# Classification of simple 0-dimensional isolated complete intersection singularities 

Thuy Huong Pham, Gerhard Pfister, and Gert-Martin Greuel


#### Abstract

The aim of this article is the classification of simple 0-dimensional isolated complete intersection singularities in positive characteristic. As usual, a singularity is called simple or 0 -modal if there are only finitely many isomorphism classes of singularities into which the given singularity can deform. The notion of simpleness was introduced by V. I. Arnold and the classification of low modality singularities has become a fundamental task in singularity theory. Simple complex analytic isolated complete intersection singularities (ICIS) were classified by M. Giusti. However, the classification in positive characteristic requires different methods and is much more involved. The final result is nevertheless similar to the classification in characteristic 0 with some additional normal forms in low characteristic.

The theoretical results in this paper mainly concern families of ICIS that are formal in the fiber and algebraic in the base (formal deformation theory is not sufficient). In particular, we give a definition of modality in this situation and prove its semicontinuity.


## 1 A criterion for non-simpleness

In this section we establish the notations, prove some preliminary results and show a general criterion for non-simplicity. This criterion (Theorem 1.15) allows us to significantly reduce the number of cases to be analyzed. The classification then consists of two main tasks: a) to further reduce the list of potentially simple singularities and b) to prove that the singularities of the list are indeed simple (Theorem
2.6 and 3.4). In step a), the semicontinuity of the modality is an important tool. In both steps we have to consider small characteristics separately (where characteristic 2 is treated in a separate section) and for the computations we make substantial use of the system singular [DGPS15].

Let $K$ be an algebraically closed field of arbitrary characteristic $p \geq 0$ and denote by

$$
R=K\left[\left[x_{1}, \ldots, x_{n}\right]\right]
$$

the formal power series ring in $n$ variables $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ with maximal ideal $\mathfrak{m}=\left\langle x_{1}, \ldots, x_{n}\right\rangle$. For $f=\sum a_{\alpha} \mathbf{x}^{\alpha} \in R$, we denote by $f(\mathbf{0})$ the coefficient $a_{0}$ of $f$ and by $\operatorname{supp}(f)=\left\{\alpha \mid a_{\alpha} \neq 0\right\}$ the support of $f$. For $I \subset R$ a proper ideal, denote by $\operatorname{mng}(I)=\operatorname{dim}_{K} I / \mathfrak{m} I$ the minimal number of generators of $I$. Let $G L(m, R)$ the group of $m \times m$ invertible matrices over $R$ and $\operatorname{Aut}(R)$ the group of $K$-algebra automorphisms of $R$.

Definition 1.1. 1. A polynomial $f \in K\left[x_{1}, \ldots, x_{n}\right]$ is called quasi-homogeneous of type $(d ; \mathbf{a})=\left(d ; a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}_{+} \times \mathbb{Z}_{+}^{n}$ if $f$ is a $K$-linear combination of monomials $\mathbf{x}^{\alpha}=x_{1}^{\alpha_{1}} \ldots x_{n}^{\alpha_{n}}$ satisfying $\langle\mathbf{a}, \alpha\rangle:=\sum_{i=1}^{n} a_{i} \alpha_{i}=d$. Denote by $G_{d}^{\mathbf{a}}$ the vector space of all quasi-homogeneous polynomials of type $(d ; \mathbf{a})$. Then

$$
K\left[x_{1}, \ldots, x_{n}\right]=\underset{d \geq 0}{\oplus} G_{d}^{\mathbf{a}}
$$

We set $G_{d}^{\mathbf{a}}=\{0\}$ if $d<0$.
2. An element $f=\left(f_{1}, \ldots, f_{m}\right) \in K\left[x_{1}, \ldots, x_{n}\right]^{m}$ is called quasi-homogeneous of type $(\mathbf{d} ; \mathbf{a})=\left(d_{1}, \ldots, d_{m} ; a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}_{+}^{m} \times \mathbb{Z}_{+}^{n}$ if

$$
f_{i} \in G_{d_{i}}^{\mathrm{a}}, \quad \forall i=1, \ldots, m
$$

$a_{i}$ is called the weight of $x_{i}$ and $d_{i}$ the (weighted) degree of $f_{i}$. Set

$$
G_{\nu}^{\mathbf{d}, \mathbf{a}}:=\left\{\left(f_{1}, \ldots, f_{m}\right) \in K\left[x_{1}, \ldots, x_{n}\right]^{m} \mid f_{i} \in G_{d_{i}+\nu}^{\mathbf{a}}, i=1, \ldots, m\right\}
$$

which is a finite dimensional $K$-vector space and $K\left[x_{1}, \ldots, x_{n}\right]^{m}=\oplus_{\nu \in \mathbb{Z}} G_{d_{i}+\nu}^{\mathbf{a}}$.
Definition 1.2. Let $(\mathbf{d} ; \mathbf{a})=\left(d_{1}, \ldots, d_{m} ; a_{1}, \ldots, a_{n}\right) \in \mathbb{Z}_{+}^{m} \times \mathbb{Z}_{+}^{n}$.

1. For $f \in R \backslash\{0\}$ we define

$$
v_{\mathbf{a}}(f):=\inf \{\langle\mathbf{a}, \alpha\rangle \mid \alpha \in \operatorname{supp}(f)\}
$$

and call it the $\mathbf{a}$ - order of $f$, or just the order if $\mathbf{a}=(1, \ldots, 1)$, denoted by $\operatorname{ord}_{\mathbf{a}}(f)$ or $\operatorname{ord}(f)$. If $f=0$ we set $v_{\mathbf{a}}(f)=\infty$.
2. For $f=\left(f_{1}, \ldots, f_{m}\right) \in R^{m}$ we set

$$
v_{\mathbf{d}, \mathbf{a}}(f):=\inf _{i=1, \ldots, m}\left\{v_{\mathbf{a}}\left(f_{i}\right)-d_{i}\right\} .
$$

If $f \neq 0$ then $v_{\mathbf{d}, \mathbf{a}}(f)=\inf \left\{\nu \mid f=\sum_{\nu} f^{(\nu)}, 0 \neq f^{(\nu)} \in G_{\nu}^{\mathbf{d}, \mathbf{a}}\right\}$.
Definition 1.3. (1) Let $f=\left(f_{1}, \ldots, f_{m}\right) \in R^{m}$. Denote by

$$
T^{1}(f):=R^{m} /\left(\left\langle f_{1}, \ldots, f_{m}\right\rangle \cdot R^{m}+\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle \cdot R\right)
$$

$\frac{\partial f}{\partial x_{i}}=\left(\frac{\partial f_{1}}{\partial x_{i}}, \ldots, \frac{\partial f_{m}}{\partial x_{i}}\right) \in R^{m}$, the Tjurina module of $f$ and

$$
\tau(f):=\operatorname{dim}_{K} T^{1}(f)
$$

the Tjurina number of $f$.
(2) Two elements $f, g \in R^{m}$ are called left equivalent, if they belong to the same orbit of the left group

$$
G:=G L(m, R) \rtimes \operatorname{Aut}(R),
$$

acting on on $R^{m}$ by

$$
(U, \phi, f) \mapsto U \cdot \phi(f),
$$

with $U \in G L(m, R), \phi \in \operatorname{Aut}(R)$ and

$$
\phi(f)=\left[f_{1}(\phi(\mathbf{x})) \ldots f_{m}(\phi(\mathbf{x}))\right]^{t}
$$

where $\phi(\mathbf{x})=\left(\phi\left(x_{1}\right), \ldots, \phi\left(x_{n}\right)\right)$.
Definition 1.4. An ideal $I \subset R$ defines a complete intersection if $I$ can be gnerated by $f_{1}, \ldots, f_{m}$ with $f_{i} \in \mathfrak{m}$ for all $i$ such that $f_{i}$ is a non-zero divisor of $R /\left\langle f_{1}, \ldots, f_{i-1}\right\rangle$ for $i=1, \ldots, m$. Then $\operatorname{mng}(I)=m$ and $\operatorname{dim} R / I=n-m \geq 0$.

Let $f=\left(f_{1}, \ldots, f_{m}\right)$ and $g=\left(g_{1}, \ldots, g_{m}\right) \in R^{m}$ be such that $I=\left\langle f_{1}, \ldots, f_{m}\right\rangle$ and $J=\left\langle g_{1}, \ldots, g_{m}\right\rangle$ define complete intersections with $\operatorname{mng}(I)=\operatorname{mng}(J)=m$.

Then $f$ is called contact equivalent to $g$, denoted $f \sim g$, if $g \in G f$, i.e. there is a matrix $U \in G L(m, R)$ and an automorphism $\phi \in \operatorname{Aut}(R)$ such that

$$
\left[\begin{array}{c}
g_{1} \\
\vdots \\
g_{m}
\end{array}\right]=U \phi\left(\left[\begin{array}{c}
f_{1} \\
\vdots \\
f_{m}
\end{array}\right]\right)
$$

For simplicity of notations, we also write $g=U \phi(f)$.
Note that by [P16, Proposition 3.3.4], $f \sim g$ if and only if $I \stackrel{c}{\sim} J$, i.e. there is an automorphism $\phi \in \operatorname{Aut}(R)$ such that $\phi(I)=J$. It follows easily that if $f \sim g$ then $\tau(f)=\tau(g)$.

Definition 1.5. Let $f=\left(f_{1}, \ldots, f_{m}\right) \in R^{m}$ be such that $I=\left\langle f_{1}, \ldots, f_{m}\right\rangle$ defines a complete intersection with $\operatorname{mng}(I)=m$. We call $f$ (or $I$ ) an isolated complete intersection singularity (ICIS) if there is a positive integer $k$ such that

$$
\mathfrak{m}^{k} \subset I+I_{m}(\operatorname{Jac}(f))
$$

where $\operatorname{Jac}(f)=\left[\frac{\partial f_{i}}{\partial x_{j}}\right]_{i j}$ is the $m \times n$ Jacobian matrix of $f$ and $I_{m}(\operatorname{Jac}(f))$ denotes the ideal generated by all $m \times m$ minors of $\operatorname{Jac}(f)$. We set

$$
I_{m, n}:=\left\{f=\left(f_{1}, \ldots, f_{m}\right) \in R^{m} \mid f \text { is an ICIS with } \operatorname{mng}(I)=m\right\}
$$

Note that any ICIS $f$ is finitely determined (see Proposition 1.11), in particular, the $f_{i}$ can be chosen as polynomials (up to an automorphism of $R$ ).

Lemma 1.6. Let $k \in \mathbb{N}$ and $D=\left\langle h_{1}, \ldots, h_{k}\right\rangle \subset R^{m}$, a submodule, where $h_{i} \in G_{\nu_{i}}^{\mathbf{d}, \mathbf{a}}$, $i=1, \ldots, k$ are quasi homogeneous. Then we have an isomorphism of $K$-vector spaces

$$
R^{m} / D \cong \prod_{\nu \in \mathbb{Z}}\left(G_{\nu}^{\mathbf{d}, \mathbf{a}} / D \cap G_{\nu}^{\mathbf{d}, \mathbf{a}}\right)
$$

We omit the elementary proof.
Lemma 1.7. Let $f \in K\left[x_{1}, \ldots, x_{n}\right]^{m} \cap I_{m, n}$ be quasi homogeneous of type $(\mathbf{d} ; \mathbf{a})=$ $\left(d_{1}, \ldots, d_{m} ; a_{1}, \ldots, a_{n}\right)$. Then, as $K$-vector spaces,

$$
T^{1}(f) \cong \underset{\nu \in \mathbb{Z}}{\oplus} T_{\nu}^{1}(f)
$$

where $T_{\nu}^{1}(f)=G_{\nu}^{\mathbf{d}, \mathbf{a}} /\left(\left\langle f_{1}, \ldots, f_{m}\right\rangle \cdot R^{m}+\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle\right) \cap G_{\nu}^{\mathbf{d}, \mathbf{a}}$.

Proof. Note that if $f \in K\left[x_{1}, \ldots, x_{n}\right]^{m} \cap I_{m, n}$ is quasi homogeneous of type (d;a) then for all $i=1, \ldots, n, j=1, \ldots, m, \frac{\partial f_{j}}{\partial x_{i}}$ is quasi homogeneous of type $\left(d_{j}-a_{i} ; a\right)$ (or zero). Therefore, $f_{j} e_{k} \in G_{d_{j}-d_{k}}^{\mathbf{d , a}}$, where $\left\{e_{k}\right\}_{k=1, \ldots, m}$ is the canonical basis of $R^{m}$, and $\frac{\partial f_{j}}{\partial x_{i}} \in G_{-a_{i}}^{\mathbf{d}, \mathbf{a}}$, i.e. $D:=\left\langle f_{1}, \ldots, f_{m}\right\rangle \cdot R^{m}+\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle$ is a quasi homogeneous submodule of $R^{m}$. By Lemma 1.6,

$$
R^{m} / D \cong \prod_{\nu \in \mathbb{Z}}\left(G_{\nu}^{\mathbf{d}, \mathbf{a}} / D \cap G_{\nu}^{\mathbf{d}, \mathbf{a}}\right)
$$

Since $\tau(f)<\infty$, we have $R^{m} / D \cong \underset{\nu \in \mathbb{Z}}{\oplus}\left(G_{\nu}^{\mathbf{d}, \mathbf{a}} / D \cap G_{\nu}^{\mathbf{d}, \mathbf{a}}\right)$.
The following proposition is quite useful for the classification of singularities.
Proposition 1.8. Let $f=\left(f_{1}, \ldots, f_{m}\right) \in K\left[x_{1}, \ldots, x_{n}\right]^{m} \cap I_{m, n}$ be quasi homogeneous of type $(\mathbf{d} ; \mathbf{a}) \in \mathbb{Z}_{+}^{m} \times \mathbb{Z}_{+}^{n}$. Let $g=\left(g_{1}, \ldots, g_{m}\right) \in R^{m}$ be such that

$$
v_{\mathbf{d}, \mathbf{a}}(g)>\beta=\sup (0, \alpha),
$$

where $\alpha=\sup \left\{i \mid T_{i}^{1}(f) \neq 0\right\}$. Then $f+g \sim f$.
Proof. In the complex analytic setting a proof is given in [Giu83, Proposition 1]. Using the finite determinacy of an ICIS (see Proposition 1.11), the proof works as well for formal power series in any characteristic. We omit the details.

Definition 1.9. Let $f=\left(f_{1}, \ldots, f_{m}\right) \in I_{m, n}$. The tangent image of the action of $G$ on $R^{m}$ is defined as the module

$$
\tilde{T}_{f}(G f):=\left\langle f_{1}, \ldots, f_{m}\right\rangle R^{m}+\mathfrak{m}\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle \subset \mathfrak{m} R^{m}
$$

which has finite $K$-codimension in $R^{m}$ for an ICIS. It is easy to see that

$$
T^{1, s e c}(f):=\mathfrak{m} R^{m} / \tilde{T}_{f}(G f)
$$

has finite $K$-dimension iff this holds for $T^{1}(f)=R^{m} /\left\langle f_{1}, \ldots, f_{m}\right\rangle R^{m}+\left\langle\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle$ ([GP18, Proposition 4.2]). We denote by

$$
\tau^{s e c}(f):=\operatorname{dim}_{K} T^{1, s e c}(f)
$$

Remark 1.10. (1) The tangent image was introduced in [GP18] (in a more general context) to replace the tangent space to the orbit $G f$ in positive characteristic. In characteristic zero $\tilde{T}_{f}(G f)$ coincides with the tangent space $T_{f}(G f)$, see [GP18, Lemma 2.8]. In general we have $\tilde{T}_{f}(G f) \subset T_{f}(G f)$ and the inclusion can be strict in positive characteristic (see Example 2.9 in [GP18]).

It was shown in [GP19, Theorem 1.4 ] that an ICIS $f$ is finitely determined iff $\tau^{s e c}(f)<\infty$ (equivalently $\tau(f)<\infty$ ).
(2) Let $g_{1}, \ldots, g_{d} \in K[[\mathbf{x}]]^{m}$ represent a $K$-basis of $T^{1}(f)$, resp. of $T^{1, \text { sec }}(f)$, then

$$
F_{\mathbf{t}}(\mathbf{x}):=F\left(\mathbf{x}, t_{1}, \ldots, t_{d}\right)=f(\mathbf{x})+\sum_{i=1}^{d} t_{i} g_{i}(\mathbf{x}) \in K[[\mathbf{t}, \mathbf{x}]]
$$

represents a formally semiuniversal deformation of $f$, resp. a formally semiuniversal deformation with section of $f$. See [KS72] for the first case. The latter statement follows because every section can be trivialized (see [GLS07, Proposition II.2.2]) and then the proof given in [KS72] or [GLS07, Theorem II.1.16] can be adapted. If the $g_{i}$ represent a system of generators, then $F_{\mathbf{t}}$ represents a formally versal deformation, resp. with section.
(3) Note that the $g_{i}$ can be chosen as monomials and the $f_{i}$ as polynomials (since the ICIS $f$ is finitely determined). Thus the semiuniversal deformation, resp. with section, of $f$ has an algebraic representative $F_{\mathbf{t}}(\mathbf{x}) \in K[\mathbf{t}, \mathbf{x}]$ with $F_{\mathbf{0}}(\mathbf{x})=f(\mathbf{x})$. $F_{\mathbf{t}}(\mathbf{x})$ is called an unfolding at $0 \in T$ of $f$ over $T=K^{d}$.

The algebraic unfolding $F_{\mathbf{t}}(\mathbf{x})$ is $G$-complete or contact complete, meaning that any unfolding $H_{\mathbf{s}}(\mathbf{x})=H\left(\mathbf{x}, s_{1}, \ldots, s_{e}\right) \in K[\mathbf{s}, \mathbf{x}]$ of $f$ at $s_{0} \in S=K^{e}$ with $H_{s_{0}}(\mathbf{x})=$ $f(\mathbf{x})$ is an étale pullback of $F$, i.e. there exist an étale neighbourhood $\varphi: W, w_{0} \rightarrow$ $S, s_{0}$ and a morphism $\psi: W, w_{0} \rightarrow T, 0$ such that $H(\mathbf{x}, \varphi(w))$ is contact-equivalent to $F(\mathbf{x}, \psi(w))$ for all $w \in W$. This is important for the classification of singularities in the non-analytic case (the proof of [GNg16, Proposition 2.14] for hypersurfaces can be generalized to an ICIS).

For the classification we need to consider $k$-jets of power series. Denote by

$$
R_{k}^{m}=R^{m} / \mathfrak{m}^{k+1} R^{m}, k \geq 0
$$

the $k$ - jet space of $R^{m}$ and for $f \in R^{m}$, let $j_{k}(f)$ denote the $k$-jet of $f$, i.e., the image of $f$ in $R_{k}^{m}$. Consider the group of $k$-jets

$$
G_{k}=\left\{\left(j_{k}(U), j_{k}(\phi)\right) \mid U \in G L(m, R), \phi \in \operatorname{Aut}(R)\right\}
$$

where $\left(j_{k}(\phi)\right)\left(x_{i}\right)=j_{k}\left(\phi\left(x_{i}\right)\right), i=1, \ldots, n$. Then the action of the left group $G$ on $R^{m}$ induces the action on the $k$-jet spaces

$$
\begin{aligned}
G_{k} \times R_{k}^{m} & \rightarrow R_{k}^{m} \\
\left(j_{k}(U), j_{k}(\phi), j_{k}(f)\right) & \mapsto j_{k}(U \phi(f)),
\end{aligned}
$$

which is an algebraic action of the algebraic group $G_{k}$ on the affine space $R_{k}^{m}$.
We recall the finite determinacy and the semicontinuity of the Tjurina number of a complete intersection in arbitrary characteristic. We consider families $F_{\mathbf{t}}(\mathbf{x})=$ $F\left(\mathbf{x}, t_{1}, \ldots, t_{k}\right) \in K[\mathbf{t}][[\mathbf{x}]]$, which are polynomial in the parameter $\mathbf{t}$, and denote for fixed $\mathbf{t} \in K^{k}$ by $F_{\mathbf{t}}(\mathbf{x})$ also the power series in $K[[\mathbf{x}]]$.

Proposition 1.11. (1) Let $f \in R^{m}$ be an ICIS. Then $f$ is finitely determined, i.e. there exists a $k$ such that each $g \in R^{m}$ with $j_{k}(g)=j_{k}(f)$ is contact equivalent to $f(f$ is then called $k$-determined). Moreover, $f$ is $(2 \tau(f)-\operatorname{ord}(f)+2)$-determined and every deformation of $f$ is $(2 \tau(f)+1)$-determined.
(2) Let $F_{\mathbf{t}}(\mathbf{x}):=F\left(\mathbf{x}, t_{1}, \ldots, t_{k}\right) \in K[\mathbf{t}][[\mathbf{x}]]$ s.t. $F_{\mathbf{t}_{0}}$ is an ICIS in $I_{m, n}$ for some $\mathbf{t}_{0} \in K^{k}$. Then there exists a Zariski open set $U \subset K^{k}$ s.t. $F_{\mathbf{t}} \in I_{m, n}$ for all $\mathbf{t} \in U$. Moreover, for each $\tau \geq 0$ the sets

$$
U_{\tau}=\left\{\mathbf{t} \in U \mid \tau\left(F_{\mathbf{t}}\right) \leq \tau\right\} \text { and } U_{\tau}^{s e c}=\left\{\mathbf{t} \in U \mid \tau^{s e c}\left(F_{\mathbf{t}}\right) \leq \tau\right\}
$$

are open in $K^{k}$. In particular, if $\tau_{\min }=\min \left\{\tau\left(F_{\mathbf{t}}\right) \mid \mathbf{t} \in K^{d}\right\}$, then $U_{\tau_{\text {min }}}$ is open and dense in $K^{k}$, and similar for $\tau_{m i n}^{s e c}$.
Proof. (1) follows from [GP18, Theorem 4.6]. (2) follows for $d=1$ from [GP18, Proposition 3.4] and for an arbitrary Noetherian ring $A$ instead of $K[\mathbf{t}]$ (also for non-closed points) from [GPf21, Proposition 3.4 and Corollary 2.7].

In our classification we will consider families of $k$-jets of an ICIS. These are deformations with trivial section which are versal for sufficiently large $k$. Hence, in the following definition of simple we consider deformations with section.

Definition 1.12. Let $g_{1}, \ldots, g_{d} \in K[\mathbf{x}]^{m}$ be a set of $K$-generators of $T^{1, s e c}(f)$. An element $f \in I_{m, n}$ is called simple if there is a finite set $\left\{h_{1}, \ldots, h_{l}\right\} \subset I_{m, n}$ such that for

$$
F_{\mathbf{t}}(\mathrm{x}):=F\left(\mathrm{x}, t_{1}, \ldots, t_{d}\right)=f(\mathbf{x})+\sum_{i=1}^{d} t_{i} g_{i}(\mathbf{x})
$$

there is a Zariski open neighborhood $U$ of 0 in $K^{d}$ such that for each $\mathbf{t}=\left(t_{1}, \ldots, t_{d}\right) \in$ $U$, there is a $j \in\{1, \ldots, l\}$ such that $F_{\mathbf{t}} \sim h_{j}$.

Remark 1.13. (1) By Remark 1.10 (2) and (3) the unfolding $F_{\mathbf{t}}(\mathbf{x}) \in K[\mathbf{t}, \mathbf{x}]$ defines a formally versal deformation with section of $f$ which is contact complete. It follows that $f$ is simple iff for an arbitrary unfolding $H_{\mathbf{s}}(\mathbf{x})=H\left(\mathbf{x}, s_{1}, \ldots, s_{e}\right) \in K[\mathbf{s}, \mathbf{x}]$ of $f$ at $s_{0} \in S=K^{e}$ there is a Zariski open neighborhood $U$ of $s_{0}$ in $S$ such that for each $\mathbf{s} \in U$, there is a $j \in\{1, \ldots, l\}$ such that $H_{\mathbf{s}} \sim h_{j}$.
(2) Following [GK90], we call an element $f \in \mathfrak{m} R^{m}$ 0-modular if for each $k$ (equivalently, for sufficiently large $k$ ), there is a Zariski neighborhood $U_{k}$ of $j_{k}(f)$ in $R_{k}^{m}$ such that $U_{k}$ meets only finitely many $G_{k}$-orbits of $k$-jets. Here, an integer $k$ is sufficiently large for $f$ with respect to $G$ if there exists a neighbourhood $U$ of $j_{k}(f)$ in $\mathfrak{m} R_{k}^{m}$ such that every $g \in \mathfrak{m} R^{m}$ with $j_{k}(g) \in U$ is $k$-determined with respect to $G$. If $f$ is an ICIS, then $k=(2 \tau(f)+1)$ is sufficiently large by Proposition 1.11 (1).
(3) More generally, we define the modality of an ICIS $f$ as follows. Let $U \subset R_{k}^{m}$ an open neighbourhood of $j_{k}(f)$. We set

$$
\begin{aligned}
U(i) & :=\left\{g \in U \mid \operatorname{dim}_{g}\left(U \cap G_{k} \cdot g\right)=i\right\}, i \geq 0 \\
G_{k}-\operatorname{par}(U) & :=\max _{i \geq 0}\{\operatorname{dim} U(i)-i\}
\end{aligned}
$$

and call $G_{k}-\operatorname{par}\left(j_{k}(f)\right):=\min \left\{G-\operatorname{par}(U) \mid U\right.$ a neighbourhood of $\left.j_{k}(f)\right\}$ the number of $G_{k}$-parameters of $j_{k}(f)$ (in $R_{k}^{m}$ ). It can be shown that if $k$ is sufficiently large then $G_{k}-\operatorname{par}\left(j_{k}(f)\right)$ is independent of $k$ and denoted by $G-\operatorname{par}(f)$ or $G-\bmod (f)$ and called the $G$-modality of $f$ (this shown in [GNg16, Definition 2.3] for hypersurfaces; the proof can be adapted for an ICIS). $f$ is 0 -modular iff $G-\bmod (f)=0$.

Proposition 1.14. (1) An ICIS is simple iff it is 0-modular.
(2) The G-modality of an ICIS is semicontinuous. In particular, any deformation of a simple ICIS is again simple.

Proof. We only sketch the proof. (1) We can modify the proof in [GNg16, Proposition 2.14. and Corollary 2.17] (for hypersurfaces) and use the semi-continuity of $\tau(f)$ and $\tau^{s e c}(f)$ (see Proposition 1.11), to show that an ICIS is simple iff it is 0 modular.
(2) It follows from Definition 1.12 and the openness of versality that any deformation of a simple ICIS is again simple. More general, the $G$-modality of an ICIS is semicontinuous. This can be proved as for hypersurfaces, see [GNg16, Proposition 2.7].

The following result is the main general condition for a 0-dimensional ICIS being not simple.

Theorem 1.15. $f=\left(f_{1}, \ldots, f_{n}\right) \in I_{n, n}$ is not simple if one of the following cases occurs:
a) $n \geq 2$ and $\operatorname{ord}(f) \geq 3$.
b) $n \geq 3$ and ord $(f)=2$.

Proof. Set $l=\operatorname{ord}(f)=\min \left\{\operatorname{ord}\left(f_{i}\right)\right\}$. Then the action of $G_{l}$ on $R_{l}^{n}$ induces an algebraic action of the affine algebraic group $G^{\prime}:=G L(n, K) \times G L(n, K)$ on the affine variety $X:=\mathfrak{m}^{l} R^{n} / \mathfrak{m}^{l+1} R^{n}$.

Assume by contradiction that $f$ is simple. Since $f$ is 0 -modular, there is a Zariski neighborhood $U_{l}$ of $j_{l}(f)$ in $R_{l}^{n}$ which intersects only finitely many $G_{l}$-orbits. This implies that there is Zariski neighborhood $U$ of $j_{l}(f)$ in $X$ which intersects only finitely many $G^{\prime}$-orbits, say $U=\bigcup_{i=1}^{s}\left(U \cap O_{i}\right)$.

We get that there exists an orbit $G^{\prime} h, h \in X$, such that

$$
n\binom{n-1+l}{l}=\operatorname{dim} X=\operatorname{dim}(U \cap X)=\max _{i}\left\{\operatorname{dim}\left(O_{i} \cap U\right)\right\} \leq \operatorname{dim} G^{\prime} h
$$

(otherwise $X$ cannot be covered by finitely many orbits).
It is easy to see that the elements $\left\{\left(a^{l} E_{n},(1 / a) E_{n}\right) \mid a \in K^{*}\right\}$, with $E_{n}$ the $n \times n$ identity matrix, are in the stabilizer of $h$ under $G^{\prime}$, hence

$$
\operatorname{dim} G^{\prime} h \leq \operatorname{dim} G^{\prime}-1=2 n^{2}-1
$$

a) If $n \geq 2$ and $l=3$ then $\operatorname{dim} X=\frac{n^{2}(n+1)(n+2)}{6} \geq 2 n^{2}>2 n^{2}-1$. If $l>3$ the l.h.s. gets bigger while the r.h.s. stays constant. Hence $\operatorname{dim} X>\operatorname{dim} G^{\prime} h$ and $f$ cannot be simple for $n \geq 2$ and $l \geq 3$.
b) If $n \geq 3$ and $l=2$ then $\operatorname{dim} X=\frac{n^{2}(n+1)}{2}>2 n^{2}-1$, showing that $f$ is not simple.

For further analysis we need the following splitting lemma in any characteristic.
Definition 1.16. Let $f \in R$. We denote by

$$
H(f)=\left[\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}(0)\right]_{i, j=1, \ldots, n} \in \operatorname{Mat}(n, K)
$$

the Hessian matrix of $f$ and by

$$
\operatorname{corank}(f)=n-\operatorname{rank}(H(f))
$$

the corank of $f . f$ is called non-degenerate if $\operatorname{corank}(f)=0$, otherwise it is called degenerate.

Lemma 1.17 (Right splitting lemma in characteristic $p \neq 2$ ). Let $p \neq 2$ and $f \in \mathfrak{m}^{2}$ be such that $\operatorname{corank}(f)=k \geq 0$. Then

$$
f \stackrel{r}{\sim} g\left(x_{1}, \ldots, x_{k}\right)+x_{k+1}^{2}+\ldots+x_{n}^{2},
$$

where $g \in \mathfrak{m}^{3}$ is uniquely determined up to right equivalence. Of course, $x_{i}^{2}+$ $x_{j}^{2} \stackrel{r}{\sim} x_{i} x_{j}$.

Proof. [GNg16, Lemma 3.9].
Lemma 1.18 (Right splitting lemma in characteristic 2). Let $p=2$ and $f \in \mathfrak{m}^{2} \subset$ $K\left[\left[x_{1}, \ldots, x_{n}\right]\right], n \geq 2$. Then there is an $l, 0 \leq 2 l \leq n$, such that

$$
f \stackrel{r}{\sim} x_{1} x_{2}+x_{3} x_{4}+\ldots+x_{2 l-1} x_{2 l}+g\left(x_{2 l+1}, \ldots, x_{n}\right)
$$

where $g \in\left\langle x_{2 l+1}, \ldots, x_{n}\right\rangle^{3} \subset K\left[\left[x_{2 l+1}, \ldots, x_{n}\right]\right]$, or $g=x_{2 l+1}^{2}+h$ with $h \in\left\langle x_{2 l+1}, \ldots, x_{n}\right\rangle^{3} \subset$ $K\left[\left[x_{2 l+1}, \ldots, x_{n}\right]\right]$ if $2 l<n . g$ is uniquely determined up to right equivalence.

Proof. [GNg16, Lemma 3.12].

## 2 Simple isolated complete intersection singularities in characteristic $p \neq 2$

Let $f=\left(f_{1}, \ldots, f_{n}\right) \in I_{n, n}$ be an ICIS. If $n=1$, then $f$ defines a hypersurface singularity $\stackrel{\mathcal{c}}{\sim} x^{k+1}$, which is simple of type $A_{k}$. Hence we assume $n \geq 2$. If $n \geq 3$ then $\operatorname{ord}(f)=1$ for $f$ simple (by Theorem 1.15) and then $f \sim g \in I_{n-1, n-1}$ by the implicit function theorem. Thus we may assume $n=2$. By Theorem 1.15 an ICIS $\left(f_{1}, \ldots, f_{n}\right) \in I_{n, n}$ can be simple only if $n=2$ and $\operatorname{ord}(f)=2$, which we assume from now on. Let $R=K[[x, y]], \mathfrak{m}=\langle x, y\rangle$ its maximal ideal, and $p=\operatorname{char}(K)$.

Proposition 2.1. Let $p \geq 0$. Let $f=\left(f_{1}, f_{2}\right) \in I_{2,2}$, with $\operatorname{ord}(f)=2$ and $j_{2}\left(f_{i}\right)$ a non-degenerate quadratic form for some $i \in\{1,2\}$. Then there are $m, n \geq 2$ such that

$$
f \sim\left(x y, x^{n}+y^{m}\right)=: F_{n+m-1}^{n, m} .
$$

Proof. By Lemma 1.17 and 1.18 we may assume that $f_{1}=x y$. We can use $x y$ to kill in $f_{2}$ all terms divisible by $x y$. We may therefore assume that $f_{2}=u_{1} x^{n}+u_{2} y^{m}$ with $u_{1} \in K[[x]]$ and $u_{2} \in K[[y]]$ units $\left(u_{1}=0\right.$ or $u_{2}=0$ is not possible, since $\left(f_{1}, f_{2}\right)$ is a complete intersection). We multiply $f_{2}$ with $u_{1}^{-1} u_{2}^{-1}$ and kill with $x y$ the terms
of $u_{1}^{-1} y^{m}$ and $u_{2}^{-1} x^{n}$ and obtain $\alpha x^{n}+\beta y^{m}$ with $\alpha=u_{2}^{-1}(0)$ and $\beta=u_{1}^{-1}(0) \in K$ both different from 0 . Since $K$ is algebraically closed there exist the $n$-th root of $\alpha$ and the $m$-th root of $\beta$ and we find an automorphism mapping the ideal $\left\langle x y, f_{2}\right\rangle$ to $\left\langle x y, x^{n}+y^{m}\right\rangle$.

From now on in this section, we assume that $\operatorname{char}(K)=p \neq 2$ unless otherwise stated.

Lemma 2.2. Let $f, g \in \mathfrak{m}^{2}$. Assume that $j_{2}(f)$ is degenerate. Then there exist $s \geq 3$ and $\alpha \in\{0,1\}$ such that

$$
(f, g) \sim\left(x^{2}+\alpha y^{s}, \sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}\right)
$$

where $t \geq 2, q \geq 1$ and $a_{i}, b_{j} \in K$.
Proof. Using Lemma 1.17 we may assume that $f=x^{2}+h(y)$ with $h=0$ or $h=y^{s} \cdot e$, $s \geq 3, e$ a unit. If $h \neq 0$, we can find (since $p \neq 2$ ) $e_{1}$ with $e_{1}^{2}=e^{-1}$. Using the automorphism $\varphi$ of $K[[x, y]]$ defined by $\varphi(x)=e_{1} x, \varphi(y)=y$ we may assume that $f=x^{2}+y^{s}$. All together we have $f=x^{2}+\alpha y^{s}$ with $\alpha \in\{0,1\}$. The Weierstraß Division Theorem implies that $g=k \cdot\left(x^{2}+\alpha y^{s}\right)+\sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}$ for suitable $k \in K[[x, y]]$ and $a_{i}, b_{j} \in K$. This implies that

$$
\left(x^{2}+\alpha y^{s}, g\right) \sim\left(x^{2}+\alpha y^{s}, \sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}\right)
$$

Lemma 2.3. Let $f=\left(f_{1}, f_{2}\right)=\left(x^{2}+\alpha y^{s}, \sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}\right)$ be an ICIS such that $s \geq 3, t \geq 2, q \geq 1$ and $\alpha \in\{0,1\}$.

1. If $a_{i}=0$ for all $i$ and $b_{q} \neq 0$ then $\alpha=1$ and $f \sim\left(x^{2}+y^{s}, x y^{q}\right)$.
2. If $b_{j}=0$ for all $j$ and $a_{t} \neq 0$ then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$. If $\alpha=0$ or $t \leq s$ then $f \sim\left(x^{2}, y^{t}\right)$.

Assume now that $a_{t} b_{q} \neq 0$.
3. If $t \leq q$ then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$. If additionally $\alpha=0$ or $t \leq s$ then $f \sim\left(x^{2}, y^{t}\right)$.
4. If $t>q$ and $\alpha=0$ then $f \sim\left(x^{2}, y^{t}+x y^{q}\right)$.
5. Let $t>q$ and $\alpha=1$.

If $2 t-2 q-s \neq 0^{1}$ and ( $p=0$ or $p \nmid 2 t-2 q-s$ ) then $f \sim\left(x^{2}+y^{s}, y^{t}+e x y^{q}\right)$, $e \in K[[y]]$ with $e(0)=1$.
Otherwise $f \sim\left(x^{2}+y^{s}, y^{t}+e x y^{q}\right)$ for a suitable unit $e \in K[[y]]$.
6. If $t=q+1$ and $p \nmid t$ then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$.

Proof. (1): $f_{2}=x \sum_{j \geq q} b_{j} y^{j}=u x y^{q}$, $u$ a unit. We get $\left\langle f_{1}, f_{2}\right\rangle=\left\langle x^{2}+\alpha y^{s}, x y^{q}\right\rangle$. Since $f$ is an ICIS $\alpha$ must be 1 .
(2): $f_{2}=\sum_{i \geq t} a_{i} y^{i}=u y^{t}, u$ a unit. This implies $\left\langle f_{1}, f_{2}\right\rangle=\left\langle x^{2}+\alpha y^{s}, y^{t}\right\rangle$. If $\alpha=0$ or $t \leq s$ then obviously $\left\langle x^{2}+\alpha y^{s}, y^{t}\right\rangle=\left\langle x^{2}, y^{t}\right\rangle$.
(3):If $t \leq q$ then $\sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}=y^{t}$. unit, implying $\left\langle f_{1}, f_{2}\right\rangle=\left\langle x^{2}+\alpha y^{s}, y^{t}\right\rangle$. If $\alpha=0$ or $t \leq s$ then $\left\langle x^{2}+\alpha y^{s}, y^{t}\right\rangle=\left\langle x^{2}, y^{t}\right\rangle$.
(4): Multiplying $\sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}$ by the inverse of $\sum_{i \geq t} a_{i} y^{i-t}$ we may assume that $f=\left(x^{2}, y^{t}+e \cdot x y^{q}\right)$ with $e=\sum_{j \geq q} b_{j} y^{j-q}$ a unit. Consider the automorphism $\varphi$ of $K[[x, y]]$ defined by $\varphi(x)=e^{-1} x$ and $\varphi(y)=y$ then $\varphi(f)=\left(e^{-2} x^{2}, y^{t}+x y^{q}\right) \sim$ $\left(x^{2}, y^{t}+x y^{q}\right)$.
(5): Multiplying $\sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}$ by the inverse of $\sum_{i \geq t} a_{i} y^{i-t}$ we may assume that $f=\left(x^{2}+y^{s}, y^{t}+e \cdot x y^{q}\right)$ with $e=\sum_{j \geq q} b_{j} y^{j-q}$ a unit. Consider the automorphism $\varphi$ of $K[[x, y]]$ defined by $\varphi(x)=a x$ and $\varphi(y)=b y$ with units $a, b \in K[[x, y]]$. Then $\varphi(f)=\left(a^{2} x^{2}+b^{s} y^{s}, b^{t} y^{t}+\varphi(e) a b^{q} x y^{q}\right)$. We have to choose $a$ and $b$ such that

- $a^{2}=b^{s}$,
- $b^{t}=e(0) a b^{q}$.

[^0]We get the equation

$$
b^{t-q-\frac{s}{2}}=e(0)
$$

which can be solved if $p \nmid 2 t-2 q-s \neq 0$.
(6): If $\alpha=0$, using (4) we may assume that $f=\left(x^{2}, y^{t}+x y^{t-1}\right)$. Now $y^{t}+x y^{t-1}=$ $\left(y+\frac{1}{2} x\right)^{t}+x^{2} h$ for a suitable $h \in K[[x, y]]$. This implies that $\left\langle x^{2}, y^{t}+x y^{t-1}\right\rangle=$ $\left\langle x^{2},\left(y+\frac{1}{2} x\right)^{t}\right\rangle$. Using the automorphism $\varphi$ of $K[[x, y]]$ definde by $\varphi(x)=x$ and $\varphi(y)=y-\frac{1}{2} x$ we obtain the result.
If $\alpha=1$, using (5) we may assume that $f=\left(x^{2}+y^{s}, y^{t}+e x y^{t-1}\right), n \geq 3$, for a suitable unit $e \in K[[x, y]]$. Now $y^{t}+e x y^{t-1}=\left(y+\frac{1}{t} e x\right)^{t}+x^{2} h$ for a suitable $h \in \mathfrak{m}^{t-2}$. Using the automorphism $\varphi$ of $K[[x, y]]$ definde by $\varphi(x)=x$ and $\varphi(y)=y-\frac{1}{t} x$ we obtain $f \sim\left(x^{2}+\left(y-\frac{1}{t} e x\right)^{s}, y^{t}+x^{2} \bar{h}\right)$ with $\bar{h} \in \mathfrak{m}^{t-2}$. Now $\left\langle x^{2}+\left(y-\frac{1}{t} e x\right)^{s}, y^{t}+\right.$ $\left.\left.x^{2} \bar{h}\right\rangle=\left\langle x^{2}+\left(y-\frac{1}{t} e x\right)^{s}, y^{t}-\left(y-\frac{1}{t} e x\right)^{n} \bar{h}\right)\right\rangle=\left\langle x^{2}+y^{s}+\beta x y^{s-1}+\tilde{k}_{1}, y^{t}+\tilde{h}_{1}\right\rangle=$ $\left\langle\left(x+\frac{1}{2} \beta y^{n-1}\right)^{2}+y^{s}+\tilde{k}_{1}, y^{t}+\tilde{h}_{1}\right\rangle$ with $\beta \in K, \bar{k}_{1}, \tilde{k}_{1} \in \mathfrak{m}^{n+1}$ and $\tilde{h}_{1} \in \mathfrak{m}^{t+1}$. Using the automorphism $\tilde{\varphi}$ of $K[[x, y]]$ definde by $\tilde{\varphi}(x)=x-\frac{1}{2} \beta y^{s-1}$ and $\tilde{\varphi}(y)=y$ we obtain $f \sim\left(x^{2}+y^{s}+k_{1}, y^{t}+h_{1}\right)$ with $k_{1} \in \mathfrak{m}^{s+1}$ and $h_{1} \in \mathfrak{m}^{t+1}$. Iterating this process we obtain $f \sim\left(x^{2}+y^{s}, y^{t}\right)$.

Proposition 2.4. Let $\operatorname{char}(K)=p$.

1. Assume that $p \neq 2$. Let $f=\left(x^{2}+\alpha y^{s}, \sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}\right), \alpha \in\{0,1\}$. If $s \geq 4, t \geq 5$ and $q \geq 3$ then $f$ is not simple.
2. Assume that $p=2$. Then $\left(x^{2}+a x y, h\right), a \in K$, is not simple for $h \in K[[x, y]]$ with $\operatorname{ord}(h) \geq 3$.

Proof. The following Singular computation shows that $\left(x^{2}+y^{4}, y^{5}+a x y^{3}\right)$ is not equivalent to $\left(x^{2}+y^{4}, y^{5}+b x y^{3}\right)$, except $a^{2}=b^{2}$. In particular $\left(x^{2}+y^{4}, y^{5}+a x y^{3}\right)$ is not simple for each $a \in K$. We show below that this implies that $f$ is not simple if $\alpha=1$. $f$ with $\alpha=0$ is not simple (by Proposition 1.14 (2)), since it deforms to $f$ with $\alpha=1$ and $s \geq 4$.

Singular computation to show that $\left(x^{2}+y^{4}, y^{5}+a x y^{3}\right)$ is not simple:

```
ring R=(0,a,b,l,m0,m1,m2,m3,m4,m5,m6,m7,n0,n1,n2,n3,n4,n5,n6,n7,o,s0,s1,
    t,u0,u1,v),(x,y),dp;
int i;
```

```
poly f=x2+y4;
poly g=y5+a*xy3;
poly h=x2+y4;
poly k=y5+b*xy3;
map phi=R,m0*y+m1*y2+m2*x+m3*xy+m4*y3+m5*y4+m6*x2+m7*xy2,
                                    n0*y+n}1*y2+n2*x+n3*xy+n4*y3+n5*y4+n6*x2+n7*xy2
poly F=jet(phi(f)-(s0+s1*y)*h-t*k,5);
poly G=jet(phi(g)-(u0+u1*y)*h-v*k,5);
// R ist graded with deg (x)=2, deg(y)=1, we compute up to degree 5
matrix M1=coef(F,xy);
matrix M2=coef(G,xy);
ideal I;
for(i=1;i<=ncols(M1);i++){I[size(I)+1]=M1[2,i];}
for(i=1;i<=ncols(M2);i++){I[size(I)+1]=M2[2,i];}
ring S=0, (a,b,l,m0,m1,m2,m3,m4,m5,m6,m7,n0,n1,n2,n3,n4,n5,n6,n7,o,s0,s1,
    t,u0,u1,v,u,z),dp;
ideal I=imap(R,I);
I=I,(m0*n2-n0*m2)*o-1,(s0*v-u0*t)*z-1; //to make matrix and map invertible
ideal F=eliminate(I,
l*m0*m1*m2*m3*m4*m5*m6*m7*n0*n1*n2*n}3*\textrm{n}4*\textrm{n}5*\textrm{n}6*\textrm{n}7*o*\textrm{s}0*\textrm{s}1*\textrm{t}*\textrm{u}0*\textrm{u}1*\textrm{v}*o*\textrm{z})
F;
> F[1]=a^2-b^2
```

Let $s \geq 4, t \geq 5$ and $q \geq 3$. Set $f_{a}:=\left(x^{2}+y^{4}, y^{5}+a x y^{3}\right), a \in K$, and $F_{\lambda}:=f+\lambda f_{a} . \quad F_{\lambda}$ is a deformation of $f$ and we show that $F_{\lambda}$ is not simple for almost all $\lambda$ (i.e. all except finitely many). Then, by the semicontinuity of the modality (Proposition 1.14), $f$ is not simple.
We have

$$
F_{\lambda}=\left(x^{2}(1+\lambda)+y^{4}\left(\lambda+\alpha y^{s-4}\right), y^{5}\left(\lambda+\sum_{i \geq t-5} a_{i} y^{i}\right)+x\left(\lambda a y^{3}+\sum_{j \geq q} b_{j} y^{j}\right)\right.
$$

For general $\lambda$ this is equivalent to

$$
\left(x^{2}+y^{4}, y^{5}+x y^{3}\left(b a+\sum_{j \geq q}-3 b_{j}^{\prime} y^{j}\right)\right)
$$

for some $b \neq 0$. In fact, replace $x$ by $\sqrt{1+\lambda}^{-1} x$, multiply $F_{\lambda, 1}$ with the inverse of $v=\left(\lambda+\alpha y^{s-4}\right)$ and replace $x$ by $\sqrt{v} x$ and, finally, multipy $F_{\lambda, 2}$ with the inverse of $\lambda+\sum_{i \geq t-5} a_{i} y^{i}$. With $a \neq 0$ the factor of $x y^{3}$ is a unit and we get $\left(x^{2}+y^{4}, y^{5}+\right.$ $x y^{3} b_{0} a+g$ ), with $b_{0} \neq 0$ and $g=\sum_{j \geq 4} b_{j}^{\prime} y^{j}$. By the SingULAR computation above $h_{0}:=\left(x^{2}+y^{4}, y^{5}+x y^{3} b_{0} a\right)$ is not simple.

To see that $h_{g}:=\left(x^{2}+y^{4}, y^{5}+x y^{3} b_{0} a+g\right)$ is not simple, we cannot use the semicontinuity since $h_{g}$ is a deformation of $h_{0}$ with a priori smaller modality. But we can prove the non-simpleness in two ways:
a) It suffices that the 5 -jet of $h_{g}$ is not simple. This follows from a similar (but longer ${ }^{2}$ ) Singular computation as above, which shows that $\left(x^{2}+y^{4}, y^{5}+a x y^{3}+\right.$ $\left.c x y^{4}\right)$ is not equivalent to $\left(x^{2}+y^{4}, y^{5}+b x y^{3}+d x y^{4}\right)$, except $a^{2}=b^{2}$ for arbitrary $c, d$.
b) We notice that $h_{0}$ is quasi-homogeneous of type $(4,5 ; 1,2)$ and it suffices to show that the weighted 5 -jet of $h_{g}$ is not simple. We make an "Ansatz" with a general (not quasi-homogeneous) coordinate transformation up to weighted degree 5, apply it to $(f, g)_{a, c}=\left(x^{2}+y^{4}, y^{5}+a x y^{3}+c x y^{4}\right)$ and compare it with $(h, k)=:(f, g)_{b, d}$ :

```
map phi=R,m0*y+m1*y2+m2*x+m3*xy+m4*y3+m5*y4+m6*x2+m7*xy2,
    n}0*\textrm{y}+\textrm{n}1*\textrm{y}2+\textrm{n}2*\textrm{x}+\textrm{n}3*\textrm{xy}+\textrm{n}4*\textrm{y}3+\textrm{n}5*\textrm{y}4+\textrm{n}6*\textrm{x}2+\textrm{n}7*\textrm{xy}2
poly F=jet(phi(f)-(s0+s1*y)*h-t*k,5);
poly G=jet(phi(g)-(u0+u1*y)*h-v*k,5);
```

Then $(f, g)_{a, c} \sim(f, g)_{b, d}$ implies $F=G=0$. Looking at $F$, we see that m0 must be 0 and hence phi $\left(\mathrm{cxy}^{4}\right)$ has weighted degree bigger than 5 . Therefore $c x y^{4}$ can be deleted and the first Singular computation above suffices to show that $h_{g}$ is not simple.

In characteristic 2 the following SINGULAR computation shows that $\left(x^{2}+a x y, c x y^{2}+\right.$ $\left.d y^{3}\right) \sim\left(x^{2}+b x y, c x y^{2}+d y^{3}\right)$ implies $a^{4} b^{2} c^{4} d+a^{2} b^{4} c^{4} d+a^{4} c^{2} d^{3}+b^{4} c^{2} d^{3}=0$, which happens, for fixed $a, c, d$, only for finitely many $b$.
It follows that $\left\langle x^{2}+a x y, j_{3}(h)\right\rangle=\left\langle x^{2}+a x y, c x y^{2}+d y^{3}\right\rangle, c, d$ suitable, is not simple. Hence $\left(x^{2}+a x y, h\right)$ is not simple.

```
ring R=(2,a,b,c,d,m0,m1,n0,n1,o,s,t,u,v),(x,y),dp;
int i;
poly f=x2+a*xy; //a the parameter of the family
poly g=c*xy2+d*y3; //generic element of degree 3
```

[^1]```
poly h=x2+b*xy;
poly k=c*xy2+d*y3;
map phi=R,m0*y+m1*x,n0*y+n1*x;
poly F=phi(f)-s*h-t*k;
poly G=phi(g)-u*h-v*k;
matrix M1=coef(F,xy);
matrix M2=coef(G,xy);
ideal I;
for(i=1;i<=ncols(M1);i++){I[size(I)+1]=M1[2,i];}
for(i=1;i<=ncols(M2);i++){I[size(I)+1]=M2[2,i];}
ring S=2,(a,b,c,d,m0,m1,n0,n1,o,s,t,u,v,u,z),dp;
ideal I=imap (R,I);
I=I,(m0*n1-n0*m1)*o-1,(s*v-u*t)*\textrm{z}-1;
ideal F=eliminate(I,m0*m1*n0*n1*o*S*t*u*V*o*z);
F;
```



Proposition 2.5. If $p \neq 2$, then the following ICIS are the only candidates for being simple:

1. $F_{s+m-1}^{s, m}:\left(x y, x^{s}+y^{m}\right), s, m \geq 2$,
2. $G_{5}^{0}:\left(x^{2}, y^{3}\right)$,
3. $G_{5}^{1}:\left(x^{2}, x y^{2}+y^{3}\right)$ in char $p=3$,
4. $G_{7}:\left(x^{2}, y^{4}\right)$,
5. $H_{s+3}:\left(x^{2}+y^{s}, x y^{2}\right), s \geq 3$,
6. $I_{2 t-1}^{0}:\left(x^{2}+y^{3}, y^{t}\right), t \geq 4$,
7. $I_{2 t-1}^{1}:\left(x^{2}+y^{3}, y^{t}+x y^{t-1}\right), t \geq 4$ if $p \mid t$,
8. $I_{2 q+2}^{0}:\left(x^{2}+y^{3}, x y^{q}\right), q \geq 3$,
9. $I_{2 q+2}^{1}:\left(x^{2}+y^{3}, x y^{q}+y^{q+2}\right), q \geq 3$ if $p \mid 2 q+3$.

Proof. Let $f=\left(f_{1}, f_{2}\right)$ be an ICIS.
(I) If $j_{2}\left(f_{i}\right)$ is non-degenerate for some $i$ we obtain $f \sim F_{s+m-1}^{s, m}$ for suitable $s, m \geq 2$, using Proposition 2.1.
(II) If $j_{2}\left(f_{i}\right)$ is degenerate for some $i$ we obtain using Lemma 2.2 that

$$
\begin{aligned}
& f \sim\left(x^{2}+\alpha y^{s}, \sum_{i \geq t} a_{i} y^{i}+x \sum_{j \geq q} b_{j} y^{j}\right), a_{i}, b_{j} \in K, \\
& \quad \text { with } s \geq 3, t \geq 2, q \geq 1 \text { and } \alpha \in\{0,1\} .
\end{aligned}
$$

Using Proposition 2.4 and that $\operatorname{ord}(f)=2$, we obtain that if $f$ is simple, then

$$
s=3 \text { or } 2 \leq t \leq 4 \text { or } 1 \leq q \leq 2 .
$$

Now we use these inequalities and Lemma 2.3.

1. Let $a_{i}=0$ for all $i$ and $b_{q} \neq 0$ (this we may assume since $f$ is a complete intersection). Then $f \sim\left(x^{2}+y^{s}, x y^{q}\right)(\alpha=1$ since $f$ is a complete intersection $)$. For $f$ to be simple we have $s=3$ and then we get $I_{2 q+2}^{0}$ or we have $q=2$ and then we get $H_{s+3}$.
2. Let $b_{j}=0$ for all $j$ and $a_{t} \neq 0$. Then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$.

- If $\alpha=0$ or $t \leq s$ then $f \sim\left(x^{2}, y^{t}\right)$. Then $t$ must be 2,3 or 4 . For $t=2$ we get $\left(x^{2}, y^{2}\right) \sim\left(x y, x^{2}+y^{2}\right)=F_{3}^{2,2}$. For $t=3,4$ we obtain $G_{5}^{0}, G_{7}$.
- If $\alpha=1$ and $t>s$ then $s=3$, and we obtain $I_{2 t-1}^{0}$.

3. Assume now that $a_{t} b_{q} \neq 0$.

Lemma 2.3 implies:
(a) If $t \leq q$ then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$.

- If $\alpha=0$ or $t \leq s$ then $f \sim\left(x^{2}, y^{t}\right)$. We have $2 \leq t \leq 4$ and we obtain $F_{3}^{2,2}$ or $G_{5}^{0}$ or $G_{7}$.
- If $\alpha=1$ and $t>s$ then $s=3$, and we obtain $I_{2 t-1}^{0}$.
(b) If $t>q$ then, by Lemma 2.3 (4) and (5) and using the automorphism $\varphi$ of $K[[x, y]]$ defined by $\varphi(x)=e^{-1} x$ and $\varphi(y)=y$, we obtain

$$
\begin{aligned}
& f \sim\left(x^{2}, y^{t}+x y^{q}\right), \text { if } \alpha=0, \\
& f \sim\left(x^{2}+y^{s}, y^{t}+e x y^{q}\right) \sim\left(x^{2}+e^{2} y^{s}, y^{t}+x y^{q}\right), \text { if } \alpha=1,
\end{aligned}
$$

where $e \in K[[y]]$ a unit with $e(0)=1$ if $2 t-2 q-s \neq 0$ and $(p=0$ or $p \nmid 2 t-2 q-s)$.
i. Let $t=q+1$.
(i.1) If $p \nmid t$, then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}\right)$, by Lemma 2.3 (6).

- If $\alpha=0$ or $t \leq s$ then $f \sim\left(x^{2}, y^{t}\right), 2 \leq t \leq 4$, and we obtain $F_{3}^{2,2}$ or $G_{5}^{0}$ or $G_{7}$.
- If $\alpha=1$ and $t>s$ then $s=3$, and we obtain $I_{2 t-1}^{0}$.
(i.2) If $p \mid t$, then $f \sim\left(x^{2}+\alpha y^{s}, y^{t}+e x y^{t-1}\right)$ with $e \in K[[y]]$ a unit.
- If $\alpha=0$ then $p=3$ and $t=3$ and by Lemma 2.3 (4) we get $G_{5}^{1}$.
- If $\alpha=1$ then either $s=3$ or $(t=3$ and $p=3)$.
(i.2.1) For the first case $(s=3)$ we obtain from Lemma 2.3 (5)

$$
f \sim\left(x^{2}+y^{3}, y^{t}+e x y^{t-1}\right) \sim\left(x^{2}+e^{2} y^{3}, y^{t}+x y^{t-1}\right)
$$

$e \in K[[y]]$ with $e(0)=1$ (since $p \mid t$ we have $p \nmid 2 t-2 q-s$ since $s=3$ ). Let $e^{2}=\sum_{j} a_{j} y^{j}$ with $a_{0}=1$. Define $F(z):=z \sum_{j} a_{j} y^{j} z^{j}-1$. $F$ has the following properties:

- $F(1) \in\langle y\rangle$,
- $F^{\prime}(1)$ is a unit.

The implicit function theorem implies that there is a unit $z(y)$ such that

$$
z(y) e^{2}(z(y) x, z(y) y)=z(y) \sum_{j} a_{j} y^{j} z(y)^{j}=1
$$

The automorphism $\varphi$ defined by $\varphi(x)=z(y) x$ and $\varphi(y)=z(y) y$ gives the equivalence $\left(x^{2}+e^{2} y^{3}, y^{t}+x y^{t-1}\right) \sim\left(x^{2}+y^{3}, y^{t}+x y^{t-1}\right)$. If $p=t=3$ we get $G_{5}^{1}$. If $t \geq 4$ we get $I_{2 t-1}^{1}$.
(i.2.2) To see the second case $(s>3, t=3$ and $p=3)$ we start with $f=\left(f_{1}, f_{2}\right) \sim\left(x^{2}+e^{2} y^{s}, y^{3}+x y^{2}\right)$. Substracting $e^{2} y^{s-3} f_{2}$ from $f_{1}$ we get

$$
\begin{aligned}
& \left.\left\langle x^{2}+e^{2} y^{s}, y^{3}+x y^{2}\right)\right\rangle=\left\langle x^{2}-e^{2} x y^{s-1}, y^{3}+x y^{2}\right\rangle \\
& \quad=\left\langle\left(x-\frac{1}{2} e^{2} y^{s-1}\right)^{2}-\frac{1}{4} e^{4} y^{2 s-2}, y^{3}+x y^{2}\right\rangle .
\end{aligned}
$$

Using the automorphism $\varphi(x)=x+\frac{1}{2} e^{2} y^{s-1}$ and $\varphi(y)=y$ we obtain

$$
\left(x^{2}-\frac{1}{4} e^{4} y^{2 s-2}, y^{3}+x y^{2}+\frac{1}{2} e^{2} y^{s+1}\right)=\left(x^{2}-\frac{1}{4} e^{4} y^{2 s-2}, u y^{3}+x y^{2}\right)
$$

with $u=1+\frac{1}{2} e^{2} y^{s-2}$ a unit.
Setting $\varphi(x)=u x$ and $\varphi(y)=y$ we obtain

$$
\left(u^{2} x^{2}-\frac{1}{4} e^{4} y^{2 s-2}, u y^{3}+u x y^{2}\right) \sim\left(x^{2}-\frac{1}{4} u^{-2} e^{4} y^{2 s-2}, y^{3}+x y^{2}\right)
$$

which raises the power $s$ of $y$ to $2 s-2$. Iterating this process we obtain $f \sim\left(x^{2}, y^{3}+x y^{2}\right)$ which is $G_{5}^{1}$.
ii. Now assume that $t>q+1$.
(ii.1) If Let $\alpha=0$. Then Proposition 2.4 implies $q \leq 2$.

- If $q=1$, we obtain $\left(x^{2}, y^{t}+x y\right)$. Similar as below with $q=2$ this is equivalent to $\left(x^{2}+y^{2 t-2}, x y\right)$, which is $F_{2 t-1}^{2,2 t-2}, t \geq 2$.
- If $q=2$ we obtain $\left(x^{2}, y^{t}+x y^{2}\right)$. But

$$
\begin{gathered}
\left(x^{2}, y^{t}+x y^{2}\right)=\left(x^{2}, y^{2}\left(y^{t-2}+x\right)\right) \sim\left(\left(x-y^{t-2}\right)^{2}, x y^{2}\right) \\
=\left(x^{2}-2 x y^{t-2}+y^{2 t-4}, x y^{2}\right) \sim\left(x^{2}+y^{2 t-4}, x y^{2}\right), \text { since } t \geq 4 .
\end{gathered}
$$

We get $H_{2 t-1}$.
(ii.2) Let $\alpha=1$. Using 3.(b) above we can start with $\left(x^{2}+y^{s}, y^{t}+\right.$ $e x y^{q}$ ), with $e=e(y)=e_{0}+e_{k} y^{k}+$ higher terms, where $e_{0}$ and $e_{k}$ are non-zero.
Then Proposition 2.4 implies $s=3$ or $2 \leq t \leq 4$ or $1 \leq q \leq 2$.
(ii.2.1) Let $s \geq 3$ and $q \leq 2$

- Let $q=1$. Set $m=: \min \{s, 2 t-2\}$ if $s \neq 2 t-2$ or if $s=2 t-2$ and $e^{-2}(0) \neq-1$. If $s=2 t-2$ and $e^{-2}(0)=-1$ set $m=: s+k$.
We get
$\left(x^{2}+y^{s}, y^{t}+e x y\right) \sim\left(x^{2}+y^{s}, y\left(e^{-1} y^{t-1}+x\right)\right.$
$\left.\sim\left(x-e^{-1} y^{t-1}\right)^{2}+y^{s}, x y\right) \sim\left(x^{2}+e^{-2} y^{2 t-2}+y^{s}, x y\right)$
$\sim\left(x^{2}+y^{m} u, x y\right) \sim\left(x^{2} u^{-1}+y^{m}, x y\right), u(y)$ a unit.
We get $F_{m+1}^{2, m}$ since $p>2$. If $s=3$ we get $F_{4}^{2,3}$.
- Let $q=2$. Consider $\left(x^{2}+y^{s}, y^{t}+e x y^{2}\right), e(y)=e_{0}+e_{k} y^{k}+$ higher terms, a unit. Set $m=: \min \{s, 2 t-4\}$ if $s \neq 2 t-4$ or if $s=2 t-4$ and $e^{-2}(0) \neq-1$. If $s=2 t-4$ and $e^{-2}(0)=-1$ set $m=: s+k$.
We get $\left(x^{2}+y^{s}, y^{t}+e x y^{2}\right) \sim\left(x^{2}+y^{m}, x y^{2}\right)$, i.e. $H_{m+3}$.
This follows from

$$
\begin{gathered}
\left(x^{2}+y^{s}, y^{t}+e x y^{2}\right)=\left(x^{2}+y^{s}, y^{2}\left(e^{-1} y^{t-2}+x\right)\right) \sim \\
\left(\left(x-e^{-1} y^{t-2}\right)^{2}+y^{s}, x y^{2}\right)=\left(x^{2}-2 e^{-1} x y^{t-2}+e^{-2} y^{2 t-4}+y^{s}, x y^{2}\right) \sim \\
\left(x^{2}+e^{-2} y^{2 t-4}+y^{s}, x y^{2}\right) \sim\left(x^{2}+u y^{m}, x y^{2}\right) \sim\left(u^{-1} x^{2}+y^{m}, x y^{2}\right), u \text { a } \\
\text { unit, } \sim\left((v x)^{2}+y^{m}, x y^{2}\right)\left(v^{2}=u^{-1}\right) \sim\left(x^{2}+y^{m}, v^{-1} x y^{2}\right) \\
\sim\left(x^{2}+y^{m}, x y^{2}\right) .
\end{gathered}
$$

(ii.2.2) Let $s=3$ and $t \geq q+3$. We have

$$
y^{t}+e x y^{q}=y^{t-3}\left(x^{2}+y^{3}\right)+\left(e-x y^{t-q-3}\right) x y^{q},
$$

with $\left(e-x y^{t-q-3}\right)$ a unit. Hence $\left(x^{2}+y^{3}, e x y^{q}+y^{t}\right) \sim\left(x^{2}+y^{3}, x y^{q}\right)$.
(ii.2.3) Let $s=3$ and $t=q+2$.

We consider $(f, g)=\left(x^{2}+y^{3}, x y^{q}+e y^{q+2}\right)$ and, using Lemma 2.3 (5),
we may assume that $e(0)=1$. We make the Ansatz

$$
\begin{gathered}
\varphi(x)=x+\alpha y^{2} \text { and } \varphi(y)=y+\beta x \\
\varphi(f)=x^{2}+y^{3}+(2 \alpha+3 \beta) x y^{2}+\left(3 \beta^{2} y+\beta^{3} x\right) x^{2}+\alpha^{2} y^{4} \\
\varphi(g)=x y^{q}+q \beta x^{2} y^{q-1}+(\alpha+1) y^{q+2}+\text { terms of w-deg }>2 q+4,
\end{gathered}
$$

with $\mathrm{w}(x)=3, \mathrm{w}(y)=2$. Then
$\varphi(g)-q \beta y^{q-1} \varphi(f)=x y^{q}+(\alpha+1-q \beta) y^{q+2}+$ terms of $\mathrm{w}-\mathrm{deg}>2 q+4$.
We want to remove the term $x y^{2}$ from $\varphi(f)$ and $y^{q+2}$ from the last equation.

If $p$ does not divide $2 q+3$ then this is possible choosing

$$
\alpha=-\frac{3}{2 q+3} \text { and } \beta=\frac{2}{2 q+3} .
$$

We obtain

$$
\begin{gathered}
(f, g) \sim\left(x^{2}+y^{3}+\left(3 \beta^{2} y+\beta^{3} x\right) x^{2}+\alpha^{2} y^{4}, x y^{q}+\text { terms of w-deg }>2 q+4\right) \\
\sim\left(x^{2}+\bar{e} y^{3}, x y^{q}+\text { terms of } \mathrm{w}-\operatorname{deg}>2 q+4\right)
\end{gathered}
$$

for a suitable unit $\bar{e}(x, y)$ with $\bar{e}(0)=1$.
Now consider the polynomial $F(z)=\bar{e}(z x, z y) z-1$. We have

$$
F(1) \in\langle x, y\rangle \text { and } \frac{d F}{d z}(1)=\bar{e}(z x, z y)(1)+z \frac{d \bar{e}(z x, z y)}{d z}(1) \in 1+\langle x, y\rangle .
$$

The Implicit Function Theorem implies that there exists an unit $z(x, y) \in K[[x, y]]$ such that

$$
F(z(x, y))=\bar{e}(z(x, y) x, z(x, y) y) z(x, y)-1=0
$$

This implies

$$
\begin{gathered}
\left(x^{2}+\bar{e} y^{3}, x y^{q}+\text { terms of w-deg }>2 q+4\right) \\
\sim\left(x^{2}+y^{3}, x y^{q}+\text { terms of w-deg }>2 q+4\right) \\
\sim\left(x^{2}+y^{3}, x y^{q}+\gamma y^{l}\right)
\end{gathered}
$$

for a suitable $\gamma$ and $l \geq q+3$. This case is already settled (see ii.2.2).
If $p \mid 2 q+3$ we consider for $(f, g)=\left(x^{2}+y^{3}, x y^{q}+e y^{q+2}\right)$ with $e(0)=1$ the polynomial $F(z)=z e\left(z^{3} x, z^{2} y\right)-1$. We have
$F(1) \in\langle x, y\rangle$ and $\frac{d F}{d z}(1)=e\left(z^{3} x, z^{2} y\right)(1)+z \frac{d e\left(z^{3} x, z^{2} y\right)}{d z}(1) \in 1+\langle x, y\rangle$. The Implicit Function Theorem implies that there exists an unit $z(x, y) \in K[[x, y]]$ such that

$$
F(z(x, y))=e\left(z(x, y)^{3} x, z(x, y)^{2} y\right) z(x, y)-1=0
$$

The map $\varphi: K[[x, y]] \longrightarrow K[[x, y]]$ define by $\varphi(x)=z(x, y)^{3} x$ and $\varphi(y)=z(x, y)^{2} y$ applied to $(f, g)$ gives

$$
\begin{aligned}
\left(z(x, y)^{6} x^{2}+z(x, y)^{6} y^{3}\right. & \left., z(x, y)^{2 q+3} x y^{q}+e\left(z(x, y)^{3} x, z(x, y)^{2} y\right) z(x, y)^{2 q+4} y^{q+2}\right) \\
& \sim\left(x^{2}+y^{3}, x y^{q}+y^{q+2}\right)
\end{aligned}
$$

Theorem 2.6. Let $p=\operatorname{char}(K) \neq 2$. An ICIS $f \in K[[x, y]]^{2}$ is simple iff it is contact equivalent to one of the normal forms in Table 2.

| Type | Normal form of $f$ |
| :---: | :---: |
| $F_{m+n-1}^{m, n}, m, n \geq 2$ | $\left(x y, x^{m}+y^{n}\right)$ |
| $G_{5}$ | $G_{5}^{0} \quad\left(x^{2}, y^{3}\right)$ and additionally <br> $G_{5}^{1} \quad\left(x^{2}, x y^{2}+y^{3}\right)$ if $p=3$ |
| $G_{7}$ | $\left(x^{2}, y^{4}\right)$ |
| $H_{n+3}, n \geq 3$ | $\left(x^{2}+y^{n}, x y^{2}\right)$ |
| $I_{2 t-1}, \quad t \geq 4$ | $I_{2 t-1}^{0}\left(x^{2}+y^{3}, y^{t}\right)$ and additionally $I_{2 t-1}^{1}\left(x^{2}+y^{3}, y^{t}+x y^{t-1}\right)$ if $p \mid t$ |
| $I_{2 q+2}, q \geq 3$ | $\begin{aligned} & I_{2 q+2}^{0}\left(x^{2}+y^{3}, x y^{q}\right) \text { and additionally } \\ & I_{2 q+2}^{1}\left(x^{2}+y^{3}, x y^{q}+y^{q+2}\right) \text { if } p \mid 2 q+3 \end{aligned}$ |

Table 2
Proof. By Proposition 2.5 we have only to show that the normal forms $f$ in Table 2 are simple. By Definition 1.12 we have to consider the semiuniversal deformation (with section) of $f=\left(f_{1}, f_{2}\right)$,

$$
F_{\mathbf{t}}(\mathbf{x}):=F\left(\mathbf{x}, t_{1}, \ldots, t_{d}\right)=f(\mathbf{x})+\sum_{i=1}^{d} t_{i} g_{i}(\mathbf{x})
$$

where $g_{1}, \ldots, g_{d}$ is a $K$-basis of

$$
T^{1, s e c}(f)=\mathfrak{m} K[[x, y]]^{2} /\left\langle f_{1}, f_{2}\right\rangle K[[x, y]]^{2}+\mathfrak{m}\langle\partial f / \partial x, \partial f / \partial y\rangle
$$

$f$ is simple if the set $\left\{F_{\mathbf{t}}(\mathbf{x}), \mathbf{t} \in K^{d}\right\}$ decomposes into only finitely many contact classes. Moreover, by the argument at the beginning of Section 2 we need to consider only those $g_{i}$ with $\operatorname{ord}\left(g_{i}\right) \geq 2$ (if $\operatorname{ord}\left(g_{i}\right)=1$ for some $i$ we get an $A_{k}$ hypersurface singularity with bounded $k$ ).

We have the following bases of $T^{1, \text { sec }}(f)$ :

1. $F_{m+n-1}^{m, n}, m, n \geq 2$

- $p \nmid m$ or $p \nmid n:(x, 0),(y, 0),(0, x), \ldots,\left(0, x^{m-1}\right),(0, y), \ldots,\left(0, y^{n-1}\right)$.
- $p|m, p| n$ and $m>n:(0, x), \ldots,\left(0, x^{m}\right),(0, y), \ldots,\left(0, y^{n-1}\right)$.
- $p \mid m$ and $m=n:(x, 0),(y, 0),(0, x), \ldots,\left(0, x^{m-1}\right),(0, y), \ldots,\left(0, y^{n}\right)$.
$F_{m+n-1}^{m, n}$ is of quasi-homogeneous type $(m+n, m n ; m, n)$.

2. $G_{5}^{0}$

- $p=3:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y),(0, x y),\left(0, y^{2}\right),\left(0, x y^{2}\right)$.
- $p \geq 5:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y),(0, x y),\left(0, y^{2}\right)$.
$G_{5}^{1}$
$(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y),(0, x y),\left(0, y^{2}\right)$.
$G_{5}$ is of quasi-homogeneous type $(2,3 ; 1,1)$.

3. $G_{7}$
$(x, 0),(y, 0),\left(y^{2}, 0\right),\left(y^{3}, 0\right),(0, x),(0, y),(0, x y),\left(0, y^{2}\right),\left(0, x y^{2}\right),\left(0, y^{3}\right)$.
$G_{7}$ is of quasi-homogeneous type $(2,4 ; 1,1)$.
4. $H_{n+3}, n \geq 3$
$(x, 0),(y, 0), \ldots,\left(y^{n-1}, 0\right),(0, x),(0, y),(0, x y),\left(0, y^{2}\right),\left(0, y^{3}\right)$.
$H_{n+3}$ is of quasi-homogeneous type $(2 n, n+4 ; n, 2)$.
5. a) $I_{2 t-1}^{0}, t \geq 4$.

- $p \nmid t:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, y), \ldots,\left(0, y^{t-1}\right),(0, x),(0, x y), \ldots,\left(0, x y^{t-2}\right)$
- $p \mid t:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, y), \ldots,\left(0, y^{t-1}\right),(0, x),(0, x y), \ldots,\left(0, x y^{t-1}\right)$.
b) $I_{2 t-1}^{1}, t \geq 4$.
- $p \mid t:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, y), \ldots,\left(0, y^{t-1}\right),(0, x),(0, x y), \ldots,\left(0, x y^{t-2}\right)$.
$I_{2 t-1}^{0}$ is of quasi-homogeneous type $(6,2 t ; 3,2)$.

6. a) $I_{2 q+2}^{0}, q \geq 3$

- $p \nmid 2 q+3:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y), \ldots,\left(0, y^{q+1}\right)$, $(0, x y),\left(0, x y^{2}\right), \ldots,\left(0, x y^{q-1}\right)$
- $p \mid 2 q+3:(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y), \ldots,\left(0, y^{q+2}\right)$,

$$
(0, x y),\left(0, x y^{2}\right), \ldots,\left(0, x y^{q-1}\right)
$$

b) $I_{2 q+2}^{1}, q \geq 3, p \mid 2 q+3$.
$(x, 0),(y, 0),\left(y^{2}, 0\right),(0, x),(0, y), \ldots,\left(0, y^{q+1}\right),(0, x y),\left(0, x y^{2}\right), \ldots,\left(0, x y^{q-1}\right)$
$I_{2 q+2}^{0}$ is of quasi-homogeneous type $(6,2 q+3 ; 3,2)$.

1. We prove that $F_{m+n-1}^{m, n}=\left(x y, x^{m}+y^{n}\right)$ is simple:

Since $j_{2}\left(f_{1}\right)$ is non-degenerate, this holds also for any deformation, which is then of type $F_{p+q-1}^{p, q}$ for some $p, q$ by Proposition 2.5. Since $\tau^{\text {sec }}(f)$ depends only on $n$ and $m$ (by (1) above) and since $\tau^{s e c}$ is semicontinuous by Proposition 1.11 there are only finitely many $F_{p+q-1}^{p, q}$ into which $F_{m+n-1}^{m, n}$ deforms. Hence $f$ is simple.
2. a) We prove that $G_{5}^{0}=\left(x^{2}, y^{3}\right)$ is simple:

- $p=3$ : We have to consider

$$
g=\left(x^{2}+a y^{2}, b y^{2}+y^{3}+c x y+d x y^{2}\right)
$$

where $a, b, c, d \in K$.
i) $c \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $c=0$ :

1) $a \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $a=0$ :
2.1) $b \neq 0: g=\left(x^{2}, b y^{2}+y^{3}+d x y^{2}\right)$. By Lemma 2.3,,$g \sim\left(x y, x^{2}+y^{2}\right)=F_{3}^{2,2}$.
2.2) $b=0: g=\left(x^{2}, y^{3}+d x y^{2}\right)$.
2.2.1) $d=0: g=G_{5}^{0}$.
2.2.2) $d \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}+x y^{2}\right)=G_{5}^{1}$.

- $p \geq 5$ : Consider

$$
g=\left(x^{2}+a y^{2}, b y^{2}+y^{3}+c x y\right)
$$

where $a, b, c \in K$.
i) $c \neq 0$ : By Proposition 2.1, there are $m, n \geq 2$ such that $g \sim F_{m+n-1}^{m, n}$.
ii) $c=0$ :

1) $a \neq 0$ : By Proposition 2.1, there are $s, t \geq 2$ such that $g \sim F_{s+t-1}^{s, t}$.
2) $a=0$ :
2.1) $b \neq 0$ : Since $p \neq 2$, let $u \in K[[y]]$ be such that $u^{2}=y+b$. Consider the automorphism $\phi: R \rightarrow R, x \mapsto x, y \mapsto u y$. Then

$$
\phi^{-1}(g)=\phi^{-1}\left(x^{2}, y^{2} u^{2}\right)=\left(x^{2}, y^{2}\right) \sim\left(x y, x^{2}+y^{2}\right)=F_{3}^{2,2} .
$$

2.2) $b=0: g \sim G_{5}^{0}$.
2. b) We prove $G_{5}^{1}=\left(x^{2}, x y^{2}+y^{3}\right)$ is simple. Consider

$$
g=\left(x^{2}+c y^{2}, a y^{2}+y^{3}+b x y+x y^{2}\right),
$$

where $a, b, c \in K$.
i) $c \neq 0$ : By Proposition 2.1, there are $m, n \geq 2$ such that $g \sim F_{m, n}^{m, n}$.
ii) $c=0: g=\left(x^{2}, a y^{2}+y^{3}+b x y+x y^{2}\right)$

1) $a \neq 0$ :
1.1) $b \neq 0$ : By Proposition 2.1, there are $s, t \geq 2$ such that $g \sim F_{s+t-1}^{s, t}$.
1.2) $b=0: g=\left(x^{2}, a y^{2}+y^{3}+x y^{2}\right)$
1.2.1) $a=0: g=G_{5}^{1}$.
1.2.2) $a \neq 0$ : By Lemma 2.3, $g \sim F_{3}^{2,2}$.
2) $a=0: g=\left(x^{2}, y^{3}+b x y+x y^{2}\right)$
2.1) $b=0: g=G_{5}^{1}$.
2.2) $b \neq 0$ : By Proposition 2.1, there are $k, l \geq 2$ such that $g \sim F_{k+l-1}^{k, l}$.
3. We prove $G_{7}=\left(x^{2}, y^{4}\right)$ is simple. Consider

$$
g=\left(x^{2}+c y^{2}+d y^{3}, a_{2} y^{2}+a_{3} y^{3}+y^{4}+b_{1} x y+b_{2} x y^{2}\right),
$$

where $a_{i}, b_{i}, c, d \in K$.
i) $b_{1} \neq 0$ : By Proposition 2.1, there are $m, n \geq 2$ such that $g \sim F_{m+n-1}^{m, n}$.
ii) $b_{1}=0$ :

1) $c \neq 0$ : By Proposition 2.1, there are $s, t \geq 2$ such that $g \sim F_{s+t-1}^{s, t}$.
2) $c=0: g=\left(x^{2}+d y^{3}, a_{2} y^{2}+a_{3} y^{3}+y^{4}+b_{2} x y^{2}\right)$.
2.1) $d \neq 0$ : We may assume $d=1$.
2.1.1) $a_{2} \neq 0$ : By Lemma 2.3, we get $g \sim F_{3}^{2,2}$.
2.1.2) $a_{2}=0: g=\left(x^{2}+y^{3}, a_{3} y^{3}+y^{4}+b_{2} x y^{2}\right)$.
2.1.2.1) $a_{3} \neq 0$ :
2.1.2.1.1) $b_{2} \neq 0$ : By the proof of Proposition 2.5, we get $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$, and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.1.2.1.2) $b_{2}=0$ : By Lemma 2.3, we have $g \sim\left(x^{2}, y^{3}\right)$.
2.1.2.2) $a_{3}=0: g=\left(x^{2}+y^{3}, y^{4}+b_{2} x y^{2}\right)$.
2.1.2.2.1) $b_{2} \neq 0$ : By Lemma 2.3, we get

$$
g \sim H_{6}=\left(x^{2}+y^{3}, x y^{2}\right) .
$$

2.1.2.2.2) $b_{2}=0: g=\left(x^{2}+y^{3}, y^{4}\right)=I_{7}^{0}$.
2.2) $d=0: g=\left(x^{2}, a_{2} y^{2}+a_{3} y^{3}+y^{4}+b_{2} x y^{2}\right)$.
2.2.1) $b_{2} \neq 0$ :
2.2.1.1) $a_{2} \neq 0$ : By Lemma 2.3, $g \sim F_{3}^{2,2}$.
2.2.1.2) $a_{2}=0: g=\left(x^{2}, a_{3} y^{3}+y^{4}+b_{2} x y^{2}\right)$.
2.2.1.2.1) $a_{3} \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$ and

$$
g \sim\left(x^{2}, y^{3}+x y^{2}\right) \text { if } p=3 .
$$

2.2.1.2.2) $a_{3}=0$ : By the proof of Proposition 2.5, there is $r \geq 4$ such that

$$
g=\left(x^{2}, y^{4}+b_{2} x y^{2}\right) \sim\left(x^{2}+y^{r}, x y^{2}\right)=H_{r+3} .
$$

2.2.2) $b_{2}=0: g=\left(x^{2}, a_{2} y^{2}+a_{3} y^{3}+y^{4}\right)$.
2.2.2.1) $a_{2} \neq 0$ : By Lemma 2.3, $g \sim F_{3}^{2,2}$.
2.2.2.2) $a_{2}=0$ :
2.2.2.2.1) $a_{3} \neq 0$ : By Lemma 2.3, $g \sim\left(x^{2}, y^{3}\right)$.
2.2.2.2.2) $a_{3}=0: g=G_{7}=\left(x^{2}, y^{4}\right)$.
4. We prove $H_{n+3}=\left(x^{2}+y^{n}, x y^{2}\right), n \geq 3$, is simple. Consider

$$
g=\left(x^{2}+c_{2} y^{2}+\ldots+c_{n-1} y^{n-1}+c_{n} y^{n}, a_{2} y^{2}+a_{3} y^{3}+b x y+x y^{2}\right)
$$

where $c_{n}=1$.
i) $b \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $b=0$ :

1) $c_{2} \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $c_{2}=0: g=\left(x^{2}+c_{3} y^{3}+\ldots+c_{n-1} y^{n-1}+y^{n}, a_{2} y^{2}+a_{3} y^{3}+x y^{2}\right)$.
2.1) If $n=3$ then $g=\left(x^{2}+y^{3}, a_{2} y^{2}+a_{3} y^{3}+x y^{2}\right)$. If $a_{2} \neq 0$, by Lemma 2.3, $g \sim F_{3}^{2,2}$. If $a_{2}=0$ and $a_{3} \neq 0$, by the proof of Proposition $2.5, g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$ and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$. If $a_{2}=a_{3}=0$ then $g=H_{6}$.
2.2) If $n \geq 4$, set

$$
i=\min \left\{j \mid c_{j} \neq 0\right\} \geq 3
$$

Set $u=c_{i}+c_{i+1} y+\ldots+y^{n-i}$. Then $g \sim\left(u^{-1} x^{2}+y^{i}, a_{2} y^{2}+a_{3} y^{3}+x y^{2}\right)$. Let $v \in K[[y]]$ be such that $v^{2}=u^{-1}$. Then $v$ is a unit. Consider the automorphism $\phi: R \rightarrow R, x \mapsto v x, y \mapsto y$. Then

$$
g \sim\left(x^{2}+y^{i}, \phi^{-1}\left(a_{2} y^{2}+a_{3} y^{3}+x y^{2}\right)\right)=\left(x^{2}+y^{i}, a_{2} y^{2}+a_{3} y^{3}+\sum_{i \geq 2} d_{j} x y^{j}\right)
$$

where $d_{j} \in K$ and $d_{2} \neq 0$.
2.2.1) $a_{2} \neq 0$ : By Lemma 2.3, $g \sim F_{3}^{2,2}$.
2.2.2) $a_{2}=0$ :
2.2.2.1) $a_{3} \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$ and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.2.2.2) $a_{3}=0:$ By Lemma 2.3, $g \sim H_{i+3}$.
5. a) We prove that $I_{2 t-1}^{0}=\left(x^{2}+y^{3}, y^{t}\right), t \geq 4$, is simple.

The proofs for the cases $p \nmid t$ and $p \mid t$ are put together as below.
Consider the unfolding

$$
g=\left(x^{2}+y^{3}+c y^{2}, a_{2} y^{2}+\cdots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{1} x y+\cdots+b_{t-1} x y^{t-1}\right),
$$

where $a_{t}=1$, and $b_{t-1}=0$ if $p \nmid t$.
i) $b_{1} \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $b_{1}=0$ :

1) $c \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $c=0: g=\left(x^{2}+y^{3}, a_{2} y^{2}+\ldots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{2} x y^{2}+\ldots+b_{t-1} x y^{t-1}\right)$.
2.1) $a_{2} \neq 0$ : By Lemma 2.3, $g \sim\left(x^{2}, y^{2}\right) \sim F_{3}^{2,2}$.
2.2) $a_{2}=0: g=\left(x^{2}+y^{3}, a_{3} y^{3}+\ldots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{2} x y^{2}+\ldots+b_{t-1} x y^{t-1}\right)$.
2.2.1) $a_{3} \neq 0$ :
2.2.1.1) $b_{2} \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$, and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.2.1.2) $b_{2}=0$ : By Lemma 2.3, $g \sim\left(x^{2}, y^{3}\right)$.
2.2.2) $a_{3}=0$ : Set $T=\min \left\{i \mid a_{i} \neq 0\right\}$. Then $4 \leq T \leq t$.
2.2.2.1) $b_{2} \neq 0$ : By the proof of Proposition 2.5, $g \sim H_{r+3}$ for some $r \geq 3$.
2.2.2.2) $b_{2}=0:$

If $b_{j}$ are not all 0 then we set

$$
Q=\min \left\{j \mid b_{j} \neq 0\right\}
$$

Then $Q \geq 3$. We consider the following cases:
2.2.2.2.1) $T \leq Q$ : by Lemma 2.3, $g \sim\left(x^{2}+y^{3}, y^{T}\right)=I_{2 T-1}^{0}$.
2.2.2.2.2) $T=Q+1$ : if $p \nmid T$, by Lemma $2.3, g \sim I_{2 T-1}^{0}$. If $p \mid T$, by the proof of Proposition 2.5, $g \sim I_{2 T-1}^{1}$.
2.2.2.2.3) $T \geq Q+3$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$.
2.2.2.2.4) $T=Q+2$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$ if $p \nmid 2 Q+3$, and $g \sim I_{2 Q+2}^{1}$ if $p \mid 2 Q+3$. If $b_{j}=0$ for all $j$ then by Lemma 2.3, $g \sim I_{2 T-1}^{0}$.
5. b) We prove that $I_{2 t-1}^{1}=\left(x^{2}+y^{3}, y^{t}+x y^{t-1}\right), t \geq 4, p \mid t$, is simple.

Consider the unfolding

$$
g=\left(x^{2}+y^{3}+c y^{2}, a_{2} y^{2}+\cdots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{1} x y+\cdots+b_{t-1} x y^{t-1}\right),
$$

where $a_{t}=b_{t-1}=1$.
i) $b_{1} \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $b_{1}=0$ :

1) $c \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $c=0: g=\left(x^{2}+y^{3}, a_{2} y^{2}+\ldots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{2} x y^{2}+\ldots+b_{t-1} x y^{t-1}\right)$.
2.1) $a_{2} \neq 0$ : By Lemma 2.3, $g \sim\left(x^{2}, y^{2}\right) \sim F_{3}^{2,2}$.
2.2) $a_{2}=0: g=\left(x^{2}+y^{3}, a_{3} y^{3}+\ldots+a_{t-1} y^{t-1}+a_{t} y^{t}+b_{2} x y^{2}+\ldots+b_{t-1} x y^{t-1}\right)$.
2.2.1) $a_{3} \neq 0:$
2.2.1.1) $b_{2} \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$, and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.2.1.2) $b_{2}=0$ : By Lemma 2.3, $g \sim\left(x^{2}, y^{3}\right)$.
2.2.2) $a_{3}=0$ : Set

$$
T=\min \left\{i \mid a_{i} \neq 0\right\}
$$

Then $4 \leq T \leq t$.
2.2.2.1) $b_{2} \neq 0$ : By the proof of Proposition 2.5, $g \sim\left(x^{2}+y^{r}, x y^{2}\right)=H_{r+3}$ for some $r \geq 3$.
2.2.2.2) $b_{2}=0$ : Set

$$
Q=\min \left\{j \mid b_{j} \neq 0\right\}
$$

Then $Q \geq 3$. We consider the following cases:
2.2.2.2.1) $T \leq Q$ : by Lemma 2.3, $g \sim\left(x^{2}+y^{3}, y^{T}\right)=I_{2 T-1}^{0}$.
2.2.2.2.2) $T=Q+1$ : if $p \nmid T$, by Lemma $2.3, g \sim I_{2 T-1}^{0}$. If $p \mid T$, by the proof of Proposition 2.5, $g \sim I_{2 T-1}^{1}$.
2.2.2.2.3) $T \geq Q+3$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$.
2.2.2.2.4) $T=Q+2$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$ if $p \nmid 2 Q+3$, and $g \sim I_{2 Q+2}^{1}$ if $p \mid 2 Q+3$.
6. a) We prove $I_{2 q+2}^{0}=\left(x^{2}+y^{3}, x y^{q}\right), q \geq 3$, is simple.

The proofs for the cases $p \mid 2 q+3$ and $p \nmid 2 q+3$ are put together as below.
Consider the unfolding

$$
g=\left(x^{2}+y^{3}+c y^{2}, a_{2} y^{2}+\ldots+a_{q+2} y^{q+2}+b_{1} x y+\ldots+b_{q-1} x y^{q-1}+b_{q} x y^{q}\right)
$$

where $b_{q}=1$, and $a_{q+2}=0$ if $p \nmid 2 q+3$.
i) $b_{1} \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $b_{1}=0$ :

1) $c \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $c=0$ :
2.1) $a_{2} \neq 0$ : by Lemma 2.3, $g \sim F_{3}^{2,2}$.
2.2) $a_{2}=0$ : we consider the following cases:
2.2.1) $a_{3} \neq 0$ :
2.2.1.1) $b_{2} \neq 0$ : by the proof of Proposition 2.5, $g \sim\left(x^{2}, y^{3}\right)$ if $p \geq 5$, and $g \sim\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.2.1.2) $b_{2}=0$ : by Lemma 2.3, $g \sim\left(x^{2}, y^{3}\right)$.
2.2.2) $a_{3}=0$ :
2.2.2.1) $b_{2} \neq 0$ : by Lemma 2.3 and the proof of Proposition 2.5, $g \sim H_{r+3}=$ $\left(x^{2}+y^{r}, x y^{2}\right)$ for some $r \geq 3$.
2.2.2.2) $b_{2}=0$ : Set $Q=\min \left\{j \mid b_{j} \neq 0\right\}$. Then $3 \leq Q \leq q$.

If there is $i \geq 4$ such that $a_{i} \neq 0$ then we set

$$
T=\min \left\{i \mid a_{i} \neq 0\right\}
$$

Then $T \geq 4$.
2.2.2.2.1) $T \leq Q$ : by Lemma 2.3, $g \sim I_{2 T-1}^{0}=\left(x^{2}+y^{3}, y^{T}\right)$.
2.2.2.2.2) $T=Q+1$ : by the proof of Proposition 2.5, $g \sim I_{2 T-1}^{0}$ if $p \nmid T$, and $g \sim\left(x^{2}+y^{3}, x^{T}+x y^{T-1}\right)=I_{2 T-1}^{1}$ if $p \mid T$.
2.2.2.2.3) $T \geq Q+3$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$.
2.2.2.2.4) $T=Q+2$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}=\left(x^{2}+y^{3}, x y^{Q}\right)$ if $p \nmid 2 Q+3$, and $g \sim I_{2 Q+2}^{1}=\left(x^{2}+y^{3}, x y^{Q}+y^{Q+2}\right)$ if $p \mid 2 Q+3$.

If $a_{i}=0$ for all $i$ then by Lemma 2.3, we obtain $g \sim I_{2 Q+2}^{0}$.
6. b) We prove $I_{2 q+2}^{1}=\left(x^{2}+y^{3}, x y^{q}+y^{q+2}\right), q \geq 3, p \mid 2 q+3$ is simple.

Consider the unfolding

$$
g=\left(x^{2}+y^{3}+c y^{2}, a_{2} y^{2}+\ldots+a_{q+2} y^{q+2}+b_{1} x y+\ldots+b_{q-1} x y^{q-1}+b_{q} x y^{q}\right)
$$

where $a_{q+2}=b_{q}=1$.
i) $b_{1} \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
ii) $b_{1}=0$ :

1) $c \neq 0$ : By Proposition 2.1, $g$ is of type $F$.
2) $c=0$ :
2.1) $a_{2} \neq 0$ : by Lemma 2.3, $g \sim\left(x^{2}, y^{2}\right) \sim F_{3}^{2,2}$.
2.2) $a_{2}=0$ : we consider the following cases:
2.2.1) $a_{3} \neq 0$ :
2.2.1.1) $b_{2} \neq 0$ : by the proof of Proposition 2.5, $g \sim G_{5}^{0}=\left(x^{2}, y^{3}\right)$ if $p \geq 5$, and $g \sim G_{5}^{1}=\left(x^{2}, y^{3}+x y^{2}\right)$ if $p=3$.
2.2.1.2) $b_{2}=0$ : by Lemma 2.3, $g \sim\left(x^{2}, y^{3}\right)$.
2.2.2) $a_{3}=0$ : Set

$$
T=\min \left\{i \mid a_{i} \neq 0\right\}
$$

Then $T \geq 4$.
2.2.2.1) $b_{2} \neq 0$ : by the proof of Proposition 2.5, $g \sim H_{r+3}=\left(x^{2}+y^{r}, x y^{2}\right)$ for some $r \geq 3$.
2.2.2.2) $b_{2}=0$ : Set $Q=\min \left\{j \mid b_{j} \neq 0\right\}$. Then $3 \leq Q \leq q$.
2.2.2.2.1) $T \leq Q$ : by Lemma 2.3, $g \sim I_{2 T-1}^{0}=\left(x^{2}+y^{3}, y^{T}\right)$.
2.2.2.2.2) $T=Q+1$ : by the proof of Proposition 2.5, $g \sim I_{2 T-1}^{0}$ if $p \nmid T$, and $g \sim\left(x^{2}+y^{3}, x^{T}+x y^{T-1}\right)=I_{2 T-1}^{1}$ if $p \mid T$.
2.2.2.2.3) $T \geq Q+3$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}$.
2.2.2.2.4) $T=Q+2$ : by the proof of Proposition 2.5, $g \sim I_{2 Q+2}^{0}=\left(x^{2}+y^{3}, x y^{Q}\right)$ if $p \nmid 2 Q+3$, and $g \sim I_{2 Q+2}^{1}=\left(x^{2}+y^{3}, x y^{Q}+y^{Q+2}\right)$ if $p \mid 2 Q+3$.

## 3 Simple isolated complete intersection singularities of $I_{2,2}$ in characteristic 2

In this section, we assume that the field $K$ has characteristic $p=2$.
Lemma 3.1. Let $f \in I_{2,2}$ be such that $\operatorname{ord}(f)=2$. Then either there is a $g_{2} \in \mathfrak{m}^{2}$ such that

$$
f \sim\left(x y, g_{2}\right)
$$

or there are $h_{1} \in \mathfrak{m}^{3}$ and $g_{2} \in \mathfrak{m}^{2}$ such that

$$
f \sim\left(x^{2}+h_{1}, g_{2}\right)
$$

If $g_{2} \in \mathfrak{m}^{3}$ then $f$ is not simple.

Proof. Let $f=\left(f_{1}, f_{2}\right)$. We may assume that ord $\left(f_{1}\right)=2$. By Lemma 1.18, either there is $\phi \in \operatorname{Aut}(R)$ such that $\phi\left(f_{1}\right)=x y$ or there is $\varphi \in \operatorname{Aut}(R)$ such that $\varphi\left(f_{1}\right)=x^{2}+h_{1}$ for some $h_{1} \in \mathfrak{m}^{3}$. Therefore, either $f \sim\left(x y, \phi\left(f_{2}\right)\right)$ or $f \sim\left(x^{2}+h_{1}, \varphi\left(f_{2}\right)\right)$. Proposition 2.4 implies that $f$ is not simple if $g_{2} \in \mathfrak{m}^{3}$. This proves the lemma.

Proposition 3.2. Let $g=\left(x y, g_{2}\right) \in I_{2,2}$, where $g_{2} \in \mathfrak{m}^{2}$. Then there are $m, n \geq 2$ such that $g \sim\left(x y, x^{n}+y^{m}\right)=F_{n+m-1}^{n, m}$.

Proof. The proof is similar to a part of the proof of Proposition 2.1, which does not depend on characteristic.

Proposition 3.3. Let $g=\left(x^{2}+h_{1}, g_{2}\right) \in I_{2,2}$, where $h_{1} \in \mathfrak{m}^{3}$ and $j_{2}\left(g_{2}\right)=$ $a x^{2}+b x y+c y^{2} \neq 0$, where $a, b, c \in K$.
a) If $b=0$ and $c \neq 0$ then $g \sim\left(x^{2}, y^{2}\right)$.
b) If $b \neq 0$ and $c \neq 0$ then $g \sim\left(x^{2}, x y+y^{2}\right)$.

Proof. The proposed normal forms are of quasi homogeneous type $(\mathbf{d} ; \mathbf{a})=(2,2 ; 1,1)$.
a) If $b=0$ and $c \neq 0$ then $g \sim\left(x^{2}+h_{1}, y^{2}+h_{2}\right), h_{i} \in \mathfrak{m}^{3}$. Setting

$$
f=\left(x^{2}, y^{2}\right), \quad h=\left(h_{1}, h_{2}\right),
$$

we have $v_{\mathbf{d}, \mathbf{a}}(h) \geq 1$ and $T^{1}(f)=R^{2} /\left\langle x^{2}, y^{2}\right\rangle R^{2}$. It is easy to check $T_{\nu}^{1}(f)=0$ for all $\nu \geq 1$. This implies $\sup \left\{i \mid T_{i}^{1}(H) \neq 0\right\} \leq 0<v_{\mathbf{d}, \mathbf{a}}(h)$ and by Proposition 1.8 we deduce $f \sim f+h \sim g$.
b) If $b \neq 0$ and $c \neq 0$ then $g \sim\left(x^{2}+h_{1}, x y+y^{2}+h_{2}\right), h_{i} \in \mathfrak{m}^{3}$. Setting

$$
f=\left(x^{2}, x y+y^{2}\right), \quad h=\left(h_{1}, h_{2}\right) .
$$

we have $v_{\mathbf{d}, \mathbf{a}}(h) \geq 1$. Moreover, $T^{1}(f)=R^{2} / D$, with

$$
D:=\langle(0, y),(0, x)\rangle+\left\langle x^{2}, x y+y^{2}\right\rangle R^{2} .
$$

Then $\left(x^{2}, 0\right),\left(x y^{2}, 0\right),\left(y^{3}, 0\right),\left(0, x y^{2}\right),\left(0, y^{3}\right) \in D$. This implies $T_{\nu}^{1}(f)=0$ for all $\nu \geq 1$. By Proposition 1.8, $f \sim f+h \sim g$.

Theorem 3.4. All ICIS of the table 3 are simple and these are the only ones if $p=$ $\operatorname{char}(K)=2$.

| Type | Equation of $f$ |
| :--- | :--- |
| $F_{m+n-1}^{m, n} m, n \geq 2$ | $\left(x y, x^{m}+y^{n}\right)$ |
| $F_{3}^{2,2 ; 0}$ | $\left(x^{2}, y^{2}\right)$ |
| $F_{3}^{2,2 ; 1}$ | $\left(x^{2}, x y+y^{2}\right)$ |

Table 3
Proof. To prove that the $f$ from table 3 are simple, we proceed as in the proof of Theorem 2.6. We have the following bases of $T^{1, \text { sec }}(f)$ :

1. $F_{m+n-1}^{m, n}, m, n \geq 2$

- $m$ or $n$ odd: $(x, 0),(y, 0),(0, x), \ldots,\left(0, x^{m-1}\right),(0, y), \ldots,\left(0, y^{n-1}\right)$.
- $m$, $n$ even, $m>n:(x, 0),(y, 0),(0, x), \ldots,\left(0, x^{m}\right),(0, y), \ldots,\left(0, y^{n-1}\right)$.
- $m=n$ even: $(x, 0),(y, 0),(0, x), \ldots,\left(0, x^{m-1}\right),(0, y), \ldots,\left(0, y^{n}\right)$.
$F_{m+n-1}^{m, n}$ is of quasi-homogeneous type $(m+n, m n ; m, n)$.

2. $F_{3}^{2,2 ; 0}$
$(x, 0),(y, 0),(x y, 0),(0, x),(0, y),(0, x y)$.
$F_{3}^{2,2 ; 1}$
$(x, 0),(y, 0),\left(y^{2}, 0\right)$.
$F_{3}^{2,2}$ is of quasi-homogeneous type $(2,2 ; 1,1)$.
3. The proof is similar to the proof of Proposition 2.1, which works also for $p=2$.
4. We prove that $F_{3}^{2,2 ; 0}=\left(x^{2}, y^{2}\right)$ is simple.

Consider $g=\left(x^{2}+a x y, y^{2}+b x y\right)$ with $a, b \in K$.
i) $a=0: g=\left(x^{2}, y^{2}+b x y\right)$

1) $b=0: g=\left(x^{2}, y^{2}\right)$.
2) $b \neq 0$ : Using the automorphism $\phi: x \mapsto b^{-1} x, y \mapsto y$ we get $\phi(g)=\left(b^{-2} x^{2}, x y+y^{2}\right) \sim\left(x^{2}, x y+y^{2}\right)$.
ii) $a \neq 0$ : Using the automorphism $\phi: x \mapsto x, y \mapsto-a^{-1} x+a^{-1} y$, we get $\phi(g)=\left(x y,\left(a^{-2}-a^{-1} b\right) x^{2}+a^{-1} b x y+a^{-2} y^{2}\right) \sim\left(x y,\left(a^{-2}-a^{-1} b\right) x^{2}+a^{-2} y^{2}\right)$.
3) $a^{-1} \neq b: g \sim\left(x y, x^{2}+y^{2}\right)=F_{3}^{2,2}$.
4) $a^{-1}=b: g \sim\left(x y, y^{2}\right)$ is not a complete intersection.

We prove that $F_{3}^{2,2 ; 1}=\left(x^{2}, x y+y^{2}\right)$ is simple.
Consider $g=\left(x^{2}+a y^{2}, x y+y^{2}\right)$, where $a \in K$. Using $\phi: x \mapsto x-\sqrt{a} y, y \mapsto y$ we get $\phi(g)=\left(x^{2},(1-\sqrt{a}) y^{2}+x y\right)$.
i) $a=1: g$ is not a complete intersection.
ii) $a \neq 1$ : Let $c=1-\sqrt{a}$. Using the automorphism $\varphi: x \mapsto \sqrt{c} x, y \mapsto c^{-\frac{1}{2}} y$ we get $g \sim \varphi(\phi(g))=\left(c x^{2}, x y+y^{2}\right) \sim\left(x^{2}, x y+y^{2}\right)$.

Acknowledgements: This research is funded by Vietnam Ministry of Education and Training (MOET) under grant number B2024-DQN-02. The latter author would like to thank the Mathematical Institute of the University of Freiburg for its hospitality.

## References

[DGPS15] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, Hans Schönemann, Singular 4-0-2 - A computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2015.
[Giu83] Marc Giusti, Classification des singularités isolées simples d'intersections complètes, Proceedings of Symposia in Pure Mathematics, 40 (1983), Part 1.
[GLS07] Greuel, Gert-Martin; Lossen, Christoph; Shustin, E., Introduction to singularities and deformations. Springer Monographs in Mathematics. Berlin: Springer (ISBN 978-3-540-28380-5/hbk). xii, 471 p. (2007).
[GNg16] Gert-Martin Greuel, Hong Duc Nguyen, Right simple singularities in positive characteristic, J. Reine Angew. Math. 712, 81-106, 2016.
[GPf21] Greuel, Gert-Martin; Pfister, Gerhard: Semicontinuity of singularity invariants in families of formal power series. Fernández de Bobadilla, Javier (ed.) et al., Singularities and their interaction with geometry and low dimensional topology. In honor of András Némethi on the occasion of his 60th birthday. Selected papers based on the presentations at the conference "Némethi60: geometry and topology of singularities", Budapest, Hungary, May 27-31, 2019. Basel: Birkhäuser/Springer. Trends Math., 207-245 (2021).
[GP18] Gert-Martin Greuel, Thuy Huong Pham, On finite determinacy for matrices of power series, Math. Z. 290 (3), 759-774, 2018.
[GP19] Gert-Martin Greuel, Thuy Huong Pham, Finite determinacy of matrices and ideals, J. of Algebra, 530, 195-214, 2019.
[GK90] Gert-Martin Greuel, Heike Kröning, Simple singularities in positive characteristic, Math. Z. 203 (1990), no. 2, 339-354. Zbl 0715.14001
[KS72] Arnold Kas, Michael Schlessinger, On the versal deformation of a complex space with an isolated singularity, Math. Ann.,196 (1972), 23-29.
[P16] Thuy Huong Pham, On finite determinacy of hypersurface singularities and matrices in arbitrary characteristic, Ph.D. thesis, TU Kaiserslautern, 2016.

Thuy Huong Pham
Department of Mathematics and Statistics, Quy Nhon University, 170 An Duong Vuong, Quy Nhon, Vietnam
Email address: phamthuyhuong@qnu.edu.vn
Gerhard Pfister
Department of Mathematics, University of Kaiserslautern, Germany
Email address: pfister@mathematik.uni-kl.de
Gert-Martin Greuel
Department of Mathematics, University of Kaiserslautern, Germany
Email address: greuel@mathematik.uni-kl.de


[^0]:    ${ }^{1}$ Note that for $f=\left(x^{2}+y^{4}, y^{5}+a x y^{3}\right)$ we have $2 t-2 q-s=0$ and the parameter $a$ cannot be avoided by the proof of Proposition 2.4.

[^1]:    ${ }^{2}$ On a MacBook Air with M1 chip about $4,5 \mathrm{~h}$ and about 5 sec without the term $x y^{4}$.

