Infinitesimal module deformations in the Thom–Sebastiani Problem

To the memory of Hideyuki Matsumura

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Introduction

Let K be a field, $K[[x]]$ (respectively $K[[y]]$), be the formal power series in the variables $x = (x_1, \ldots, x_n)$ (respectively $y = (y_1, \ldots, y_m)$) and $f \in K[[x]]$, $g \in K[[y]]$ two non–zero, non–invertible formal power series. The problem of Thom–Sebastiani type for the categories of maximal Cohen–Macaulay modules (see [HP]) means studying the category of maximal Cohen–Macaulay modules over $K[[x,y]]/(f+g)$ in connection with the categories of maximal Cohen–Macaulay modules over $K[[x]]/(f)$ and $K[[y]]/(g)$, respectively. A very special case of this type of problem appears in Knörrer's paper [Kn], where the hypersurface singularities of type $f + y^2$, that is $g = y^2$, are studied. If char $K \neq 2$, then every maximal Cohen–Macaulay $R'' = K[[x,y]]/(f+y^2)$–module is a direct summand in the first syzygy over R'' of a certain maximal Cohen–Macaulay $R := K[[x]]/(f)$–module (see [Kn]). This result was very useful for proving that a hypersurface over an algebraically closed field of characteristic $\neq 2$ is simple if and only if it has a finite Cohen–Macaulay representation type (see [Kn], [BGS]).

Let $s \geq 2$ be an integer which is not a multiple of the characteristic of K. Then every maximal Cohen–Macaulay $R' := K[[x,y]]/(f+y^s)$–module N is a direct summand of the first syzygy $\Omega^1_{R'}(T)$ of $T := N/y^{s-1}N$ over R' (see [Po] or [HP](2.8)). Clearly T is a deformation of the maximal Cohen–Macaulay R–module $M := N/yN$ to $\widetilde{R} := R[y]/y^{s-1}) \cong R'/(y^{s-1})$. Thus, we may describe the maximal Cohen–Macaulay R'–modules by taking the first syzygies over R' of deformations of maximal Cohen–Macaulay R–modules to $\widetilde{R}$.

The purpose of this paper is to give some applications of the above results, when $s = 3$. (We also suppose K to be an algebraically closed field in some proofs.) In this situation, we know from [HP] (3.1) that it is enough to consider only those indecomposable infinitesimal deformations of M which are liftable to $R'/(y^4)$. Section 1 gives a homological characterization of this liftability and shows its connection with the characterization stated in [HP] (3.4). We also show that the correspondence $T \longrightarrow \Omega^1_{R'}(T)$ is "almost" injective and "almost" preserves indecomposability (see Theorem 1.7). In the following two sections we take $f = x^t$ and we are mainly interested in describing the maximal Cohen–Macaulay modules N over $K[[x,y]]/(x^t + y^3) =: R'$ such that $N/yN \cong P_i^d$ (where $P_i = K[[x]]/(x^i)$, $1 \leq i < t$, are all non–free indecomposable modules over $R = K[[x]]/(x^t)$ and $d \geq 1$ is an integer). We obtain a precise description (see Theorem 3.1), which completes the preliminary results from [HP] (4.2). To this intent we find in Section 2 all infinitesimal deformations of P_i^d, which are liftable to $R'/(y^4)$. Our Theorem 3.2 gives a necessary condition for an R–module M to have the form $M \cong N/yN$ for a certain maximal Cohen–Macaulay R'–module N.

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1 Liftability of infinitesimal deformations

Let K be a field of characteristic $\neq 3$, $K[[x]]$ be the formal power series ring over K in $x = (x_1, \ldots, x_n)$, $f \in K[[x]]$, $f \neq 0$ a non–invertible formal power series,

$$R := K[[x]]/(f),\ R' := K[[x,y]]/(f + y^3),\ \widetilde{R} := R[y]/(y^2) \simeq R'/(y^2).$$

Let M be a maximal Cohen–Macaulay R–module and $\xi \in \mathrm{Ext}^1_R(M,M)$ be represented by

$$0 \longrightarrow M \xrightarrow{i} T \xrightarrow{p} N \longrightarrow 0.$$

Then T has an $\widetilde{R}$–module structure given by $y \cdot 1_T = i \circ p$, and the complex $T \longrightarrow T \longrightarrow T \xrightarrow{p} M \longrightarrow 0$ is exact, that is T is an infinitesimal deformation in the sense of [HP]. T is liftable to $R'/(y^j)$, $j \geq 2$, if there exists an $R'/(y^j)$–module E such that $E/y^2E \cong T$ and the complex

$$E \xrightarrow{y^2} E \xrightarrow{y^{j-2}} E \xrightarrow{y^2} E$$

is exact (it follows $\mathrm{Tor}_\ell^{R/(y^j)}(\widetilde{R}, E) = 0$ for all $\ell \geq 1$).

Let $\mathcal{L}_M \subset \mathrm{Ext}^1_R(M,M)$ be the subset of those ξ represented by

$$0 \longrightarrow M \xrightarrow{i} T \xrightarrow{p} M \longrightarrow 0$$

for which the infinitesimal deformation T of M is liftable to $R'/(y^4)$. We are interested in studying $\mathcal{L}_M$, mainly because of the following basical result from [HP] (3.1) (see also [Po] and [HP] (2.8)).

Theorem 1.1
For every indecomposable maximal Cohen–Macaulay R'–module N there exists an indecomposable $\widetilde{R}$–module T such that:

(i) T is an infinitesimal deformation of a maximal Cohen–Macaulay R–module M to $\widetilde{R}$;

(ii) T is liftable to $R'/(y^4)$;

(iii) N is a direct summand of the first syzygy $\Omega^1_{R'}(T)$ of T over R'.

In the same paper it was also given a characterization of the liftability of an infinitesimal deformation T to $R'/(y^4)$ (see [HP] (3.4)). An homological characterization of the liftability to $R'/(y^4)$ is given in this section. We begin by introducing some notations, which will be useful also for the following sections.

Let $(A, \underline{a})$ be a noetherian local ring, $g \in \underline{a}$ a non–zero divisor of A and $B := A/(g)$. A pair of square d–matrices (φ, ψ) with entries in A satisfying $\varphi\psi = gI_d$, where I_d is the $d \times d$–unit matrix, is called a matrix factorization of g (we call (φ, ψ) reduced if the entries of φ, ψ are all in $\underline{a}$). Since g is a non–zero divisor, we have $\varphi\psi = gI_d$ if and only if $\psi\varphi = gI_d$. If $A = K[[x]]$, $g = f$ the matrix factorizations of f describe the maximal Cohen–Macaulay R–modules.

For every such non–free R–module M there exists a reduced matrix factorization (φ, ψ) of f which defines a minimal free resolution of M

$$R^d \xrightarrow{\varphi} R^d \xrightarrow{\psi} R^d \xrightarrow{\varphi} R^d \longrightarrow M \longrightarrow 0.$$

This is an important factor, which makes it easier to study the maximal Cohen–Macaulay R–modules over hypersurfaces (see [Ei] and [Yo], Ch. 4). If $A = K[[x,y]]/(y^2)$, $g = f$ the matrix factorizations of f describe (by [PP] (2.8) and also [Ru] (2.5.4)) the infinitesimal deformations of maximal Cohen–Macaulay R–modules in a similar way.

If (φ, ψ) is a reduced matrix factorization of f corresponding to the maximal Cohen–Macaulay R–module M, then a reduced matrix factorization of f over $K[[x,y]]/(y^2)$ corresponding to an infinitesimal deformation T of M has the form $(\varphi + y\alpha, \psi - y\beta)$ for some $\alpha, \beta \in M(d \times d, K[[x]])$ satisfying $\alpha\psi = \varphi\beta$ or, equivalently, $\psi\alpha = \beta\varphi$, in other words (α, β) is a morphism $(\psi, \varphi) \longrightarrow (\varphi, \psi)$ of matrix factorizations corresponding to a morphism $u : \Omega^1_R(M) \longrightarrow M$. It is easy to see that $\xi = \mathrm{Ext}^1_R(M, u)(\eta)$, η defined by

$$0 \longrightarrow \Omega^1_R(M) \longrightarrow L \longrightarrow M \longrightarrow 0,$$

the beginning of a minimal free resolution of M. Since $\alpha\beta(\varphi + y\alpha) = \alpha\psi\alpha + y\alpha\beta\alpha = (\varphi + y\alpha)\beta\alpha$ we have the following

Lemma 1.2
The pair $(\alpha\beta, \beta\alpha)$ is an endomorphism of the matrix factorization $(\varphi + y\alpha, \psi - y\beta)$, which corresponds to an $\widetilde{R}$–endormorphism θ of T.

The map θ is uniquely given by ξ as follows below. Let $\Omega^1_{R'}(T)$ be the first syzygy of T over R' and $0 \longrightarrow \Omega^1_{R'}(T) \longrightarrow F \longrightarrow T \longrightarrow 0$ a part from a minimal free R'–resolution of T. Tensorizing by $\widetilde{R} \otimes_{R'} -$ we obtain the following exact sequence

1) $\qquad 0 \longrightarrow T \longrightarrow \widetilde{R} \otimes_{R'} \Omega^1_{R'}(T) \longrightarrow \widetilde{R} \otimes_{R'} F \longrightarrow T \longrightarrow 0$

which yields the extension

(ε) $\qquad 0 \longrightarrow T \xrightarrow{v} \widetilde{R} \otimes_{R'} \Omega^1_{R'}(T) \longrightarrow \Omega^1_{\widetilde{R}}(T) \longrightarrow 0.$

The minimal free resolution of T over $\widetilde{R}$

2) $\qquad \ldots \longrightarrow \widetilde{L} \xrightarrow{\varphi + y\alpha} \widetilde{L} \xrightarrow{\psi - y\beta} \widetilde{L} \xrightarrow{\varphi + y\alpha} \widetilde{L} \xrightarrow{q} T \longrightarrow 0$

gives the following extension

(ν) $\qquad 0 \longrightarrow T \simeq \Omega^2_{\widetilde{R}}(T) \xrightarrow{w} \widetilde{L} \longrightarrow \Omega^1_{\widetilde{R}}(T) \longrightarrow 0$

(note that $\widetilde{R} \otimes_R F \simeq \widetilde{L}$).

Lemma 1.3
$\mathrm{Ext}^1_{\widetilde{R}}(\Omega^1_{\widetilde{R}}(T), \theta + y \cdot 1_T)(\nu) = \varepsilon.$

Proof: It is enough to show that there exists a cocartesian square

$$\begin{array}{ccc} T & \xrightarrow{w} & \widetilde{L} \\ {\scriptstyle \theta+y\cdot 1_T}\downarrow & & \downarrow \\ T & \xrightarrow{v} & \widetilde{R}\otimes_R \Omega^1_{R'}(T). \end{array}$$

By [HP] (3.1),

$$\tau = \begin{pmatrix} \psi - y\beta & 0 \\ \alpha\beta + yI_d & \varphi + y\alpha \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} \varphi + y\alpha & 0 \\ -(\beta\alpha + yI_d) & \psi - y\beta \end{pmatrix}$$

form a matrix factorization of f corresponding to $\widetilde{R}\otimes_R \Omega^1_{R'}(T)$ and if $j_2 \cdot \widetilde{L} \longrightarrow \widetilde{L}^2$ is the injection on the second summand $\widetilde{L}$, the pair (j_2, j_2) gives a matrix factorization morphism $(\varphi + y\alpha, \psi - y\beta) \longrightarrow (\tau, \sigma)$ corresponding to v.

Thus, we have:

$$\begin{aligned} \widetilde{R}\otimes_R \Omega^1_{R'}(T) &\cong \widetilde{L}^2/\langle(\psi - y\beta)(a),\ (\alpha\beta + yI_d)(a) + ((\varphi + y\alpha)(b))|a, b \in \widetilde{L}\rangle \\ &\cong \widetilde{L}\oplus T/\langle(\psi - y\beta)(a),\ (\theta + y1_T)(q(a))|a \in \widetilde{L}\rangle, \end{aligned}$$

which is enough. $\square$

We have a canonical isomorphism $\underline{\mathrm{Hom}}_{\widetilde{R}}(\Omega^2_{\widetilde{R}}(T), T) \xrightarrow{\sim} \mathrm{Ext}^2_{\widetilde{R}}(T, T)$ given by $\alpha \longrightarrow \mathrm{Ext}^2_{\widetilde{R}}(\Omega^2_{\widetilde{R}}(T), \alpha)(\tilde{\varepsilon})$ where $\tilde{\varepsilon}$ is defined by $0 \longrightarrow \Omega^2_{\widetilde{R}}(T) \longrightarrow \widetilde{L}_1 \longrightarrow \widetilde{L}_0 \longrightarrow T \longrightarrow 0$, is the beginning of the minimal free resolution of T over $\widetilde{R}$, and we denote by $\underline{\mathrm{Hom}}_{\widetilde{R}}$ the usual Hom in $\mathrm{Mod}\widetilde{R}/\{\widetilde{R}\}$ (for notations cf. [Yo]). Of course, in our case $\Omega^2_{\widetilde{R}}(T) \cong T$ and using the above Lemma, we obtain the following:

Proposition 1.4
The map induced in $\underline{End}_{\widetilde{R}}(T)$ by $\theta + y\cdot 1_T$ corresponds via the canonical isomorphism $\underline{End}_{\widetilde{R}}(T) \cong \mathrm{Ext}^2_{\widetilde{R}}(T, T)$ to 1) above. In particular, θ depends only on T.

Corollary 1.5

(i) $\xi \in \mathcal{L}_M$ (that is T is liftable to $R'/(y^4)$) if and only if $\theta + y1_T$ factorizes through a free $\widetilde{R}$–module.

(ii) T is liftable to $R'/(y^3)$ if and only if $R\otimes_{\widetilde{R}}\theta$ factorizes through a free R–module.

Proof:

(i) T is liftable to $R'/(y^4)$ if and only if ε splits (see [ADS] (1.5)), that is 1) is zero in $\mathrm{Ext}^2_{\widetilde{R}}(T, T)$. Now apply Proposition 1.4.

(ii) Tensorizing ε by $R\otimes_{\widetilde{R}} -$ we obtain the extension

$$0 \longrightarrow M \longrightarrow R\otimes_{R'} \Omega^1_{R'}(T) \longrightarrow M \longrightarrow 0.$$

$(\mathrm{Tor}_1^{\widetilde{R}}(R, \Omega^1_{\widetilde{R}}(T)) = 0)$, which splits if and only if T is liftable to $R'/(y^3)$ by [ADS] (1.5). The proof goes now as in (i). $\square$

Remark 1.6
Starting with Proposition 3.2 [HP], we can obtain another proof of the **Corollary 1.5**. We sketch this proof only for (i) (the proof of (ii) is similar).

In addition to the above notations, we take $S := K[[x,y]]/(y^2)$ and $\tilde{\varphi} := \varphi + y\alpha$, $\tilde{\psi} := \psi - y\beta$, such that $(\tilde{\varphi}, \tilde{\psi})$ gives a matrix factorization over S, corresponding to T.

Let $(\tilde{\varphi}_1, \tilde{\psi}_1)$ and $(\tilde{\varphi}_2, \tilde{\psi}_2)$ be two arbitrary matrix factorizations corresponding to T_1 and T_2, respectively ($\widetilde{T}_1, \widetilde{T}_2$ are both infinitesimal deformations of maximal Cohen–Macaulay modules). For a morphism (p,q) between $(\tilde{\varphi}_1, \tilde{\psi}_1)$ and $(\tilde{\varphi}_2, \tilde{\psi}_2)$ we denote by Coker (p,q) the canonical morphism induced between T_1 and T_2 (see [Yo], Chapter 7). From Proposition 3.2 [HP] we know that

(*) T is liftable to $R'/(y^4)$ if and only if there are $\gamma, \tau \in \mathcal{M}(d \times d, S)$ such that $\alpha\beta + yI_d = \tilde{\varphi}\tau + \gamma\tilde{\psi}$. Suppose T is liftable to $R'/(y^4)$.

We have $\theta + y \cdot 1_T = \mathrm{Coker}(\alpha\beta + y1_T, \beta\alpha + y \cdot 1_T)$ and, by (*), $\mathrm{Coker}(\alpha\beta + y \cdot 1_T, \beta\alpha + y \cdot 1_T) = \mathrm{Coker}(\gamma \cdot \tilde{\psi}, \tilde{\psi}\gamma)$ (notice that $T \simeq \mathrm{Coker}\,\tilde{\varphi}!$). Moreover, we have the following two commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & S^d \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{\psi}\gamma"'] & S^d \arrow[r] \arrow[d, "\gamma\tilde{\psi}"] & T \arrow[r] \arrow[d, "\theta + y \cdot 1_T"] & 0 \\
0 \arrow[r] & S^d \arrow[r] & S^d \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & S^d \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & S^d \arrow[r] \arrow[d, "\tilde{\psi}"] & T \arrow[r] \arrow[d, "\mathrm{Coker}(\tilde{\psi}, \mathrm{id})"] & 0 \\
0 \arrow[r] & S^d \arrow[r] \arrow[d, "\tilde{\psi}\gamma"'] & S^d \arrow[r] \arrow[d, "\gamma"] & \widetilde{R}^d \arrow[r] \arrow[d, "\mathrm{Coker}(\gamma, \tilde{\psi}\gamma)"] & 0 \\
0 \arrow[r] & S^d \arrow[r] & S^d \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

Comparing the above diagrams, we see that $\theta + y \cdot 1_T$ factorizes through a free $\widetilde{R}$–module. For the converse implication, let $\widetilde{L}$ be a free module and $T \xrightarrow{h} \widetilde{L} \xrightarrow{g} T$ two maps such that $\theta + y \cdot 1_T = g \circ h$ we have two commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & S^d \arrow[r, "\tilde{\varphi}"] \arrow[d, "h''"'] & S^d \arrow[r] \arrow[d, "h'"] & T \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L \arrow[r, "f \cdot 1_L"] & L \arrow[r] & \widetilde{L} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f \cdot 1_L"] \arrow[d, "g''"'] & L \arrow[r] \arrow[d, "g'"] & \widetilde{L} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & S^d \arrow[r] & S^d \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

with L a free S–module, and so we deduce

$$\begin{aligned} h''\tilde{\psi} &= h' \\ g'' &= \tilde{\psi} g' \end{aligned}$$

(notice that f is a non–zero divisor in S).

Unifying these diagrams, we find $\gamma \in M(d \times d, S)$ given by $g'h''$ such that $\theta + y \cdot 1_T =$ Coker $(\gamma\tilde{\psi}, \tilde{\psi}\gamma)$ which is enough, because of (*), for the liftability of T to $R'/(y^4)$.

Let now $\mathcal{T}_M \subset \mathcal{L}_M$ be the subset of those ξ

$$0 \longrightarrow M \longrightarrow T_\xi \longrightarrow M \longrightarrow 0$$

for which T_ξ is indecomposable.

We end this section by proving a result which shows the role of the sets $\mathcal{T}_M$ in the description of maximal Cohen–Macaulay modules over R'.

Theorem 1.7
The following statements hold for $\xi, \xi' \in \mathcal{T}_M$:

(i) $\Omega^1_{R'}(T_\xi) \cong \Omega^1_{R'}(T_{\xi'})$ *as R'–modules, if and only if either $T_\xi \cong T_{\xi'}$ or $T_\xi \cong \Omega^1_{\widetilde{R}}(T_\xi)$ as $\widetilde{R}$–modules (the last case can only appear if $\Omega^1_R(M) \cong M$).*

(ii) $\Omega^1_{R'}(T_\xi)$ *is either indecomposable, or $\Omega^1_{R'}(T_\xi)$ is a direct sum of two indecomposable maximal Cohen–Macaulay R'–modules N, N' such that $\widetilde{R} \otimes_{R'} N \simeq T_\xi$ and $\widetilde{R} \otimes_{R'} N' \simeq \Omega^1_R(T_\xi)$.*

(iii) If $\Omega^1_{\widetilde{R}}(T_\xi)$ and T_ξ are not isomorphic as $\widetilde{R}$–modules, then there exists at most one maximal Cohen–Macaulay R'–module N such that $\widetilde{R} \otimes_{R'} N \cong T_\xi$.

Proof:

(i) If $\Omega^1_{R'}(T_\xi) \cong \Omega^1_{R'}(T_{\xi'})$ for some $\xi, \xi' \in \mathcal{T}_M$ then $T_\xi \oplus \Omega^1_{\widetilde{R}}(T_\xi) \cong \Omega^1_{R'}(T_\xi)/y^2\Omega^1_{R'}(T_\xi) \cong \Omega^1_{R'}(T_{\xi'})/y^2\Omega^1_{R'}(T_{\xi'}) \cong T_{\xi'} \oplus \Omega^1_{\widetilde{R}}(T_{\xi'})$, by [ADS] (1.5), since $\xi, \xi' \in \mathcal{L}_M$. Since $T_\xi, T_{\xi'}$ are indecomposable it follows that $\Omega^1_{\widetilde{R}}(T_\xi)$, $\Omega^1_{\widetilde{R}}(T_{\xi'})$ are indecomposable ($\widetilde{R}$ is Gorenstein!). Thus, $T_\xi \cong T_{\xi'}$, or $T_\xi \cong \Omega^1_{\widetilde{R}}(T_{\xi'})$ by Krull–Schmidt Theorem.

(ii) If $\Omega^1_{R'}(T_\xi) = N \oplus N'$ is a non–trivial decomposition of $\Omega'_{R'}(T_\xi)$ then, by Nakayama's Lemma, we have a non–trivial decomposition $\Omega^1_{R'}(T_\xi)/y^2\Omega^1_{R'}(T_\xi) \simeq N/y^2N \oplus N'/y^2N'$. As above it holds that $\Omega^1_{R'}(T_\xi)/y^2\Omega^1_{R'}(T_\xi) \simeq T_\xi \oplus \Omega^1_{\widetilde{R}}(T_\xi)$ and so $T_\xi \cong N/y^2N$, $\Omega^1_{\widetilde{R}}(T_\xi) \cong N'/y^2N'$ or conversely.

(iii) Suppose $T_\xi \cong N/y^2N \cong N'/y^2N'$ for two maximal Cohen–Macaulay R'–modules N, N'. By [Po] or [HP] (2.8) it follows

$$N \oplus \Omega^1_{R'}(N) \cong \Omega^1_{R'}(T_\xi) \cong N' \oplus \Omega^1_{R'}(N').$$

By Nakayama's Lemma $N, N', \Omega^1_{R'}(N)$, $\Omega^1_{R'}(N')$ are indecomposable and so either $N \cong N'$ or $N \cong \Omega^1_{R'}(N')$. The last case implies $T_\xi \cong \widetilde{R} \otimes_{R'} N \cong \widetilde{R} \otimes_{R'} \Omega^1_{R'}(N') \cong \Omega^1_{\widetilde{R}}(\widetilde{R} \otimes_{R'} N') \cong \Omega^1_{\widetilde{R}}(T_\xi)$, which is a contradiction. $\square$

Remark 1.8
The description of maximal Cohen–Macaulay R'–modules is complete if we are able to describe the set $\{\Omega^1_{R'}(T_\xi)\}_{\xi \in \mathcal{T}_M}$ by Theorem 1.1. Using Theorem 1.7 and [HP] (3.2) the last set is almost completely described if we are able to describe the isomorphism classes of $T_\xi, \xi \in \mathcal{T}_M$. By [HP] (1.1) $T_\xi \cong T_{\xi'}$ as $\widetilde{R}$–modules if and only if there exists an R–automorphism σ of M such that $\mathrm{Ext}^1_R(\sigma, M)(\xi) = \mathrm{Ext}^1_R(M, \sigma)(\xi)$. This defines an equivalence relation "$\sim$" on $\mathcal{T}_M$ and it remains to study $\mathcal{T}_M/\sim$.

2 Infinitesimal deformations of modules over $K[[x]]/(x^t)$

Let K be an algebraically closed field with char $K \neq 3$, x, y two variables, $R := K[[x]]/(x^t)$, $\widetilde{R} := R[y]/(y^2)$, $R' := K[[x,y]]/(x^t + y^3)$.

The non–free indecomposable R–modules are $P_i := K[[x]]/(x^i)$, $1 \leq i < t$. The reduced matrix factorization of P_i is (x^i, x^{t-i}) and P_{t-i} is the first syzygy over R of P_i.

In the spirit of the Thom–Sebastiani problems, we study the maximal Cohen–Macaulay R'–modules in connection with the infinitesimal deformations of the modules over $K[[x]]/(x^t)$ (which, of course, have the following general form $P_{i_1}^{d_1} \oplus \cdots \oplus P_{i_s}^{d_s}$, $1 \leq i_1 < \cdots < i_s < t$, $1 \leq s < t$, $d_j \in \mathbb{N}$, if they do not have free direct summands).

In this section, by different methods from those of [HP], we are going to study the existence of infinitesimal deformations which are liftable to $R'/(y^4)$.

Our next theorem completes [HP] (4.1).

Theorem 2.1
Let i, d be two positive integers. Then

(i) P_i^d has infinitesimal deformations liftable to $R'/(y^4)$ if and only if t, d satisfy one of the following conditions :

(a) $t = 2i$ and d is even,

(b) $t = 3i$,

(c) $2t = 3i$.

(ii) If $t = 3i$ (respectively $2t = 3i$), then every infinitesimal deformation of P_i^d liftable to $R'/(y^4)$ is a direct sum of d copies of three types of cyclic infinitesimal deformations given by the following matrix factorizations $(\tau_s, \sigma_s)_{1 \leq s \leq 3}$ (respectively $(\sigma_s, \tau_s)_{1 \leq s \leq 3}$), where $\tau_s = x^i + y \cdot p_s$, $\sigma_s = x^{2i} - x^i y p_s$ and p_1, p_2, p_3 are the third roots of the unity.

(iii) If $t = 2i$, $d = 2q$, $q \in \mathbb{N}$, then there exists a unique infinitesimal deformation of P_i^d liftable to $R'/(y^4)$ and its corresponding matrix factorization is a direct sum of q–copies of $\left(\begin{pmatrix} x^i & 0 \\ y & x^i \end{pmatrix}, \begin{pmatrix} x^i & 0 \\ -y & x^i \end{pmatrix} \right)$.

Proof: We may suppose $t \geq 2i$ otherwise $t < 2i$ and so $2(t-i) < t$ and we may treat the case P_{t-i}^d the first syzygy $\Omega_R^1(P_i^d)$ of P_i^d, because the correspondence $T \longrightarrow \Omega_{\widetilde{R}}^1(T)$ given by taking the first syzygy over $\widetilde{R}$ defines a bijection between the infinitesimal deformations of P_i^d and the infinitesimal deformations of P_{t-i}^d.

Suppose there exists an infinitesimal deformation T of P_i^d liftable to $R'/(y^4)$. We know that a matrix factorization $(\tilde{\varphi}, \tilde{\psi})$ of x^t over $K[[x,y]]/(y^2)$ has the form $\tilde{\varphi} = \varphi +$

$y\alpha,\ \tilde{\psi} = \psi - y\beta$, with $\varphi, \psi, \alpha, \beta \in M(d \times d, K[[x]])$, $\varphi\beta = \alpha\psi$, $\beta\varphi = \psi\alpha$, $(\varphi, \psi) = (x^i \cdot I_d, x^{t-i} \cdot I_d)$ being a matrix factorization corresponding to $M = P_i^d$.

From [HP] (3.2) or from Remark 1.6 we deduce that there exist

$$\gamma, \tau \in M(d \times d, K[[x, y]]/(y^2))$$

such that

$$\alpha\beta + yI_d = \tilde{\varphi}\tau + \gamma\tilde{\psi} = (\varphi + y\alpha)\tau + \gamma(\psi - y\beta).$$

Writing $\gamma = \gamma_1 + y\gamma_2,\ \tau = \tau_1 + y\tau_2$ with $\gamma_i, \tau_i \in M(d \times d, K[[x]])$, $i = 1, 2$.

We have

$$\begin{cases} \alpha\beta & = \psi\gamma_1 + \tau_1\psi \\ I_d & = \alpha\gamma_1 + \varphi\gamma_2 + \tau_2\psi - \tau_1\beta. \end{cases} \tag{1}$$

From $\alpha\psi = \varphi\beta$ we can write $\alpha x^{t-i} = x^i\beta$.

So we have $\alpha \cdot x^{t-2i} = \beta$ and from (1) we deduce

$$\begin{cases} \alpha^2 \cdot x^{t-2i} & = x^i\gamma_1 + \tau_1 x^{t-i} \\ I_d & = \alpha\gamma_1 + x^i\gamma_2 + x^{t-i}\tau_2 - x^{t-2i} \cdot \tau_1 \cdot \alpha. \end{cases} \tag{2}$$

If $t > 3i$ we have

$$\begin{cases} \alpha^2 \cdot x^{t-3i} & = \gamma_1 + \tau_1 \cdot x^{t-2i} \\ I_d & = \alpha(\alpha^2 \cdot x^{t-3i} - \tau_1 \cdot x^{t-2i}) + x^i\gamma_2 + x^{t-i}\tau_2 - x^{t-2i}\tau_1\alpha \end{cases}$$

and because $i \geq 1,\ t > 3i$ we obtain $I_d \equiv 0 (\mathrm{mod}\ x)$. So $3i \geq t$ ($t \geq 2i$ by our assumption!).

If $3i > t$ we can write by (2)

$$\begin{cases} \alpha^2 & = x^{3i-t}(\gamma_1 + \tau_1 \cdot x^{t-2i}) \\ I_d & = x^i(\gamma_2 + x^{t-2i} \cdot \tau_2) - x^{t-2i}\tau_1\alpha + \alpha\gamma_1. \end{cases}$$

So $I_d \equiv \alpha\gamma_1 (\mathrm{mod}\ x)$. This implies $\alpha \equiv \alpha^2\gamma_1 (\mathrm{mod}\ x)$.

But $\alpha^2 \equiv 0 (\mathrm{mod}\ x)$, because $3i > t$ and so $\alpha \equiv \alpha^2\gamma_1 \equiv 0 (\mathrm{mod}\ x)$.

We obtain a contradiction from $I_d \equiv \alpha\gamma_1 \equiv 0 (\mathrm{mod}\, x)$. Thus, the cases (b) $t = 3i$ and (a) $t = 2i$ (in order to obtain d even) remain to be studied.

The case (c) $2t = 3i$ will follow from (b) applied to the first syzygy of T over $\widetilde{R}$ (as we have seen before).

We need the following lemma, which will be proved later.

Lemma 2.2
Let (φ, ψ) be a d–matrix factorization of x^t, U, V two invertible d–matrices over $K[[x]]$ such that $\varphi = U\varphi V$ and α, β two d–matrices over $K[[x]]$ defining an infinitesimal deformation T of Coker φ to $\widetilde{R}$. Then $\alpha' := U\alpha V$, $\beta' := V^{-1}\beta U^{-1}$ give also a matrix factorization of T.

Applying this lemma for $U, V = U^{-1}$ we see that modulo such transformations we may suppose that α modulo x is in the Jordan form (in our case $\varphi = x^i \cdot I_d$ commutes with every U!), let us say $\alpha \equiv \overset{e}{\underset{j=1}{\oplus}} \varepsilon_j \bmod x$,

$$\text{where } \varepsilon_j = \begin{pmatrix} \lambda_j & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ 0 & \dots & \lambda_j & 0 \\ 0 & \dots & 1 & \lambda_j \end{pmatrix} \quad (K \text{ being algebraically closed}),$$

is a s_j–Jordan cell.

From (2) we have $\alpha^2 = \gamma_1 + \tau_1 \cdot x^i$ and $I_d = \alpha\gamma_1 + x^i\gamma_2 + x^{2i}\tau_2 - x^i\tau_1\alpha$ implies

$$I_d \equiv \alpha^3 (\operatorname{mod} x^i). \tag{3}$$

By (3) we see that $\lambda_j \neq 0$, $\lambda_j^3 = 1$ and $s_j = 1$, $e = d$. Thus

$$\alpha = \varepsilon + x\theta, \text{ for } \varepsilon = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}.$$

We show by induction on r, $0 \leq r < i$ that $\theta \equiv 0 (\operatorname{mod} x^r)$ the case $r = 0$ being obvious.

Suppose $0 < r \leq i - 1$. By induction hypothesis we have $\theta \equiv 0 (\operatorname{mod} x^{r-1})$ and so by (3)

$$I_d \equiv \alpha^3 \equiv \varepsilon^3 + 3\varepsilon^2 x\theta \operatorname{mod} x^{r+1}.$$

It follows $3\varepsilon^2 x\theta \equiv 0 \operatorname{mod} x^{r+1}$ and so $\theta \equiv 0 \operatorname{mod} x^r$. Hence, $\alpha \equiv \varepsilon \operatorname{mod} x^i$ and it is enough to apply the following lemma, since $\alpha - \varepsilon \in \varphi M(d \times d,\ K[[x]])$.

Lemma 2.3
Let (φ, ψ), α, β be as in Lemma 2.2.

If $\alpha' := \alpha + \eta\varphi + \varphi w$, $\beta' := \beta + w\psi + \psi\eta$ for two d–matrices η, w over $K[[x]]$, then (α', β') defines also T.

Proof: We have $(I_d + \eta y)(\varphi + y\alpha)(I_d + wy) \equiv \varphi + y\alpha' \operatorname{mod} y^2$ and $(I_d - wy)(\psi - y\beta)(I_d - \eta y) \equiv \psi - y\beta' \operatorname{mod} y^2$. Thus, $(\varphi + y\alpha',\ \psi - y\beta')$ defines also T since $I_d + \eta y,\ I_d + wy$ are invertible. □

Proof of Lemma 2.2: Clearly $\varphi + y\alpha' = U(\varphi + y\alpha)V,\ \psi - y\beta' = V^{-1}(\psi - y\beta)U^{-1}$. Thus, $(\varphi + y\alpha', \psi - y\beta')$ defines also T, since U, V are invertible. $\square$

We continue the proof of 2.2 with the case (a) $t = 2i$.

Thus, $\varphi = \psi = x^i \cdot I_d$ and $\alpha = \beta$. We also have

$$\begin{cases} a^2 & = x^i(\gamma_1 + \tau_1) \\ I_d & = x^i(\gamma_2 + \tau_2) + \alpha\tau_1 - \tau_1\alpha. \end{cases} \tag{4}$$

Thus we have $\alpha^2 \equiv 0 (\operatorname{mod} x)$.

As above, we can consider $\alpha = \alpha_0 + x\theta$ with $\alpha_0 \in M(d \times d, K),\ \alpha_0 = \bigoplus_{j=1}^{q} \varepsilon_j$, where ε_j is a s_j–Jordan cell $\varepsilon_j = \begin{pmatrix} \lambda_j & & & \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_j \end{pmatrix}$, $\lambda_j \in K, \sum_{j=1}^{q} s_j = d$ and $\alpha_0^2 = 0$.

Because of the particular form of α_0, $\alpha_0^2 = 0$ implies $s_j \leq 2$ and $\lambda_j = 0$ for all $1 \leq j \leq q$. In order to obtain d even we shall show that $s_j = 2$ for any $1 \leq j \leq q$. Suppose that there exists $j \in \{1, \ldots, q\}$ such that $s_j = 1$. Of course, we may suppose that $j = 1$.

From (4) we have

$$I_d = x^i(\gamma_2 + \tau_2) + (\alpha_0 + x\theta)\gamma_1 - \tau_1(\alpha_0 + x\theta).$$

So

$$I_d \equiv \alpha_0\gamma_1 - \tau_1\alpha_0 (\operatorname{mod} x). \tag{5}$$

But $s_1 = 1$ implies that $\alpha_0 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right)$.

Looking at the entry $(1,1)$ of I_d we see that this is a contradiction by (5). Thus, $s_j = 2$ for all $1 \leq j \leq q$ and $d = 2q$.

Let V_{jp} be the elementary matrix given by $V_{jp} = (v_{rs})_{1 \leq r,s \leq d}$,

$$v_{rs} = \begin{cases} 1 & \text{if } r = s \text{ and } r \neq j, p \\ 1 & \text{if } (r,s) = (j,p), \text{ or } (r,s) = (p,j), \\ 0 & \text{otherwise.} \end{cases}$$

Changing α by $V_{jp}\alpha V_{jp}$(notice $V_{jp}^{-1} = V_{jp}$) for some j, p (that is permuting some lines and some corresponding columns of α) we may suppose $\alpha_0 = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_q & 0 \end{array}\right)$ by Lemma 2.2.

By (4) we have

$$x(\alpha_0\theta + \theta\alpha_0) + x^2\theta^2 = \alpha^2 \equiv 0 \bmod x^i. \tag{6}$$

Express θ in blocks, let us say $\theta = \begin{pmatrix} \varepsilon & \nu \\ \eta & \delta \end{pmatrix}$, where $\varepsilon, \eta, \nu, \delta \in M\ (q \times q, K[[x]])$. We may suppose $\eta = 0$ because

$$\tilde{\varphi} = \begin{pmatrix} x^i I_q + yx\varepsilon & yx\nu \\ y(I_q + x\eta) & x^i I_\varepsilon + yx\delta \end{pmatrix}$$

is equivalent with

$$\hat{\varphi} = \begin{pmatrix} x^i I_q + yx\varepsilon & yx\nu(I_q + x\eta) \\ yI_q & x^i I_q + yx\delta \end{pmatrix}$$

(multiply the second line by $(I_q + x\eta)^{-1}$ and the second column by $I_q + x\eta$). Note that the matrix $\alpha_0\theta + \theta\alpha_0 + x\theta^2 \equiv 0 \bmod x^{i-1}$ has the form $\left(\begin{array}{c|c} * & * \\ \hline \varepsilon + \delta & * \end{array}\right)$ because $\eta = 0$. It follows $\varepsilon + \delta \equiv 0 \bmod x^{i-1}$. Clearly $\hat{\varphi}$ is equivalent to

$$\begin{pmatrix} x^i I_q & yx\nu - yx^2\varepsilon\delta - x^{i+1}(\varepsilon + \delta) \\ yI_q & x^i I_q \end{pmatrix}.$$

Since $x^{i+1}(\varepsilon + \delta) \equiv 0 \bmod x^t$ $(t = 2i)$ we may suppose

$$\tilde{\varphi} = \begin{pmatrix} x^i I_q & yx\nu' \\ yI_q & x^i I_q \end{pmatrix}$$

for $\nu' = \nu - x\varepsilon\delta$, that is we may take $\varepsilon = \delta = \eta = 0$. Then $\theta^2 = 0$ and by (6) it follows $\nu \equiv 0 \bmod x^{i-1}$. Using Lemma 2.3 we may suppose also $\nu = 0$. Then $\tilde{\varphi}$ is a direct sum of copies of 2×2–matrices $\begin{pmatrix} x^i & 0 \\ y & x^i \end{pmatrix}$ which are indecomposable. $\square$

In the final part of this section we are going to state a result about the existence of infinitesimal deformations liftable to $R'/(y^4)$.

Proposition 2.4
Let $M = \bigoplus_{j=1}^{s} P_{i_j}^{d_j}$, with $s \geq 2, 1 \leq i < \cdots < i_s < t$, $d_j \in \mathbb{N}$. If $t > 2i_s + i_1$ or $2t < 2i_1 + i_s$, there are no infinitesimal deformations of M to $\widetilde{R}$ liftable to $R'/(y^4)$.

Proof: We may suppose $t > 2i_s + i_1$ because otherwise we have $2(t-i_1)+(t-i_s) < t$ and so $\Omega_R^1(M)$ has no infinitesimal deformations to $\widetilde{R}$ liftable to $R'/(y^4)$, that is M has none either. We use the same method as in Theorem 2.1. A matrix factorization for M is

$$\left(\varphi = \begin{pmatrix} x^{i_1} I_{d_1} & & 0 \\ & \ddots & \\ 0 & & x^{i_s} I_{d_s} \end{pmatrix}, \psi = \begin{pmatrix} x^{t-i_1} I_{d_1} & & 0 \\ & \ddots & \\ 0 & & x^{t-i_s} I_{d_s} \end{pmatrix} \right)$$

and let $(\tilde{\varphi} = \varphi + y\alpha, \tilde{\psi} = \psi - y\beta)$ be a matrix factorization for an infinitesimal deformation T. Thus, we have

$$\alpha\psi = \varphi\beta \tag{7}$$

writing

$$\alpha = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1s} \\ \vdots & & \vdots \\ \alpha_{s1} & \dots & \alpha_{ss} \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_{11} & \dots & \beta_{1s} \\ \vdots & & \vdots \\ \beta_{s1} & \dots & \beta_{ss} \end{pmatrix}$$

with $\alpha_{ij}, \beta_{ij} \in M(d_i \times d_j, K[[x]])$ we obtain by (7) $\alpha_{rk}x^{t-i_k} = x^{i_r} \cdot \beta_{rk}$, for every $1 \le r, k \le s$, thus $\alpha_{rk} \cdot x^{t-i_k-i_r} = \beta_{rk}$ because of the hypothesis.

Let us write the relations which characterize the liftability of T to $R'/(y^4)$ (as in the Proof of 2.1):

$$\begin{cases} \alpha\beta & = \varphi\gamma_1 + \tau_1\psi & (*) \\ I_d & = \alpha\gamma_1 + \varphi\gamma_2 - \tau_1\beta + \tau_2\psi & (**) \end{cases}$$

with $\gamma_i, \tau_i \in M(d \times d, K[[x]])$, and $d = \sum_{j=1}^{s} d_j$.

From $(*)$ we obtain $\sum_{p=1}^{s} \alpha_{rp} \cdot \beta_{pk} = x^{i_r} \cdot \gamma^1_{rk} + x^{t-i_k}\tau^1_{rk}$ for every $r, k \in \{1, \dots, s\}$.

Here we set $\gamma_1 = (\gamma^1_{rk})_{1 \le r,k \le s}$ and $\tau_1 = (\tau^1_{rk})_{1 \le r,k \le s}$. But $\beta_{pk} = x^{t-i_k-i_p} \cdot \alpha_{pk}$, thus

$$\sum_{p=1}^{s} \alpha_{rp}\alpha_{pk} \cdot x^{t-i_k-i_p} = x^{i_r} \cdot \gamma^1_{rk} + x^{t-i_k}\tau^1_{rk}. \tag{8}$$

Now we look to $(**)$.
We have $\varphi\gamma_2 \equiv 0(\operatorname{mod} x)$, $\tau_2\psi \equiv 0(\operatorname{mod} x)$ and because $t > 2i_s + i_1$ we also have $\beta \equiv 0(\operatorname{mod} x)$. So we obtain $\tau_1 \cdot \beta \equiv 0(\operatorname{mod} x)$. From $(**)$ we deduce that $I_d \equiv \alpha\gamma_1(\operatorname{mod} x)$, in particular

$$I_{d_1} \equiv \sum_{p=1}^{s} \alpha_{1p}\gamma^1_{p1}(\operatorname{mod} x). \tag{9}$$

Writing (8) for $k = 1$ we obtain $\gamma^1_{r1} \equiv 0(\operatorname{mod} x)$ $(t > 2i_s + i_1!)$, for every $r \in \{1, \dots, s\}$.

Introducing this in (9) we obtain $I_{d_1} \equiv 0(\operatorname{mod} x)$, which is a contradiction.

In conclusion, we have shown that if $t > 2i_s + i_1$ there is no infinitesimal deformation T of M, liftable to $R'/(y^4)$. $\square$

Remark 2.5

(i) Proposition 2.4 implies the similar result obtained by [HP] (4.1) under hypothesis $i_s < \frac{t}{3}$ or $i > \frac{2t}{3}$.

(ii) If $s = 2, t = 2i_2 + i_1, i_2 > i_1$ there exists an infinitesimal deformation T of $P_{i_1} \oplus P_{i_2}^2$ given by the matrix factorization

$$\left(\begin{pmatrix} x^{i_1} & -y & 0 \\ 0 & x^{i_2} & -y \\ y & 0 & x^{i_2} \end{pmatrix}, \begin{pmatrix} x^{2i_2} & x^{i_2}y & 0 \\ 0 & x^{i_1+i_2} & x^{i_1}y \\ -x^{i_2}y & 0 & x^{i_1+i_2} \end{pmatrix}\right),$$

which is liftable to $R'/(y^4)$, that is the conditions from Proposition 2.4 are sharp. Indeed, if N is the R'–module from Remark 3.3, then clearly N/y^4N is a lifting of T to $R'/(y^4)$.

3 Maximal Cohen–Macaulay modules over $K[[x, y]]/(x^t + y^3)$

Let K, $R, \widetilde{R}$ and R' be as in Section 2.

In this section we shall apply the results obtained in Section 2 in order to describe the maximal Cohen–Macaulay R'–modules N with the property $N/yN \simeq P_i^d$ for i, d positive integers, $1 \leq i < t$, $P_i = K[[x]]/(x^i)$.

Theorem 3.1
Let i, d be two positive integers, $1 \leq i < t$. Then

(i) there exists a maximal Cohen–Macaulay module N such that N/yN is a direct sum of d–copies of $P_i := K[[x]]/(x^i)$ if and only if either $t = 3i$ or $2t = 3i$, or $t = 2i$ and d is even.

(ii) If $t = 3i$ then each maximal Cohen–Macaulay R'–module N such that $N/yN \cong P_i^d$ is a direct sum of d–copies of some from the following three cyclic maximal Cohen–Macaulay R'–modules

$$Q_j = K[[x, y]]/(x^i + p_j y), \ 1 \leq j \leq 3 \text{ and } p_j^3 = 1.$$

(iii) If $2t = 3i$, then each maximal Cohen–Macaulay R'–module N such that $N \cong P_i^d$ is a direct sum of d–copies of some from the following three cyclic maximal Cohen–Macaulay R'–modules

$$Q_j' = K[[x, y]]/(x^{2i} - p_j x^i y + p_j^2 y^2), \ 1 \leq j \leq 3.$$

(iv) If $t = 2i$ and $d = 2q$, $q \in \mathbb{N}$, then each maximal Cohen–Macaulay R'–module N such that $N/yN \cong P_i^d$ is a direct sum of q–copies of the ideal $(x^i, y)R'$.

Proof: If there exists a maximal Cohen–Macaulay R'–module N such that $N/yN \cong P_i^d$, then $T = N/y^2N$ is an infinitesimal deformation of P_i^d liftable to $R'/(y^4)$. Thus, the necessity from (i) follows from 2.1.

To prove (ii) let $t = 3i$ and N be a maximal Cohen–Macaulay R'–module such that $N \cong P_i^d$.

Then $N \oplus \Omega'_{R'}(N) \cong \Omega^1_{R'}(T), T = N/y^2N$ by [Po] or [HP] (2.8). Since T is liftable to $R'/(y^4)$ it must be a direct sum of d–cyclic deformations corresponding to $(x^i + p_j y, x^{2i} - p_j x^i y)$, $1 \leq j \leq 3$ by Theorem 2.1.

Thus, $\Omega^1_{R'}(T)$ is a direct sum of d–copies of maximal Cohen–Macaulay R'–modules corresponding to the matrix factorizations

$$(\gamma_j, \varepsilon_j) = \left(\left(\begin{array}{c|c} x^{2i} - p_j x^i y & -y^2 \\ \hline p_j^2 x^i + y & x^i + p_j y \end{array} \right), \left(\begin{array}{c|c} x^i + p_j y & y^2 \\ \hline -(p_j^2 x^i + y) & x^{2i} - p_j x^i y \end{array} \right) \right)$$

$1 \leq j \leq 3$. But

$$\gamma_j \sim \left(\begin{array}{c|c} x^{2i} - p_j x^i y + p_j^2 y^2 & -y^2 \\ \hline 0 & x^{i_1} + i_2 \end{array} \right)$$

and

$$-3p_j^2 y^2 = (x^i - 2p_j y)(x^i + p_j y) - (x^{2i} - p_j x^i + p_j^2 y^2).$$

It follows

$$\gamma_j \sim \left(\begin{array}{c|c} x^{2i} - p_j x^i y + p_j^2 y^2 & 0 \\ \hline 0 & x^i + p_j y \end{array} \right),$$

that is, $\Omega^1_{R'}(T)$ is the direct sum of d–copies of $\{Q_j \oplus Q'_j\}_{1 \leq j \leq 3}$, $Q'_j = \Omega^1_{R'}(Q_j)$. Since $Q'_j / y Q'_j$ is not a direct summand of P_i^d we are done.

The case (iii) is similar using 2.1.

In case (iv) T is a direct summand of q–copies of the infinitesimal deformation of P_i^2 given by $(\begin{pmatrix} x^i & 0 \\ y & x^i \end{pmatrix}, \begin{pmatrix} x^i & 0 \\ -y & x^i \end{pmatrix})$ (see 2.1).

Then $\Omega^1_{R'}(T)$ is a direct sum of q–copies of the maximal Cohen–Macaulay R'–module corresponding to the matrix factorization (τ, σ) given by

$$\tau = \left(\begin{array}{cc|cc} x^i & 0 & -y^2 & 0 \\ -y & x^i & 0 & -y^2 \\ \hline y & 0 & x^i & 0 \\ 0 & y & y & x^i \end{array} \right)$$

which is equivalent to $\tau' \oplus \tau''$ for $\tau' = \begin{pmatrix} x^i & -y^2 \\ y & x^i \end{pmatrix}$, $\tau'' = \begin{pmatrix} x^i & y^2 \\ -y & x^i \end{pmatrix}$. Clearly $\tau' \sim \tau''$ and the ideal (x^i, y) corresponds to (τ', σ'). Thus, $\Omega^1_{R'}(T)$ (respectively N) is a direct sum of d–copies (respectively q–copies) of (x^i, y). $\square$

Theorem 3.2
Let $M = \bigoplus_{j=1}^{s} P_{i_j}^{d_j}$ with $1 \leq i_1 < \cdots < i_s < t$, $s \geq 2$ and $t > 2i_s + i_1$ or $t < 2i_1 + i_s$. Then there is no maximal Cohen–Macaulay R'–module N such that $N/yN \cong M$.

Proof: Suppose there is a maximal Cohen–Macaulay R'–module N such that $N/yN \cong M$ and apply 2.4 for $T = N/y^2 N$. $\square$

Remark 3.3
If $s = 2, t = 2i_2 + i_1, i_2 > i_1$ there exists a maximal Cohen–Macaulay R'–module N given by the matrix factorization

$$\left(\begin{pmatrix} x^{i_1} & -y & 0 \\ 0 & x^{i_2} & -y \\ y & 0 & x^{i_2} \end{pmatrix}, \begin{pmatrix} x^{2i_2} & x^{i_2} y & y^2 \\ -y^2 & x^{i_1+i_2} & x^{i_1} y \\ -x^{i_2} y & -y^2 & x^{i_1+i_2} \end{pmatrix} \right)$$

such that $N/yN \cong P_{i_1} \oplus P_{i_2}^2$. Thus, the conditions from theorem 3.2 are sharp.

References

[ADS] Auslander, M.; Ding, S.; Solberg, Ø: Liftings and weak liftings of modules. J. of Algebra, **156**, 273 – 317 (1993).

[BGS] Buchweitz, R.-O.; Greuel, G.-M.; Schreyer, F.-O.: Cohen–Macaulay modules over hypersurface singularities II, Invent. Math. **88**, 165 – 182 (1987).

[Ei] Eisenbud, D.: Homological algebra on a complete intersection with an application to group representations. Trans. AMS, **260**, 1, 35 – 64 (1980).

[HP] Herzog, J.; Popescu, D.: Thom–Sebastiani problems for maximal Cohen–Macaulay modules. Preprint, Göttingen (1995).

[Kn] Knörrer, H.: Cohen–Macaulay modules on hypersurface singularities I. Invent. Math. **88**, 153 – 164 (1987).

[Po] Popescu, D.: Maximal Cohen–Macaulay modules and their deformations. Analele Stiintifice Constanta, v II, 112 – 119 (1994).

[PP] Pfister, G.; Popescu, D.: Deformations of maximal Cohen–Macaulay modules. Preprint 257, Kaiserslautern, (1994), to appear in Math. Z.

[Ru] Runar, I.: Non–commutative deformation theory. Thesis, Oslo (1990).

[Yo] Yoshino, Y.: Cohen–Macaulay modules over Cohen–Macaulay rings, London Math. Soc. Lecture Notes, Ser.; 146, Cambridge (1990).