BETTI NUMBERS FOR $p-$STABLE IDEALS

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Introduction

Let k be an infinite field of characteristic $p > 0$, $R = k[x_1, \cdots, x_n]$ the polynomial ring with the standard grading $R = \bigoplus_{d \geq 0} R_d$.

Let $I \subset R$ be a monomial ideal and $G(I)$ a minimal set of monomials generating I. Let u be a monomial. Denote by $m(u)$ the maximum integer i such that $x_i|u$. If x_i^s divides u but x_i^{s+1} does not, we write $x_i^s \parallel u$. An ideal I is *stable* if it is monomial and for every monomial $u \in I$ and $i < m(u)$ it holds $x_i\left(u/x_{m(u)}\right) \in I$. These ideals become well known after Eliahou - Kervaire described their minimal free resolutions (see[EK]) . The generic initial ideals are Borel fixed ideals (see[Ei]) , which are stable in characteristic zero. When $p = char\ k > 0$, they are not stable and they are called $p-$Borel ideals, their study being very difficult (see [AH_2]) . But the real analog of stable ideals in characteristic $p > 0$ are the ideals which we introduce below:

We consider on $\mathbb{N}$ the following partial order: we say that $k \leq_p s$ if the binomial coefficient $\binom{s}{k}$ is not divisible by p. Of course, $\leq_0$ is the usual order $\leq$ on $\mathbb{N}$. For a prime number p there is the following explicit description: $k \leq_p s$ if and only if $k_j \leq s_j$ for all $j \geq 0$, where k_j, s_j are defined by the $p-$adic expansion $k = \sum\limits_{j\geq 0} k_j p^j$, $s = \sum\limits_{j\geq 0} s_j p^j$ of k, respectively s (see [Ei]).

Definition. A monomial ideal I is called $p-stable$ if $x_i^k \left(u/x_{m(u)}^k\right) \in I$ for all monomial $u \in I$, $1 \leq i < m(u)$ and positive integers $k \leq_p s$, s being given by $x_{m(u)}^s \parallel u$.

In the first section we give some properties of $p-$stable ideals and describe the saturated $p-$stable ideals. In the Section 2 we compute the Koszul homology of a subclass of $p-$stable ideals, which are called *acceptable* $p-$stable ideals (see Definition 2.1 and Theorem 2.5.). This was done applying the method used by Aramova and Herzog to compute the Koszul homology for Cohen - Macaulay $p-$Borel principal ideals [AH_2]. As an application of Theorem 2.5 the Betti numbers, the Castelnuovo regularity and the Poincaré series of acceptable $p-$stable ideals are computed. Our formula for the Castelnuovo regularity is on the line given by Pardue Conjecture (see [Pa]) and it might hold for all products of $p-$powers of stable ideals as we asked in Question 3.8.

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1. $p-$stable ideals

As already mentioned in Introduction one considers the following partial order on $\mathbb{N}$:
$k \leq_p s$ if $\binom{s}{k}$ is not divisible by p, which is equivalent with the condition: $k_j \leq s_j$ for all $j \geq 0$, where $k = \sum_{j\geq 0} k_j p^j$ and $s = \sum_{j \geq 0} s_j p^j$ are the $p-$adic expansion of k, respectively s.

Definition 1.1. A monomial ideal $I \subset R$ is called $p-stable$ if every monomial $u \in I$ satisfies the following condition:
$x_i^k \left(u/x_{m(u)}^k\right) \in I$ for any $1 \leq i < m(u)$ and $k \leq_p s$, s being given by $x_{m(u)}^s \parallel u$ (shortly we say S_u^I holds).

Definition 1.2. A set $J \subset R_d$ of monomials is called $p-stable$ if for every monomial $u \in J$, S_u^J holds, that is:
$x_i^k \left(u/x_{m(u)}^k\right) \in J$ for any $1 \leq i < m(u)$ and $k \leq_p s$, s being given by $x_{m(u)}^s \parallel u$.

In this case we say that $< J >$, the $k-$space of R_d spanned by J, is $p-stable$ too. Clearly, a monomial ideal I is $p-stable$ if all its homogeneous components I_d have this property.

We start with a simple property of the above order on $\mathbb{N}$ which is very useful in the study of $p-stable$ ideals.

Lemma 1.3. *Let k, a, b be three positive integers such that $k \leq_p a+b$. Then there exists an integer c, $0 \leq c \leq k$, such that $k-c \leq_p a$ and $c \leq_p b$.*

Proof. It is enough to look at the coefficient of x^k in the expansion

$$(1+x)^{a+b} = (1+x)^a (1+x)^b . \qquad \square$$

Lemma 1.4. *Let $J \subset R_d$ be a $p-$ stable set of monomials. Then $J' := \{x_1, \cdots, x_n\}J \subset R_{d+1}$ is $p-$ stable.*

Proof. Let $v \in J$ be a monomial. Then S_v^J holds. One has to show that $S_{x_j v}^{J'}$ holds for every $1 \leq j \leq n$. Fix v, j and let s be given by $x_{m(x_j v)}^s \| x_j v$, $i < m(x_j v)$ and $k \leq_p s$. If $m(v) > j$ then $m(x_j v) = m(v)$ and $x_{m(v)}^s \parallel v$. Since S_v^J holds it follows $x_i^k \left(v/x_{m(v)}^k\right) \in J$ and so $x_i^k \left(x_j v/x_{m(v)}^k\right) \in J'$. If $m(v) < j$ then $m(x_j v) = j$, $s = k = 1$ and $x_i \left(x_j v/x_j\right) = x_i v \in J'$. Now suppose $m(v) = j$. Then $m(x_j v) = j$, $x_j^{s-1} \parallel v$. Apply Lemma 1.3 for the case $a = s-1$, $b = 1$. Then $k \leq_p a+b$ and there exists c such that $c \leq_p 1$ and $k - c \leq_p s-1$. Thus $c = 0$ or $c = 1$. If $c = 0$ then $x_i^k \left(x_j v/x_j^k\right) = x_j \; x_i^k \left(v/x_j^k\right) \in x_j \; J \subseteq J'$ because S_v^J holds. If $c = 1$ then $x_i^k \left(x_j v/x_j^k\right) = x_i x_i^{k-1} \left(v/x_j^{k-1}\right) \in x_i J \subseteq J'$. Thus $S_{x_j v}^{J'}$ holds. $\square$

Lemma 1.5. *Let I be a monomial ideal. The following conditions are equivalent:*

(i) I is $p-$ stable;

(ii) S_u^I holds for each monomial $u \in G(I)$, that is $x_i^k \left(u/x_{m(u)}^k\right) \in I$ for all monomial $u \in G(I)$, $1 \leq i < m(u)$ and positive integers $k \leq_p s$, s being given by $x_{m(u)}^s \parallel u$.

Proof. To show $(ii) \Rightarrow (i)$ (the other implication is trivial) it is enough to see that if $v \in\ I$ is a monomial such that S_v^I holds then $S_{x_jv}^I$ holds for every $1 \leq j \leq n$. But this follows from the proof of Lemma 1.4. $\square$

Lemma 1.6. *A finite product of $p-$ stable ideals is $p-$ stable. In particular a product of $p-$ powers of stable ideals is $p-$ stable.*

Proof. Let $I,\ J$ be two ideals. We show that S_{uv}^{IJ} holds for u, v monomials from I, respectively J. Let s be defined by $x_{m(uv)}^s \parallel uv,\ i < m(uv)$ and $k \leq_p s$. If $m(u) \neq m(v)$, let us say $m(u) < m(v)$, then $m(uv) = m(v)$ and $x_i^k\left(uv/x_{m(uv)}^k\right) = ux_i^k\left(v/x_{m(v)}^k\right) \in IJ$ because S_v^J holds. If $m = m(u) = m(v)$ then $m(uv) = m$ and let a, b be defined by $x_m^a \parallel u,\ x_m^b \parallel v$. Thus $s = a + b$ and by Lemma 1.3. we may find $0 \leq c \leq k$ such that $c \leq_p b,\ k - c \leq_p a$. Thus $x_i^k\left(uv/x_m^k\right) = x_i^{k-c}\left(u/x_m^{k-c}\right) x_i^c\left(v/x_m^c\right) \in IJ$ because $S_u^I,\ S_v^J$ hold. Thus S_{uv}^{IJ} holds. For the second statement we only note that a $p-$power $T^{[p]}$ of a stable ideal T is $p-$stable ($T^{[p^e]}$ denotes the ideal generated by f^{p^e} for $f \in T$). Let $u \in T$ and s an integer defined by $x_{m(u)}^s \parallel u$. Then $x_{m(u)}^{ps} \parallel u^p$. Let $i < m(u)$ and $k \leq_p ps$, in particular $k = pk^{'}$ for a $k^{'} \in \mathbb{N}$. Since T is stable we get by recurence $x_i^{k^{'}}\left(u/x_{m(u)}^{k^{'}}\right) \in T$ which gives $x_i^k\left(u^p/x_{m(u)}^k\right) \in T^{[p]}$. Thus $S_{u^p}^{T^{[p]}}$ holds. $\square$

Remark 1.7. In general one can show only that a $p-$stable ideal is a union of products of $p-$powers of stable ideals. If $n = 3$, $p = 2$ and $I \subset R$ is the ideal generated by $\{x_1^2, x_1x_2, x_2^2, x_3^2\}$ then I is not a product of 2-powers of stable ideals but $I = J \cup T^{[2]}$, for $J = (x_1^2, x_1x_2),\ T = (x_1, x_2, x_3)$, stable ideals (even lex ideals).

The following Lemma is inspired by [Ei,15.24].

Lemma 1.8. *Suppose that $I \subset R$ is a $p-$ stable ideal. Then*

$(I\ : x_n^{\infty}) = (I\ : (x_1, \cdots, x_n)^{\infty})$.

The proof goes as in [Ei, 15.24.]. $\square$

Corollary 1.9. *Let I be a $p-$ stable ideal. I is saturated if and only if no generator from $G(I)$ involves x_n.*

Proof. By Lemma 1.8. the saturation, I^{sat}, of I is $(I : x_n^{\infty})$. Suppose $(I : x_n^{\infty}) = I$. If $G(I)$ contains a monomial u such that $x_n^q \parallel u$ for some $q \geq 1$ then $u/x_n^q \in (I : x_n^q) \setminus I$, so $I \subsetneq (I : x_n^{\infty})$, contradiction. Conversely, let $v \in (I : x_n^{\infty})$, that is $vx_n^s \in I$ for some integer $s \geq 0$. Thus there exists $u \in G(I)$ such that $vx_n^s = uz$. Since $x_n \nmid u$, it results $x_n^s | z$ thus $v \in I$, so $(I : x_n^{\infty}) \subseteq I$. The other inclusion is trivial, so $(I : x_n^{\infty}) = I$. $\square$

Remark 1.10. It is known that a stable ideal I has the property: $(I : x_n) = (I : (x_1, ..., x_n))$ [Gr]. For $p-$stable ideals this equality does not hold, though x_n is still a generic linear form for them by [Po]. This shows the following:

Example: Let $n = 2$, $p = 2$ and $I = \left(x_1^2,\ x_2^2\right) \subset k[x_1, x_2]$. Then $x_2 \in (I : x_2)$ but $x_2 \notin (I : (x_1, x_2))$.

2. Koszul homology of acceptable $p-$stable ideals

The homological study of $p-$stable ideals is much easier if we restrict to a subclass of $p-$stable ideals, which is defined below.

Definition 2.1. A monomial ideal $I \subset R$ is called *acceptable $p-$stable* if there exist some stable ideals $(I_t)_{1 \leq t \leq s}$ of R and some integers $(r_t)_{1 \leq t \leq s}$, $r_s > ... > r_1 \geq 0$ such that

i) $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$,

ii) I_t contains $x_{m_{t+1}}^{(p^{r_{t+1}-r_t}-1)}$ for each $1 \le t < s$, where $m_t = \max_{u \in G(I_t)} m(u)$.

Lemma 2.2. *Let J, I be two monomial ideals of R, r a positive integer and $m = \max_{u \in G(I)} m(u)$. If J is stable and $x_m^{p^r - 1} \in J$, then*

i) $J \supset (x_1, \cdots, x_m)^{p^r - 1}$,

ii) $G(J\, I^{[p^r]}) = \{uv^{p^r} | u \in G(J),\ v \in G(I)\}$.

In particular, $\left|G(J\, I^{[p^r]})\right| = |G(J)|\, |G(I)|$.

Proof. i) follows easily by $x_m^{p^r - 1} \in J$ because J is stable. The set $T = \{uv^{p^r} | u \in G(J),\ v \in G(I)\}$ generates $JI^{[p^r]}$ and it is enough to see that T is a minimal system of generators. Suppose that $uv^{p^r} \in (x_1, \cdots, x_n)JI^{[p^r]}$ for some $u \in G(J)$, $v \in G(I)$, that is

1) $uv^{p^r} = u^{'} v^{'p^r} w$

for some $u^{'} \in G(J)$, $v^{'} \in G(I)$ and w a monomial $\neq 1$. Let $1 \le i \le n$ and $\alpha_i, \alpha_i^{'}, \beta_i, \beta_i^{'}, \ \gamma_i \ge 0$ be the integers defined by $x_i^{\alpha_i} \parallel u$, $x_i^{\alpha_i^{'}} \parallel u^{'}$, $x_i^{\beta_i} \parallel v$, $x_i^{\beta_i^{'}} \parallel v^{'}$, $x_i^{\gamma_i} \parallel w$. If $v = v^{'}$ then 1) gives $u \in (x_1, \cdots, x_n)J$ which is not possible ($u \in G(J)!$) . Thus $v \neq v^{'}$and so there exist $1 \le j \le m$ such that $\beta_j < \beta_j^{'}$, otherwise it follows $v \in (x_1, \cdots, x_n)v^{'} \subset (x_1, \cdots, x_n)I$ which contradicts $v \in G(I)$. By 1) we have $\alpha_j = \alpha_j^{'} + p^r \left(\beta_j^{'} - \beta_j\right) + \gamma_j$. Since our hypothesis in the form of i) gives $x_j^{p^r(\beta_j^{'} - \beta_j)} \in (x_1, \cdots, x_n)J$ we obtain $u \in \left(x_j^{\alpha_j}\right) \subset (x_1, \cdots, x_n)J$ which contradicts $u \in G(I)$. Thus T is minimal. □

Proposition 2.3. *Let $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ be an acceptable $p-$stable ideal as in Definition 2.1. Then $G(I) = \{\prod_{t=1}^{s} u_t^{p^{r_t}} |\ u_t \in G(I_t), 1 \le t \le s\}$. In* *particular,* $|G(I)| = \prod_{t=1}^{s} |G(I_t)|$.

Proof. Apply induction on s. If $s = 1$ then clearly $G\left(I_1^{[p^{r_1}]}\right) = \{u_1^{p^r} | u_1 \in G(I_1)\}$. If $s > 1$ then denote $E = \prod_{t=2}^{s} I_t^{[p^{r_t - r_2}]}$. By 2.1. ii) we see that $m(E) := \max_{v \in E} m(v) = m_2$ and $G\left(I_1 E^{[p^{r_2 - r_1}]}\right) = \{u_1 v^{p^{r_2 - r_1}} | u_1 \in G(I_1),\ v \in G(E)\}$ by Lemma 2.2. Now apply the induction hypothesis and repeat the argument from case $s = 1$. $\square$

Lemma 2.4. *Let* I, J *be two monomial ideals of* R *such that* $\left(x_1^{p^r - 1}, \cdots, x_m^{p^r - 1}\right) \subset J$ *for a positive integer* r *and* $m = \max_{u \in G(I)} m(u)$. *Let* F *be a free* $R-$*module of basis* $(\varepsilon_u)_{u \in G(I)}$. *Then for each* $j \geq 2$ *there exists an exact sequence*

$$0 \to F \otimes_R H_j(x; R/J) \xrightarrow{\gamma} H_j\left(x; R/JI^{[p^r]}\right) \xrightarrow{\tau} H_j\left(x; R/I^{[p^r]}\right) \to 0$$

where $H_j(x; R/J)$ *denotes the Koszul homology defined by* $x = (x_1, \cdots, x_n)$ *on* R/J, τ *is canonical and* γ *is given by* $[z]\varepsilon_u \mapsto [u^{p^r} z]$ *for a cycle* $z \in Z_j(x; R/J)$.

Proof. Here we may assume $k = \mathbb{Z}/p\mathbb{Z}$. Applying the Koszul homology for the exact sequence

$$0 \to I^{[p^r]}/JI^{[p^r]} \to R/JI^{[p^r]} \to R/I^{[p^r]} \to 0$$

we obtain the exact sequence

$$\text{1)} \qquad H_{j+1}(x; R/I^{[p^r]}) \xrightarrow{\delta_{j+1}} H_j\left(x; I^{[p^r]}/JI^{[p^r]}\right) \xrightarrow{\rho_j} H_j(x; R/JI^{[p^r]}) \to$$
$$\to H_j\left(x; R/I^{[p^r]}\right) \xrightarrow{\delta_j} H_{j-1}\left(x; I^{[p^r]}/JI^{[p^r]}\right), \qquad 2 \leq j.$$

By [BH, Corollary 1.6.13] we have

$$H_j(x; R/I) \cong H_j(x_1, \cdots, x_m; R/(I,\ x_{m+1}, \cdots,\ x_n)))$$

because $\{x_{m+1}, \cdots, x_n\}$ is a regular system on R/I.

Thus a cycle $y \in Z_j\,(x; R/I)$ has the form

$$y = \sum_{\substack{\sigma = (i_1, \cdots, i_j) \\ 1 \le i_1 < i_2 < \cdots < i_j \le m}} \hat{a}_\sigma e_\sigma, \quad e_\sigma = e_{i_1} \wedge \cdots \wedge e_{i_j}, \quad a_\sigma \in R$$

and $\hat{a}_\sigma = a_\sigma \pmod{I}$. By [AH$_2$;Lemma 1.5] there exists a bijection

$$H_j\,(x;\ R/I) \to H_j\left(x; R/I^{[p]}\right)$$

acting on a basis by $[y] \to [y^p]$, where y^p denotes the cycle

$$\sum_{|\sigma| = j} \hat{a}_\sigma^p x_\sigma^{p-1} e_\sigma \in Z_j\left(x; R/I^{[p]}\right),\ x_\sigma^{p-1} = \left(x_{i_1} \cdots x_{i_j}\right)^{p-1}.$$

Using the recurrence we see that $[y^{p^r}], y \in Z_j\,(x; R/I)\,, y^{p^r} = \sum_{|\sigma| = j} \hat{a}_\sigma^{p^r} x_\sigma^{p^r - 1} e_\sigma$, form a basis in $H_j\left(x; R/I^{[p^r]}\right)$. Since y is a cycle we have $\partial_j y = 0$, that is

$$\sum_{|\tau| = j-1} \left(\sum_{\substack{\sigma \supset \tau \\ |\sigma| = j}} (-1)^{\alpha(\sigma, \sigma \setminus \tau)} x_{\sigma \setminus \tau} \hat{a}_\sigma \right) e_\tau = 0,$$

where $\alpha(\sigma, q) = |\{\lambda \in \sigma | \lambda < q\}|$. Thus

2) $\qquad \omega_\tau := \sum_{\substack{\sigma \supset \tau \\ |\sigma| = j}} (-1)^{\alpha(\sigma, \sigma \setminus \tau)} x_{\sigma \setminus \tau} a_\sigma \in I.$

It follows

$$\delta_j([y^{p^r}]) = [\partial_j(y^{p^r})] = \left[\sum_{|\tau| = j-1} \left(\sum_{\substack{\sigma \supset \tau \\ |\sigma| = j}} (-1)^{\alpha(\sigma, \sigma \setminus \tau)} x_{\sigma \setminus \tau}\ \bar{a}_\sigma^{p^r} x_\sigma^{p^r - 1} \right) e_\tau \right]$$

$$= \left[\sum_{|\tau| = j-1} \bar{\omega}_\tau^{p^r} x_\tau^{p^r - 1} e_\tau \right] = 0$$

($(\bar{\ })$ denotes the class modulo $JI^{[p^r]}$), because by 2) and the assumption we have $\omega_\tau^{p^r} x_\tau^{p^r - 1} \in JI^{[p^r]}$, $\tau = (i_1, \cdots, i_{j-1})$, $i_1 < \cdots < i_{j-1} \le m$, $j \ge 2$ (here is necessary $j \ge 2$ because for $j = 1$, $x_\tau = 1 \notin J$!). Thus $\delta_j = 0$ for $j \ge 2$, that gives the following exact sequence from 1):

3) $0 \to H_j(x; I^{[p^r]}/JI^{[p^r]}) \overset{\rho_j}{\to} H_j(x; R/JI^{[p^r]}) \to H_j(x; R/I^{[p^r]}) \to 0,\ \ j \geq 2.$

Now we claim that

4) $\qquad I^{[p^r]}/JI^{[p^r]} \cong F \otimes_R R/J.$

Let $R^{q_1} \overset{(c_{ij})}{\to} R^{q_0} \to I \to 0,\ \ q_0 = |G(I)|\,,$ be a minimal presentation of I, where the matrix (c_{ij}) has all entries in $(x_1, ..., x_m)$. Since the Frobenius homomorphism of R is a flat $k-$ endomorphism ($k = \mathbb{Z}/p\mathbb{Z}$!) of R we see by [Ku] that

5) $\qquad R^{q_1} \overset{\left(c_{ij}^{p^r}\right)}{\to} R^{q_0} \to I^{[p^r]} \to 0$

is exact. But $c_{ij}^{p^r} \in \left(x_1^{p^r}, \cdots, x_m^{p^r}\right) \subset J$ by hypothesis. Thus tensorizing 5) with $R/J \otimes_R -$ we get our claim. Moreover the isomorphism is given by $\varepsilon_u \to u^{p^r}$ and together with the canonical map ρ_j induce the wanted γ. $\square$

Theorem 2.5. *Let* $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ *be an acceptable* $p-$ *stable as in Definition 2.1. Then a basis in* $H_j\,(x; R/I)\,,\ j \geq 2$ *is given by the homology classes of the cycles of* $Z_j\,(x; R/I)$

$(\star) \qquad \prod_{t\,>\,q} u_t^{p^{r_t}} f(\sigma; u_q)^{p^{r_q}},$

where $1 \leq q \leq s$, $|\sigma| = j - 1$, $\sigma = (i_1, \cdots i_{j-1})$, $i_1 < \cdots < i_{j-1} < m(u_q)$, $f(\sigma; u_q) = (u_q/x_{m(u_q)})e_\sigma \wedge\ e_{m(u_q)},\ \ u_t \in G(I_t)$.

Proof. Apply induction on s. If $s = 1,\ r_1 = 0$ then $I = I_1$ is stable and by [AH$_1$, Proposition 2.0.1.], $H_j\,(x; R/I)\,,\ j \geq 2$, has a basis given by $f(\sigma; u_q)$. If $s = 1,\ r_1 > 0$ then $f(\sigma, u_q)^{p^{r_1}}$ form a basis as we have seen (apply [AH$_2$, 1.5]).

Suppose $s > 1$. By [AH$_2$, 1.5] we may assume $r_1 = 0$ as before. Apply Lemma 2.4. for $J = I_1,\ \tilde{I} = \prod_{t=2}^{s} I_t^{p^{r_t - r_2}}$. We have the following exact sequence

$$0 \to \bigoplus_{v\in G(\tilde I)} H_j(x;R/J)\varepsilon_v \xrightarrow{\gamma} H_j(x;R/J\tilde I^{[p^{r_2}]}) \to H_j(x;R/\tilde I^{[p^{r_2}]}) \to 0.$$

By induction hypothesis the cycles of type $(\star)$ with $q>1$ induce a basis in $H_j(x;R/\tilde I^{[p^{r_2}]})$ and it remains to show that the cycles of type $(\star)$ with $q=1$ induce a basis in $Im\ \gamma$. Since $\{f(\sigma;u_1)\}_{\sigma,u_1}$ induce a basis in $G(J)$ as in case $s=1$ and $G(\tilde I) = \{\prod_{t=2}^{s} u_t^{p^{r_t-r_2}} | u_t \in G(I_t)\}$, by Proposition 2.3, we see that indeed the cycles of type $(\star)$ with $q=1$ have the form $\{v^{p^{r_2}}f(\sigma;u_1)\}_{v\in G(\tilde I),\sigma,u_1}$ which corresponds bijectively by γ with the basis, $\{f(\sigma;u_1)\varepsilon_v\}_{v\in G(\tilde I),\sigma,u_1}$, of $\bigoplus_{v\in G(\tilde I)} H_j(x;R/J)\varepsilon_v$. $\square$

The following corollary is an extension of [AH$_2$, Theorem 1.6.].

Corollary 2.6. *Let $(q_t)_{1\le t\le s}$, $(r_t)_{1\le t\le s}$, $(a_t)_{1\le t\le s}$ be some integers such that $1\le q_s\le\cdots\le q_1\le n$, $0\le r_1<\cdots<r_s$, $0\le a_t<p^{r_{t+1}-r_t}$ for $1\le t<s$, $a_s\ge 0$, and let $I=\prod_{t=1}^{s}((x_1,\cdots,x_{q_t})^{a_t})^{[p^{r_t}]}$. Then a basis in $H_j(x;R/I)$, $j\ge 2$ is given by the homology classes of the cycles of $Z_j(x;R/I)$, $\prod_{t>c} u_t^{p^{r_t}} f(\sigma;u_c)^{p^{r_c}}$, where $1\le c\le s$, $\sigma=(i_1,...,i_{j-1})$, $1\le i_1<\cdots<i_{j-1}<m(u_c)$, $u_t\in\{x_1,\cdots,x_{q_t}\}^{a_t}$, $f(\sigma;u_c)=(u_c/x_{m(u_c)})e_\sigma\wedge e_{m(u_c)}$.*

Corollary 2.7. *Let $(q_t)_{1\le t\le s}$, $(a_t)_{1\le t\le s}$ be some integers such that $1\le q_s\le q_{s-1}\le\cdots\le q_1\le n$, $0\le a_t<p$ and I the smallest $p-$stable ideal containing $\prod_{t=1}^{s} x_{q_t}^{p^{t-1}a_t}$. Then a basis in $H_j(x;R/I)$, $j\ge 2$ is given by the homology classes of the cycles of $Z_j(x;R/I)$, $\prod_{t>c} u_t^{p^{t-1}} f(\sigma;\ u_c)^{p^{c-1}}$, where $1\le c\le s$, $\sigma=(i_1,\cdots,i_{j-1})$, $1\le i_1<\cdots<i_{j-1}<m(u_c)$, $u_t\in\{x_1,\cdots,x_{q_t}\}^{a_t}$, $f(\sigma,u_c)=(u_c/x_{m(u_c)})e_\sigma\wedge e_{m(u_c)}$.*

For the proof note that the smallest $p-$stable ideal containing $x_{q_t}^{a_t}$ is $(x_1,\cdots,x_{q_t})^{a_t}$ and so $I=\prod_{t=1}^{s}((x_1,\cdots,x_{q_t})^{a_t})^{[p^{t-1}]}$. Now we may apply 2.6.

Remark 2.8. If in Corollary 2.7, $q_1 = \cdots = q_s = n$ and $a = \sum\limits_{i=1}^{s} a_i p^{i-1}$ then we obtain a basis in $H_j(x; R/I)$, where I is the smallest $p-$stable ideal containing x_n^a. Clearly I is a $p-$Borel ideal and so Corollary 2.7 coincides in this case with [AH$_2$, Theorem 1.6].

3. Applications of Theorem 2.5

In this section $I = \prod\limits_{t=1}^{s} I_t^{[p^{r_t}]}$ is an acceptable $p-$stable ideal as in Definition 2.1. In particular, I_t is stable and contains $\left(x_1, \cdots, x_{m_{t+1}}\right)^{(p^{r_{t+1} - r_t} - 1)}$, $m_t = \max\limits_{u_t \in G(I_t)} m(u_t)$. Let $\beta_j(R/I)$, $j \in \mathbb{N}$, be the j^{-th} Betti number of R/I over R, that is

$$\beta_j(R/I) = \dim_k Tor_j^R(k, R/I) = \dim_k H_j(x; R/I).$$

Theorem 3.1.

$$\beta_j(R/I) = \sum_{q=1}^{s} \sum_{u_q \in G(I_q)} \left(\prod_{t>q} |G(I_t)| \right) \binom{m(u_q)}{j-1}$$

For the proof note that if we fix $u_q \in G(I_q)$, $1 \le q \le s$ there exist exactly $\prod\limits_{t>q} |G(I_t)| \binom{m(u_q)-1}{j-1}$ cycles of type $(\star)$ from Theorem 2.5.

Corollary 3.2.

$$\mathrm{pd}_R(R/I) = m_1 = \mathrm{pd}_R R/I_1,$$

$$\mathrm{depth}_R(R/I) = n - m_1 = \mathrm{depth}_R R/I_1.$$

For the proof note that $\beta_j(R/I) = 0$ if and only if $j > m_1$ since $m_1 \ge m_2 \ge \cdots \ge m_s$.

Theorem 3.3. *Let* $d_t = \max\limits_{u_t \in G(I_t)} \deg u_t$ *and* $\mathrm{reg}(I)$ *be the Castelnuovo regularity of* I. *Then*

$$\text{reg } I \;=\; \max_{1\leq q\leq s}\{\sum_{t>q}^{s} p^{r_t}d_t+\max_{u_q\in G(I_q)}[\,p^{r_q}\deg u_q+(p^{r_q}-1)(m(u_q)-1)]\}$$

Proof. Let a_j be the largest integer for which $H_j(x;R/I)_{a_j}\neq 0$. Thus a_j is the biggest degree possible in the cycles of type $(\star)$ from Theorem 2.5. Note that

$$\deg\; f(\sigma;u_q)^{p^{r_q}}=\deg\left(u_q/x_{m(u_q)}e_\sigma\wedge e_{m(u_q)}\right)^{p^{r_q}}=$$

$$=p^{r_q}(\deg u_q-1)+\deg\left(x_\sigma^{p^{r_q}-1}x_{m(u_q)}^{p^{r_q}-1}e_\sigma\wedge e_{m(u_q)}\right)=$$

$$=p^{r_q}(\deg u_q-1)+(p^{r_q}-1)j+j,\;\; j\leq m(u_q),$$

and so

$$a_j=\max_{1\leq q\leq s}\{\sum_{t>q}p^{r_t}d_t+p^{r_q}\max_{\substack{u_q\in G(I_q)\\ j\leq m(u_q)}}(\deg u_q-1)+(p^{r_q}-1)j+j\}\,.$$

Since reg $I\;=\;\max\limits_{j}\{a_j-j+1\}$ we are done. $\square$

Remark 3.4. i) Trying to simplify the formula of reg$I$, one might believe that reg I is reached on the cycle of type $(\star)$ with $q=1$. This is not the case, let $s=2, r_1=0, r_2=1, I_1=I_2=(x_1,\cdots,x_n)$. Then the degree of the cycles of type $(\star)$ for $q=1$ is $p+1$ and the degree of $f\left((1,\cdots,n-1);x_n\right)^p$ is $p+(p-1)(n-1)$. Thus we have to keep first "max" in the formula of reg I.

ii) Also $\max\limits_{u_q\in G(\mathrm{I}_q)}\;[p^{r_q}\;\deg u_q+(p^{r_q}-1)(m(u_q)-1)]$ in the formula of reg I is not necessarily $p^{r_q}d_q+(p^{r_q}-1)(m_q-1)$.

Indeed, if $I_q=(x_1^2,x_1x_2,x_1x_3,x_2^3)$, $r_q=1$, then the maximum is reached for $u_q=x_2^3$, namely $4p-1$.

Corollary 3.4. *Suppose* $m_s=1$ *and let* $T=\prod\limits_{t\,=\,1}^{s-1}I_t^{[p^{r_t}]}$. *Then* reg $I\;=p^{r_s}d_s+$reg T.

Proof. Denote

$$D_q(I) = \sum_{t>q}^{s} p^{r_t} d_t + \max_{u_q \in G(I_q)} \left[p^{r_q} \deg u_q + (p^{r_q} - 1)(m(u_q) - 1) \right],$$

$1 \leq q \leq s$. Since $m_s = 1$ we have $D_s(I) = p^{r_s} d_s < D_q(I)$ for all $q < s$. Thus

reg $I = \max_{1 \leq q < s} D_q(I) = p^{r_s} d_s + \max_{1 \leq q \leq s-1} D_q(T) = p^{r_s} d_s + regT.$ $\square$

Remark 3.5. The above corollary is inspired by a result from [Pa] : Let $v = \prod_{i=1}^{n} x_i^{\mu_i}$ and $J(v)$ the $p-$Borel ideal generated by v, that is the smallest $p-$Borel ideal containing v. Then reg $J(v) = \mu_1 +$reg $J(\prod_{i=2}^{n} x_i^{\mu_i})$. Using this Corollary we may assume from now on $m_s \geq 2$.

Corollary 3.6. *Let* $(m_t)_{1 \leq t \leq s}$, $(r_t)_{1 \leq t \leq s}$, $(a_t)_{1 \leq t \leq s}$ *be some integers such that* $2 \leq m_s \leq \cdots \leq m_1 \leq n$, $0 \leq r_1 < \cdots < r_s$, $0 \leq a_t < p^{r_{t+1} - r_t}$ *for* $1 \leq t < s$, $a_s \geq 0$, *and let* $I = \prod_{t=1}^{s} ((x_1, ..., x_{m_t})^{a_t})^{[p^{r_t}]}$.
Then reg $I = \max_{1 \leq q \leq s} D_q(I)$, *where* $D_q(I) = \sum_{t=q+1}^{s} p^{r_t} a_t + (p^{r_q} - 1)(m_q - 1)$.

Remark 3.7. Suppose $r_t = t$, $1 \leq t \leq s$. Then $a_t < p$ and I is the smallest $p-$*stable* ideal (also $p-$Borel), which contains the monomial $v = \prod_{t=1}^{s} x_{m_t}^{p^t a_t}$ and $(D_q(I))_q$ coincide with the integers (D_{m_t}) associated to v by Pardue (see[Pa], or [AH$_2$,Conjecture 3.1.]. Corollary 3.6 says in particular that the Pardue Conjecture holds for monomials of type $v(m_s \leq \cdots \leq m_1!)$. Moreover, if $m_1 = \cdots = m_s$ then I is a $p-$Borel principal Cohen - Macaulay ideal (see [Pa]) for which the Pardue conjecture follows also by [AH$_2$].

Hoping in an extension of the Pardue Conjecture for $p-$stable ideals, we may ask the following:

Question 3.8. Let $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ be a product of $p-$powers of stable ideals, which is not necessarily acceptable. Then is reg I exactly $\max_{1 \leq q \leq s} D_q(I)$,

where $\qquad D_q(I) = \sum\limits_{t>q}^{s} p^{r_t} d_t + \max\limits_{u_q \in G(I_q)} \left[p^{r_q} \deg u_q + (p^{r_q} - 1)(m(u_q) - 1) \right],$
$1 \leq q \leq s$, where $d_t = \max\limits_{u_t \in G(I_t)} \deg u_t$?

Now, let $P(R/I, T)$ be the Poincaré series of R/I.

Theorem 3.9.

$$P\left(R/I, T\right) = \frac{1}{(1-T)^n} + \frac{1}{(1-T)^n} \sum_{q\,=\,2}^{s} \prod_{t \geq q} \Big(\sum_{u_t \in G(I_t)} T^{p^{r_t} \deg u_t} \Big) -$$
$$- \sum_{q\,=\,1}^{s} \sum_{u_q \in G(I_q)} \prod_{t>q} \Big(\sum_{u_t \in G(I_t)} T^{p^{r_t} \deg u_t} \Big) \frac{T^{p^{r_q} \deg u_q}}{(1-T)^{n - p^{r_q}(m(u_q)-1)}}$$

Proof. Let $L_* : ... \to L_j \to L_{j-1} \to \cdots L_1 \to L_0 \to R/I \to 0$

be the minimal free resolution of R/I over R. By exactness of the resolution it results:

$$P(R/I, T) = \sum_{j \geq 0} (-1)^j P(L_j, T).$$

Clearly, $L_0 \simeq R$, so $P(L_0, T) = \frac{1}{(1-T)^n}$, and L_1 has the basis

$G(I) = \{ \prod\limits_{q\,=\,1}^{s} u_q^{p^{r_q}} | u_q \in G(I_q),\ 1 \leq q \leq s \}$, thus

$P(L_1, T) = \frac{1}{(1-T)^n} \prod\limits_{q\,=\,1}^{s} \Big(\sum\limits_{u_q \in G(I_q)} T^{p^{r_q} \deg u_q} \Big)$, since

$\deg \prod\limits_{q\,=\,1}^{s} u_q^{p^{r_q}} = \sum\limits_{q\,=\,1}^{s} p^{r_q} \deg u_q.$

For all $j \geq 2$, by Theorem 2.5, L_j has the basis

$\{ \prod\limits_{t>q} u_t^{p^{r_t}} f(\sigma; u_q)^{p^{r_q}} | 1 \leq q \leq s,\ |\sigma| = j-1,\ \ \sigma = (i_1, \cdots, i_{j-1}),$
$i_{j-1} < m(u_q),\ u_t \in G(I_t) \}.$

Since $\deg u_t^{p^{r_t}} f(\sigma; u_q)^{p^{r_q}} = p^{r_t} \deg u_t + p^{r_q} \deg u_q + (j-1) p^{r_q}$, it follows:

$$P(L_j, T) = \sum_{q\,=\,1}^{s} \sum_{u_q \in G(I_q)} \frac{1}{(1-T)^n} \binom{m(u_q)-1}{j-1} T^{p^{r_q}(j-1) + p^{r_q} \deg u_q} \times$$
$$\times \prod_{t>q} \Big(\sum_{u_t \in G(I_t)} T^{p^{r_t} \deg u_t} \Big).$$

Then we obtain:

$$P(R/I,T) = \frac{1}{(1-T)^n} - \frac{1}{(1-T)^n} \prod_{q=1}^{s} \Big(\sum_{u_q \in G(I_q)} T^{p^{r_q} \deg u_q} \Big) +$$

$$+ \sum_{q=1}^{s} \sum_{u_q \in G(I_q)} T^{p^{r_q} \deg u_q} \prod_{t>q} \Big(\sum_{u_t \in G(I_t)} T^{p^{r_t} \deg u_t} \Big) \times$$

$$\times \sum_{j \geq 2} (-1)^j \binom{m(u_q)-1}{j-1} \frac{T^{p^{r_q}(j-1)}}{(1-T)^n}.$$

But $\sum_{j\geq 2} (-1)^j \binom{m(u_q)-1}{j-1} \frac{T^{p^{r_q}(j-1)}}{(1-T)^n} = \frac{1}{(1-T)^n} - \frac{1}{(1-T)^{n-p^{r_q}(m(u_q)-1)}}$.
Replacing this sum we obtain the desired formula. $\square$

Corollary 3.10. *In the hypothesis of Corollary 2.6,*

$$P\,(R/I,T) = \frac{1}{(1-T)^n} + \frac{1}{(1-T)^n} \sum_{c=2}^{s} \prod_{t \geq c} \binom{a_t+q_t-1}{q_t-1} T^{a_t p^{r_t}} -$$

$$- \sum_{c=1}^{s} \prod_{t>c} \binom{a_t+q_t-t}{q_t-1} T^{a_t p^{r_t}} \cdot \frac{T^{a_c p^{r_c}}}{(1-T)^{n+p^{r_c}}} \sum_{l=0}^{q_c-1} (1-T)^{(l+1)p^{r_c}} \binom{a_c+l}{l}.$$

Proof. In the hypothesis of Corollary 2.6, $\deg u_t = a_t$ for all $u_t \in G(I_t)$ and $|G(I_t)| = \binom{a_t+q_t-1}{q_t-1}$. $\square$

Corollary 3.11. *In the hypothesis of Corollary 2.7,*

$$P\,(R/I,T) = \frac{1}{(1-T)^n} + \frac{1}{(1-T)^n} \sum_{c=2}^{s} \prod_{t \geq c} \binom{a_t+q_t-1q_t-1}{T}^{p^{t-1}a_t} -$$

$$- \sum_{c=1}^{s} \prod_{t>c} \binom{a_t+q_t-1q_t-1}{T}^{p^{t-1}a_t} \frac{T^{p^{c-1}a_c}}{(1-T)^{n+p^{c-1}}} \sum_{l=0}^{q_c-1} (1-T)^{(l+1)p^{c-1}} \binom{a_c+l1}{.}$$

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