 + xz + xw^3 + \varepsilon^4 A' w^6 \rangle.$$

Choosing ε such that $\varepsilon^4 A' = 1$ so we obtain

$$\langle wz + xy, y^2 + xw^3 + w^6 \rangle$$

with Tjurina number $\tau = 14$. □

Proposition 3.18. *The unimodular complete intersection surface singularity with Milnor number $\mu = 16$ and Tjurina number $\tau = 16$ is L_{16} defined by the ideal*

$$\langle wz + xy, y^2 + xz + w^5 \rangle,$$

$L_{16,1}$ with Tjurina number $\tau = 15$ is defined by the ideal

$$\langle wz + xy, y^2 + xz + w^5 + yw^4 \rangle$$

and and $L_{16,2}$ with Tjurina number $\tau = 14$ is defined by the ideal

$$\langle wz + xy, y^2 + xz + w^5 + yw^3 \rangle.$$

Summarizing the above results we have the following table.

$Name$	$Normalform$	μ	τ
$L_{10,1}$	$\langle wz + xy, y^2 + xz + w^3 + yw^2 \rangle$	10	9
$L_{11,1}$	$\langle xy + zw, y^2 + xz + xw^2 + w^4 \rangle$	11	10
$L_{1,0,1}$	$\langle xy + zw, y^2 + xz + x^2w + w^4 + w^5 \rangle$	13	12
$L_{15,1}$	$\langle xy + zw, y^2 + xz + xw^3 + w^6 \rangle$	15	14
$L_{15,2}$	$\langle xy + zw, y^2 + xz + xw^3 + w^5 \rangle$	15	13
$L_{16,1}$	$\langle xy + zw, y^2 + xz + w^5 + yw^4 \rangle$	16	15
$L_{16,2}$	$\langle xy + zw, y^2 + xz + w^5 + yw^3 \rangle$	16	14

TABLE 10

Proposition 3.19. *Let* $(V(\langle f,g\rangle),0) \subseteq (\mathbb{C}^4,0)$ *be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the 2-jet of* $\langle f,g\rangle$ *has normal form* $\langle wz+xy, y^2+xz\rangle$. $(V(\langle f,g\rangle),0)$ *is unimodular if and only if it is isomorphic to a complete intersection in Tables 9 and 10.*

Algorithm 5 `Lsingularity(I)`

Input: $I = \langle f,g\rangle \in \langle x,y,z,w\rangle^2\mathbb{C}[[x,y,z,w]]$ such that 2-jet of I has normal form $\langle wz+xy, y^2+xz+w^3\rangle$
Output: the type of the singularity

compute μ =Milnor number of I, τ =Tjurina number of I and B =the singularity type of the strict transform of I in the blowing up of $\langle x,y,z,w\rangle$
if $\mu = 10$ and $B = 1$ **then**
  if $\mu = \tau$ **then**
    return (L_{10});
  if $\mu - \tau = 1$ **then**
    return $(L_{10,1})$;
if $\mu = 11$ and $B = A[1]$ **then**
  if $\mu = \tau$ **then**
    return (L_{11});
  if $\mu - \tau = 1$ **then**
    return $(L_{11,1})$;
if $\mu = 13$ and $B = A[3]$ **then**
  if $\mu = \tau$ **then**
    return $(L_{1,0})$;
  if $\mu - \tau = 1$ **then**
    return $(L_{1,0,1})$;
if $\mu = 15$ and $B = D[5]$ **then**
  if $\mu = \tau$ **then**
    return (L_{15});
  if $\mu - \tau = 1$ **then**
    return $(L_{15,1})$;
  if $\mu - \tau = 2$ **then**
    return $(L_{15,2})$;
if $\mu = 16$ and $B = E[6]$ **then**
  if $\mu = \tau$ **then**
    return (L_{16});
  if $\mu - \tau = 1$ **then**
    return $(L_{16,1})$;
  if $\mu - \tau = 2$ **then**
    return $(L_{16,2})$;
if $\mu - \tau = 2$ and $\mu > 13$ **then**
  if $B = D[\mu - 10]$ **then**
    return $(L_{1,\mu-13})$;
  if $B = A[\mu - 10]$ **then**
    return $(L^b_{1,\mu-13})$;
return (*not unimodular*);

3.6. **M singularities.** Assume the $2-jet$ of $\langle f,g\rangle$ has normal form $\langle wy+x^2-z^2,wx\rangle$. In this case According to C.T.C.Wall's classification the unimodular surface singularities with Milnor number μ are given in the table below.

$Name$	$Normal form$	μ	τ
M_{11}	$\langle wy+x^2-z^2,wx+y^3\rangle$	11	11
$M_{1,0}$	$\langle wy+x^2-z^2,wx+y^2x+\lambda y^2z\rangle$	13	13
$M_{1,i}$	$\langle wy+x^2-z^2,wx+y^2x+zI_{i+1}(z,y)\rangle$	$13+i$	$13+i-2$
M_{15}	$\langle wy+x^2-z^2,wx+y^4\rangle$	15	15

TABLE 11

Proposition 3.20. *The unimodular complete intersection surface singularities with Milnor number $\mu=11$ are M_{11} with Tjurina number $\tau=11$ defined by the ideal*

$$\langle wy+x^2-z^2,wx+y^3\rangle$$

and $M_{11,1}$ with Tjurina number $\tau=10$ defined by the ideal

$$\langle wy+x^2-z^2,wx+y^3+y^2w\rangle.$$

Proof. In C.T.C. Wall's list M_{11} is the singularity defined by the ideal

$$\langle wy+x^2-z^2,wx+y^3\rangle$$

with Milnor number $\mu=11$ and Tjurina number $\tau=11$. The versal deformation of M_{11} is

$$\langle wy+x^2-z^2+t_1y^2+t_2y+t_3,wx+y^3+\lambda_1y^2w+\lambda_2yw+\lambda_3w+\lambda_4yz+\lambda_5z+\lambda_6y^2+\lambda_7y+\lambda_8\rangle.$$

M_{11} defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1,w_2,w_3,w_4)=(4,3,4,5)$$

and the degrees

$$(d_1,d_2)=(8,9).$$

The versal μ-constant deformation of M_{11} is given by

$$\langle wy+x^2-z^2,wx+y^3+\lambda_1y^2w\rangle.$$

Using the coordinate change $x\to\xi^4x$, $y\to\xi^3y$, $z\to\xi^4z$ and $w\to\xi^5w$ we obtain

$$\langle wy+x^2-z^2,wx+y^3+\lambda_1\xi y^2w\rangle.$$

Choosing ξ such that $\lambda_1\xi=1$ we obtain

$$\langle wy+x^2-z^2,wx+y^3+y^2w\rangle$$

with different Tjurina number $\tau=10$ from M_{11}. □

Proposition 3.21. *The unimodular complete intersection surface singularities with Milnor number $\mu=13$ are $M_{1,0}$ with Tjurina number $\tau=13$ defined by the ideal*

$$\langle wy+x^2-z^2,wx+y^2x+\lambda y^2z\rangle$$

$M_{1,0,1}$ with Tjurina number $\tau=12$ defined by the ideal

$$\langle wy+x^2-z^2,wx+y^2x+\lambda y^2z+y^3z\rangle.$$

Proof. In the list of C.T.C Wall $M_{1,0}$ is the singularity defined by the ideal

$$\langle wy + x^2 - z^2, wx + y^2x + \lambda y^2 z\rangle.$$

with Milnor number $\mu = 13$ and Tjurina number $\tau = 13$. The versal deformation of $M_{1,0}$ is

$$\langle wy + x^2 - z^2 + t_1y^2 + t_2y + t_3, wx + y^2x + \lambda y^2z + \lambda_1 yw + \lambda_2 w + \lambda_3 y^3 z + \lambda_4 y^2 z + \lambda_5 yz + \lambda_6 z + \lambda_7 y^3 + \lambda_8 y^2 + \lambda_9 y + \lambda_{10}\rangle.$$

$M_{1,0}$ defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (6, 4, 6, 8)$$

and the degrees

$$(d_1, d_2) = (12, 14)$$

The versal μ-constant deformation of $M_{1,0}$ is given by

$$\langle wy + x^2 - z^2, wx + y^2x + \lambda y^2 z + \lambda_3 y^3 z\rangle.$$

Using the coordinate change $x \to \xi^6 x$, $y \to \xi^4 y$, $z \to \xi^6 z$, $w \to \xi^8 w$ we obtain

$$\langle wy + x^2 - z^2, wx + y^2x + \lambda y^2 z + \lambda_3 \xi^4 y^3 z\rangle.$$

Choosing ξ such that $\lambda_3\xi^4 = 1$

$$\langle wy + x^2 - z^2, wx + y^2x + \lambda y^2 z + y^3 z\rangle.$$

with Tjurina number $\tau = 12$. □

Proposition 3.22. *The unimodular complete intersection surface singularities with Milnor number $\mu = 15$ are M_{15} with Tjurina number $\tau = 15$ defined by the ideal*

$$\langle wy + x^2 - z^2, wx + y^4\rangle,$$

$M_{15,1}$ with Tjurina number $\tau = 14$ defined by the ideal

$$\langle wy + x^2 - z^2, wx + y^4 + y^2w\rangle$$

and $M_{15,2}$ with Tjurina number $\tau = 13$ defined by the ideal

$$\langle wy + x^2 - z^2, wx + y^4 + y^3w\rangle.$$

Proof. In the list of C.T.C Wall M_{15} is the singularity defined by the ideal

$$\langle wy + x^2 - z^2, wx + y^4\rangle$$

with Milnor number $\mu = 15$ and Tjurina number $\tau = 15$. The versal deformation of M_{15} is

$$\langle wy + x^2 - z^2 + t_1y^2 + t_2y + t_3, wx + y^4 + \lambda_1 y^3 w + \lambda_2 y^2 w + \lambda_3 yw + \lambda_4 w + \lambda_5 y^2 z + \lambda_6 yz + \lambda_7 z + \lambda_8 y^3 + \lambda_9 y^2 + \lambda_{10} y + \lambda_{11}\rangle.$$

M_{15} defines a weighted homogenous isolated complete intersection singularity with weights

$$(w_1, w_2, w_3, w_4) = (5, 3, 5, 7)$$

and the degrees

$$(d_1, d_2) = (10, 12).$$

The versal μ-constant deformation of M_{15} is given by

$$\langle wy + x^2 - z^2, wx + y^4 + \lambda_1 y^3 w + \lambda_2 y^2 w\rangle.$$

If $\lambda_2 \neq 0$ then we have

$$I = \langle wy + x^2 - z^2, wx + y^4 + uy^2 w\rangle$$

where $u = \lambda_1 y + \lambda_2$. Using the coordinate change $x \to \xi^6 x$, $y \to \xi^4 y$, $z \to \xi^6 z$ and $w \to \xi^8 w$ we obtain

$$\langle wy + x^2 - z^2, wx + y^4 + u\xi y^2 w\rangle.$$

Choosing $u\xi^2 = 1$ we obtain

$$\langle wy + x^2 - z^2, wx + y^4 + y^2 w\rangle$$

with Tjurina number $\tau = 14$. If $\lambda_2 = 0$ then again by the same transformation we obtain

$$\langle wy + x^2 - z^2, wx + y^4 + y^3 w\rangle$$

with Tjurina number $\tau = 13$ by choosing $\lambda_1 \xi^2 = 1$. □

Summarizing the results of the above prepositions we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal forms $\langle wy + x^2 - z^2, wx\rangle$.

$Type$	$Normal form$	μ	τ
$M_{11,1}$	$\langle wy + x^2 - z^2, wx + y^3 + y^2 w\rangle$	11	10
$M_{1,0,1}$	$\langle wy + x^2 - z^2, wx + y^2 x + \lambda y^2 z + y^3 z\rangle$	13	12
$M_{15,1}$	$\langle wy + x^2 - z^2, wx + y^4 + y^3 w\rangle$	15	14
$M_{15,2}$	$\langle wy + x^2 - z^2, wx + y^4 + y^2 w\rangle$	15	13

TABLE 12

Proposition 3.23. *Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^4, 0)$ be the germ of a complete intersection surface singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle wy + x^2 - z^2, wx\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Tables 11 and 12.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.20 - 3.22 □

We summarize our approach in this case in Algorithm 6.

Algorithm 6 `Msingularity(I)`

Input: $I = \langle f, g\rangle \subseteq \langle x, y, z, w\rangle^2\mathbb{C}[[x, y, z, w]]$ such that 2-jet of I has normal form $\langle 2wy + x^2 - z^2, 2wx\rangle$

Output: the type of the singularity

compute μ =Milnor number of I;
compute τ =Tjurina number of I;
compute B =the singularity type of the strict transform of I in the blowing up of $\langle x, y, z, w\rangle$
if $\mu = 11$ and $B = 1$ **then**
    if $\mu = \tau$ **then**
        return (M_{11});
    if $\mu - \tau = 1$ **then**
        return $(M_{11,1})$;
if $\mu = 13$ and $B = A[1]$ **then**
    if $\mu = \tau$ **then**
        return $(M_{1,0})$;
    if $\mu - \tau = 1$ **then**
        return $(M_{1,0,1})$;
if $\mu = 15$ and $B = D[5]$ **then**
    if $\mu = \tau$ **then**
        return (M_{15});
    if $\mu - \tau = 1$ **then**
        return $(M_{15,1})$;
    if $\mu - \tau = 2$ **then**
        return $(M_{15,2})$;
    if $\mu \neq \tau$ **then**
        if $\mu - \tau = 2$ and $\mu > 13$ **then**
            if $B = A[\mu - 11]$ **then**
                return $(M_{1,\mu-13})$;
return (not unimodular);

4. Main Algorithm

Now we give our main Algorithm 7 by using the all Algorithms given in section 3 by which we can compute the type of the singularity.

Algorithm 7 `classifyicis2(I)` [Unimodular surface singularities]

Input: $I = \langle f,g\rangle \subseteq \langle x,y,z,w\rangle^2\mathbb{C}[[x,y,z]]$ isolated complete intersection curve singularity.

Output: The type of the singularity $(V(I),0)$.

compute I_2 the 2-jet of I;
compute $I_2 = \bigcap_{i=1}^{s} Q_i$ the irredundant primary decomposition over $\mathbb{C}$;
compute d_i =Krull dimension of $\mathbb{C}[x,y,z,w]/Q_i$;
compute $h_i \in \mathbb{Q}[t]$ the Hilbert polynomial corresponding to each Q_i;
compute t number of absolute prime ideals of I_2 and j_i the number of conjugates.
if $s = 4$ **then**
  if $d_1 = d_2 = d_3 = d_4 = 1$ **then**
    if $h_1 = h_2 = h_3 = h_4 = 1 + t$ **then**
      return `Isingularity(I)`; via Algorithm 1
if $s = 3$ **then**
  if $d_1 = d_2 = d_3 = 2$ **then**
    if $h_1 = 1 + 2t$ and $h_2 = h_3 = 1 + t$ **then**
      if $t = 3$, $j_1 = j_2 = j_3 = 1$ **then**
        if $Q_3 \subseteq Q_1 + Q_2$ *and* Q_3 *has a generator of order* 2 **then**
          return `Msingularity(I)`; via Algorithm 6
        if $Q_3 \nsubseteq Q_1 + Q_2$ *and* Q_3 *has a generator of order* 2 **then**
          return `Tsingulrity(I)`; via Algorithm 2
if $s = 2$ **then**
  if $d_1 = d_2 = 2$ **then**
    if $h_1 = 1 + t = h_2 = 1 + 3t$ **then**
      return `Lsingularity(I)`; via Algorithm 5
    if $h_1 = h_2 = 1 + 2t$ **then**
      return `Tsingularity(I)`; via Algorithm 2
    if $h_1 = h_2 = 1 + 2t$ **then**
      return `Tsingularity(I)`; via Algorithm 2
if $s = 1$ **then**
  if $d_1 = 2$ and $h_1 = 4t$ **then**
    if $t = 1$, $j_1 = 1$ **then**
      if A_2 after blowing up **then**
        return `J′singularity(I)`; via Algorithm 3
      if A_1 after blowing up **then**
        return `Tsingularity(I)`; via Algorithm 2
    if $t = 2$ and $j_1 = 1, j_2 = 1$ **then**
      return `Tsingularity(I)`; via Algorithm 2
    if $t = 1$ and $j_1 = 2$ **then**
      return `K′singularity(I)`; via Algorithm 4
return (*not unimodular*);

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