GEOMETRIC QUOTIENTS OF UNIPOTENT GROUP ACTIONS

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Introduction

This article is devoted to the problem of constructing geometric quotients of a quasiaffine scheme X over a field of characteristic 0 by a unipotent algebraic group G. This problem arises naturally if one tries to construct moduli spaces in the sense of Mumford's 'geometric invariant theory' for singularities of algebraic varieties or for modules over the local ring of such a singularity. Indeed, our results grew out of the attempt to construct and describe moduli spaces for torsion-free modules over the ring of a reduced curve singularity and they are applied to that case in **[8]**.

The theorems of this paper can be used to extend the result of **[10]** about generic moduli for plane curves with fixed semigroup $\langle p, q\rangle$ to the non-generic case by fixing a Hilbert function of the Tjurina algebra. The same method applies to the classification of semiquasihomogeneous hypersurface singularities with fixed principal part which will be presented in another article.

A general method for constructing moduli spaces is the following.

1. One starts with an algebraic family $X \to T$ with finite-dimensional base T which contains all isomorphism classes of objects to be classified. This is usually, but not always, a versal deformation of the 'worst' object.

2. In general, T will contain analytically trivial subfamilies and one tries to interpret these as orbits of the action of a Lie group or an algebraic group acting on T. In fact, we start with a (infinite-dimensional) Lie algebra which is usually the kernel of the Kodaira–Spencer map of the family $X \to T$. In many cases in singularity theory it is possible to reduce this to an action of a finite-dimensional solvable Lie algebra $\mathscr{L}$ such that the orbits of $\mathscr{L}$ (or rather of the group $G = \exp(\mathscr{L})$) are the isomorphism classes of an object.

3. If it happens that there is an algebraic structure on the orbit space $M = T/G$ such that the G-invariant functions on T are the functions on M, then M is the desired (coarse) moduli space. But usually this is not possible and one needs a stratification $T = \bigcup T_\alpha$ such that T_α/G has this property. The stratification will be defined by fixing certain invariants of the objects to be classified.

The problem of the existence of the geometric quotient T/G can in all known applications be reduced to the existence of the geometric quotient by the corresponding maximal unipotent subgroup.

Several papers exist which consider the problem of geometric quotients by non-reductive groups (for example, **[5, 6, 9, 16, 17]**) mostly for free actions. But

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we know of none which gives, for unipotent groups, a general and sufficient criterion for the existence of a geometric quotient as a separated scheme which can actually be applied in concrete situations. It is the purpose of this note to derive such criteria in characteristic zero. If $X = \operatorname{Spec} K[x_1, \ldots, x_n]$, $K[x_1, \ldots, x_n] = K[X_1, \ldots, X_n]/I$ and if $\delta_1, \ldots, \delta_m$ form a K-basis of the Lie algebra of G, then these criteria are given in terms of properties of the matrix $(\delta_i(x_j))$ which are quite easily checked if the $\delta_i(x_j)$ are sufficiently well-known. If the action of G is free (Theorem 3.10) or if G is abelian and the action is arbitrary (Theorem 4.1), these criteria are necessary and sufficient; in general they are sufficient (Theorems 3.5, 3.8, 4.7). Moreover, if X/G does not exist, we describe explicitly, using these criteria, several stratifications of X into locally closed G-stable subschemes on which the quotient exists (Theorem 4.7). These stratifications can also be described invariantly as the flattening stratifications of certain coherent sheaves on X, a fact which is important for the applications mentioned above. It is very useful to have different stratifications at hand since in different geometric situations some are more natural than others.

In § 1 we introduce the notion of a 'stable' point for the action of G on a quasiaffine scheme X. Our main observation here is that this has to be done relative to a G-equivariant embedding into an affine scheme in order to obtain a separated quotient. We actually always work with the Lie algebra of G. The next section provides many examples which demonstrate the (more or less well-known) fact of pathological behaviour of unipotent group actions. Some of these examples might be new. In each case we compute the set of stable points. In § 3 we study free actions, and the main criteria, also for later applications to general actions, are derived here. Moreover, as a corollary of these criteria we derive a positive answer to the Jacobian Umkehrproblem under additional conditions on the Jacobian matrix (Corollary 3.13). Section 4 gives criteria for general actions and describes the stratification mentioned above. These stratifications depend on filtrations of the coordinate ring of X and of the Lie algebra of G with certain properties. Such filtrations do always exist and we discuss certain variants. Finally, we describe and compare different stratifications for the examples from § 2.

The only methods we use are methods from linear algebra and localization (nevertheless the proofs are sometimes quite involved) and hence our results do hold for affine schemes $X = \operatorname{Spec} A$ where A is any noetherian K-algebra. In some cases we have to assume that A is reduced but we do not need, for instance, any normality assumption. We should like to mention that the examples coming from the geometric applications in **[8]** were quite essential for deriving the above-mentioned criteria.

We use the usual conventions of a commutative ring theory as in **[11]**. We use A to denote a commutative K-algebra, and if X is a subscheme of the affine K-scheme $\operatorname{Spec} A$, we write X_f or $D(f)$ for the open subscheme $X \cap \operatorname{Spec} A_f$ of X. If $\mathfrak{a} \subset A$ is an ideal, $V(\mathfrak{a})$ denotes the closed subscheme $\operatorname{Spec} A/\mathfrak{a}$ of $\operatorname{Spec} A$ and $D(\mathfrak{a})$ the open subscheme $\operatorname{Spec} A - V(\mathfrak{a})$.

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General notation and assumptions. The symbol K denotes a field of characteristic 0; a scheme X always means a separated scheme. We write $\kappa(x)$ for the residue field of $x \in X$. A geometric point is an F-valued point, where F is an algebraically closed field, and a geometric fibre, respectively a geometric orbit, means a fibre, respectively an orbit, of a geometric point. If G is an algebraic group over K (and hence smooth and of finite type over K) which acts via $G \times_K X \to X$ then Gx denotes the orbit of x, that is, the image of the induced map $G \times_K \kappa(x) \to X \times_K \kappa(x)$, whilst G_x denotes the stabilizer. Note that Gx is a subset of $X \times_K \kappa(x)$ while G_x is a subgroup of $G \times_K \kappa(x)$.

1. *Geometric quotients and stable points*

Let G be a unipotent algebraic group over K which acts rationally on a scheme X over K.

DEFINITION 1.1. A pair (Y, π) consisting of a scheme Y/K and a morphism π: $X \to Y$ over K is called a *geometric quotient* if

(i) π is open and surjective,

(ii) $(\pi_* \mathcal{O}_X)^G = \mathcal{O}_Y$,

(iii) π is an orbit map; that is, the geometric fibres of π are precisely the geometric orbits of G.

Since G is of finite type over K, this definition is equivalent to [**13**, Definition 0.6] and [**1**, II.6.3], and the quotient map is universally open [**13**, 0, § 2, Remark (4)]. Moreover, (iii) is equivalent to the condition that for the induced action of $\bar{G}$ on $\bar{X}$, $\bar{\pi}^{-1}(x) = \bar{G}x$ for all $\bar{K}$-rational points $x \in \bar{X}$ where $\bar{K}$ denotes an algebraic closure of K. (For any K-scheme Z let $\bar{Z} = Z \times \operatorname{Spec}(\bar{K})$ and $\bar{\pi}$ be the induced morphism.) Note that we require Y to be separated. A geometric quotient, if it exists, is uniquely determined and denoted by X/G. We express this fact by saying that $X \to X/G$ is a geometric quotient, or that X/G exists.

REMARK 1.2. Since K is of characteristic 0, G is isomorphic (as a scheme) to its Lie algebra $\operatorname{Lie} G$ [**4**, IV, § 2, 4.1; **2**, § 3]. Moreover, since a closed subgroup $H \subset G$ as well as the factor group G/H are again unipotent [**4**, IV, § 2, 2.3], it follows that if G acts on a reduced scheme X, the orbits Gx of G, with $x \in X$, are closed and isomorphic to the affine space $\mathbb{A}^n_{\kappa(x)}$, where $n = \dim G \times_K \kappa(x)/G_x$ [**1**, II.6.7; **2**, § 3]. The action is called *free* if $G_x = \{1\}$ for all $x \in X$.

DEFINITION 1.3. A geometric quotient (Y, π) is *locally trivial* if an open covering $\{V_i\}_{i \in I}$ of Y and $n_i \geq 0$ exist, such that $\pi^{-1}(V_i) \cong V_i \times \mathbb{A}^{n_i}_K$ over V_i.

Now, let X be quasi-affine, and an open subscheme of $\operatorname{Spec} A$. Assume that the action of G on X extends to an action on $\operatorname{Spec} A$. We want to describe an open invariant subset $U \subset X$ (of 'stable' points) such that U/G exists and satisfies:

(a) U is explicitly computable if the action is sufficiently well-known,

(b) U is as big as possible, subject to Condition (a).

A necessary condition for the existence of U/G is that geometric orbits in U are all closed of locally constant dimension. Recall that the geometric orbits are always closed since G is unipotent. Example 2.2 however shows that constant orbit dimension is not sufficient. Even worse, in Example 2.6 we have $X = \operatorname{Spec}(A) = X_1 \cup X_2$, where the X_i are open, affine and invariant subsets such that π: $X_i \to X_i/G$ exists and is trivial for $i = 1, 2$, but X/G does not exist. Hence, there does not exist a maximal open invariant set $U \subset X$ such that U/G exists. Since no point of X plays a preferred role, there seems to be no canonical way of defining 'stable' points: each point of X should be a stable point, at least if it is considered as a point of X_i. The following definition of stability, which we propose, overcomes this difficulty by fixing a G-equivariant embedding of the quasi-affine scheme X into a $\operatorname{Spec} A$ to which the definition refers.

DEFINITION 1.4. Let G be a unipotent algebraic group over K acting on the quasi-affine scheme X and on the affine scheme $Z = \operatorname{Spec} A$. Suppose i: $X \hookrightarrow Z$ is a G-equivariant open embedding. Let π: $X \to Y := \operatorname{Spec} A^G$ denote the canonical map. A point $x \in X$ is *stable* (with respect to A or Z, i and the action of G) if an $f \in A^G$, with $x \in X_f$, exists such that the induced map π_f: $X_f \to Y_f$ is open and an orbit map. We call x *pre-stable* (with respect to the action of G) if an open invariant neighbourhood $U \subset X$ of x exists such that π_U: $U \to Y$ is an open orbit map. We use $X^s(A) = X \cap Z^s(A)$ to denote the set of stable points of X (with respect to A and G) which depends on A but not on the embedding i. If $X = \operatorname{Spec} A$, we write X^s instead of $X^s(A)$ and call $x \in X^s$ simply stable (with respect to G).

REMARK 1.5. (1) The definitions say that π_f: $X_f \to \pi(X_f)$, and π_U: $U \to \pi_U(U)$ are geometric quotients. Example 2.6(6) shows how the stability of X depends on the embedding of X in an affine scheme. Example 2.5(3) shows that $\pi_f(X_f)$ may be a proper open subset of Y_f. However, note that $\pi^{-1}(Y_f) = X_f$, while, in the definition of pre-stable, U might not be a preimage of anything under π.

(2) The morphism π_f: $X_f \to Y_f$ is affine. Hence, our definition of stable, respectively pre-stable, is similar to Mumford's for reductive groups in Definitions (c) and (a), respectively, of [**13**, 1, § 4].

(3) If A is reduced and of finite type over K, it follows from [**5**, Proposition 2.2.2] that π_f, respectively π_U, are open if they are orbit maps. Hence, the set $\Omega_2(X, G)$ in [**5**] is our $X^s(K[X])$ in that case.

(4) The next proposition shows that $X^s(A)$ is not empty (if A is reduced) and defined by a universal property. This is what we gain at the cost of the fact that there might be larger open subsets of X where the quotient exists.

PROPOSITION 1.6. (1) *$X^s(A)$ is G-stable and open in X; it is dense in X if X is reduced.*

(2) *$X^s(A)/G$ exists, is quasi-affine and $\pi|_{X^s(A)}$: $X^s(A) \to \pi(X^s(A))$ is a geometric quotient. If X is of finite type over K and reduced, then $X^s(A)/G$ is of finite type over K.*

(3) (*Universal property*) *For each open, G-stable subset $U \subset X$ for which an open set $V \subset \operatorname{Spec}(A^G)$ exists, such that $U = \pi^{-1}(V)$ and π: $U \to V$ is a geometric quotient, we have $U \subset X^s(A)$.*

REMARKS 1.7. (1) Example 1.12 shows that $X^s(A)$ may be empty if X is not reduced. If X is reduced, we actually show that there exists an open dense subset $U = \pi^{-1}(V) \subset X$, $V \subset \operatorname{Spec} A^G$ open, such that π_U: $U \to V$ is a locally trivial geometric quotient.

(2) Proposition 1.6(1) does also hold for pre-stable points. It is not difficult to see that the geometric quotient of pre-stable points exists in the category of not necessarily separated schemes.

Before we prove the proposition, we introduce some notation which will be used in the sequel. Let Lie G denote the Lie algebra of G and recall that there is an exponential map exp: Lie $G \to G$ [4, IV, § 2, 4.1]. The action of G on Spec A induces a representation ρ: $G \to \mathrm{Aut}_K(A)$ and ρ_*: Lie $G \to \mathrm{Der}_K^{\mathrm{nil}}(A)$, fitting into a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{Aut}_K(A) \\ \exp\big\uparrow & & \big\uparrow\exp \\ \mathrm{Lie}\, G & \xrightarrow[\rho_*]{} & \mathrm{Der}_K^{\mathrm{nil}}(A) \end{array}$$

Here $\mathrm{Aut}_K(A)$ is the group of K-algebra automorphisms, and $\mathrm{Der}_K^{\mathrm{nil}}(A)$ denotes the set of nilpotent K-linear derivations of A. We say that $\delta \in \mathrm{Der}_K(A)$ is *nilpotent* if, for each $a \in A$, there is an $n(a)$ such that $\delta^{n(a)}(a) = 0$; $(\exp \delta)(a) := \sum_{i \geq 0} (1/i!)\delta^i(a)$ for $\delta \in \mathrm{Der}_K^{\mathrm{nil}}(A)$. Note that $L = \mathrm{Lie}\, G$ is a finite-dimensional Lie algebra which is nilpotent. Conversely, any representation of a finite-dimensional nilpotent Lie algebra ρ: $L \to \mathrm{Der}_K^{\mathrm{nil}}(A)$ gives rise to an action of the unipotent algebraic group $G = \exp(L)$ over K, on Spec A. In the following we work with the Lie algebra L rather than G, and write A^L, X/L, ... instead of A^G, X/G, ... We always assume the action to be non-trivial. The action of G is free if and only if the orbits of G have dimension equal to $\dim_K L$.

Proof of Proposition 1.6. Since $X^s(A)$ is the union of open subsets which are full preimages of open sets $Y = \mathrm{Spec}(A^G)$ under π, the local quotients can be glued inside Y. This implies that $X^s(A)/G$ is separated. Moreover, if X is reduced and of finite type over K, then, by [5, Proposition 2.2.2], the same holds for $X^s(A)/G$. This proves (3), (2) and the first part of (1).

To prove that $X^s(A)$ is dense if X is reduced, assume first that $X = \mathrm{Spec}\, A$. We use induction on dim L. Choose a vector field $\delta \neq 0$ in the centre of L and $a \in A$ such that $\delta(a) \neq 0$ and $\delta(a) \in A^L$. This is always possible since L consists of nilpotent derivations. Since A is reduced, $X_{\delta(a)}$ is not empty. Now $A^{\delta}_{\delta(a)}[a] = A_{\delta(a)}$ (Lemma 3.1). Thus $L/K\delta$ acts on $A^{\delta}_{\delta(a)}$. By the induction hypothesis, there are $f \in A^L$ and $x_1, \ldots, x_r \in A^{\delta}_{\delta(a)}$ such that $A^L_{f\delta(a)}[x_1, \ldots, x_r] = A^{\delta}_{f\delta(a)}$ and $x_1, \ldots, x_r$ are algebraically independent over $A^L_{f\delta(a)}$. Then $A^L_{f\delta(a)}[x_1, \ldots, x_r, a] = A_{f\delta(a)}$ and $x_1, \ldots, x_r$, are algebraically independent over $A^L_{f\delta(a)}$ (cf. Remark 3.4).

This shows that there is a maximal open subset $U = \pi^{-1}(V)$, $\varnothing \neq V \subset \mathrm{Spec}\, A^G$ open, such that π: $U \to \pi(U)$ is a locally trivial geometric quotient. Assume that $U \neq X$; then $X - \bar{U} \hookrightarrow X$ is quasi-affine and open and we can apply the same argument as above, which contradicts the maximality of U and proves the proposition if $X = \mathrm{Spec}\, A$. In general, take the intersection of the maximal U constructed for $(\mathrm{Spec}\, A)_{\mathrm{red}}$ with X.

DEFINITION 1.8. Let $X \subset Z = \mathrm{Spec}(A)$ be a G-equivariant open embedding and assume A to be reduced. Put $A_0 := A$, $Z_0 := Z$, $X_0 := X$ and, for $i \geq 1$,

$$Z_i := Z_{i-1} - Z^s_{i-1}(A_{i-1}), \quad A_i := K[Z_i], \quad X_i := X \cap Z_i,$$

Then $X_i^s(A_i)/G$ exists and the X_i define a strictly decreasing filtration $X = X_0 \supset X_1 \supset X_2 \supset \dots$, $X_i - X_{i-1} = X_i^s(A_i)$, into closed, reduced G-invariant subspaces of X. If A is noetherian, then $X = \bigcup_{\text{finite}} X_i^s(A_i)$ is a disjoint union of finitely many locally closed invariant subspaces on which the geometric quotients exist and are quasi-affine. We call this the *canonical stratification* of X (with respect to G and Z).

REMARK 1.9. The notion of stable points seems to be rather tautological and quite unworkable. In the subsequent section we shall show that this is not the case. First notice that in Definition 1.4 we may require (by shrinking Y_f) that π_f is surjective. Hence $x \in X^s(A)$ if and only if an $f \in A^L$ exists such that $x \in X_f$ and $\pi_f\colon X_f \to Y_f$ is a geometric quotient. Therefore, we have to look for criteria such that $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is a geometric quotient. Assume that $A = K[X_1, \dots, X_n]/I = K[x_1, \dots, x_n]$ and that the Lie algebra is generated over K by $\delta_1, \dots, \delta_r$. We derive several criteria for stability in terms of the matrix $(\delta_i(x_j))$ which can be checked explicitly if the $\delta_i(x_j)$ are known. The corresponding quotients are locally trivial and hence of finite type over K. If the action is free, then local triviality is automatic (cf. Theorem 3.10); we ignore the question of whether this also holds in the general case. All our examples, in particular the construction of a moduli space in [**8**], use these criteria.

2. *Examples*

EXAMPLE 2.1 (canonical stratification of Nagata's example). We first discuss Nagata's example to Hilbert's 14th problem (cf. [**14**]). Let

$$A = K[x_1, \dots, x_r, x_{r+1}, \dots, x_{2r}]$$

and

$$L = \sum_{i=4}^{r} K\delta_i, \quad \delta_i = \sum_{j=1}^{2r} h_{ij}\frac{\partial}{\partial x_j} \quad \text{and} \quad h_{ij} = \delta_i(x_j),$$

defined by the following matrix ($a_{ij} \in K$):

$$(h_{ij}) = \begin{bmatrix} 0 & \dots & 0 & a_{14}x_1 & a_{24}x_2 & a_{34}x_3 & x_4 & 0 & & 0 & \dots & 0 \\ & & & & & & 0 & x_5 & & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \ddots & & & \vdots \\ & & & & & & & & & & & 0 \\ 0 & \dots & 0 & a_{1r}x_1 & a_{2r}x_2 & a_{3r}x_3 & 0 & \cdots & & 0 & \dots & x_r \end{bmatrix}.$$

Obviously, $h_{ij} \in K[x_1, \dots, x_{2r}]^L$ and $[\delta_i, \delta_j] = 0$ for all i,j. Nagata proved that $K[x_1, \dots, x_{2r}]^L$ is not a finitely generated K-algebra provided r is large and the a_{ij} are sufficiently general. Now let $X_e = \{\mathbf{x} \in \mathbb{A}^{2r}\colon \operatorname{rank}(h_{ij}(\mathbf{x})) = e\}$ be the stratification of $\operatorname{Spec} K[x_1, \dots, x_{2r}] = \bigcup X_e$ by constant orbit dimension with respect to the action of L. An easy consequence of our results (see § 5, Examples (continuation) 5.1) will be that $X_e \to X_e/L$ is a geometric quotient and, moreover, that X_e/L is a locally closed subset in an affine space. This is the canonical stratification and it is the best result we can obtain because constant orbit dimension is necessary for the quotient to be separated.

EXAMPLE 2.2 (constant orbit dimension does not suffice). Let

$$\delta := x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} \in \mathrm{Der}_K^{\mathrm{nil}} K[x_1, x_2, x_3]$$

and $L = K\delta$. Then $\delta(x_1) = 0$, $\delta(x_2) = x_1$, $\delta(x_3) = x_2$, $\exp(L)$ is isomorphic to the additive group of K and acts on $\mathbb{A}^3$ by

$$\alpha \circ (x_1, x_2, x_3) = (x_1, x_2 + \alpha x_1, x_3 + \alpha x_2 + \tfrac{1}{2}\alpha^2 x_1).$$

It is not difficult to see, using Remark 3.2, that

(1) $K[x_1, x_2, x_3]^L = K[x_1, x_1x_3 - \frac{1}{2}x_2^2]$, and

(2) the L-invariant open set of all points of $\mathbb{A}^3$ with orbit dimension 1 is the complement of the x_3-axis and is covered by the invariant affine subsets $D(x_1)$ and $D(x_1x_3 - \frac{1}{2}x_2^2)$.

We obtain the picture of orbits shown in Fig. 1.

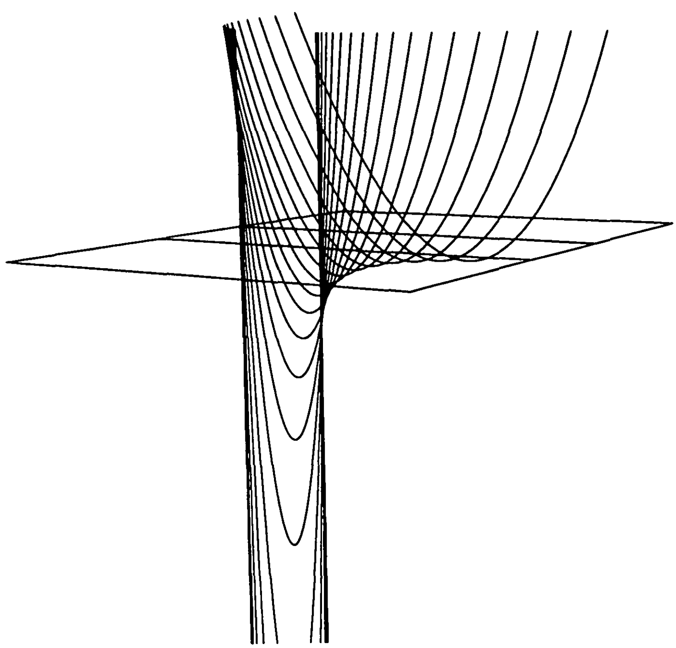

FIG. 1

Consider the map π induced by $K[\mathbf{x}]^L \subset K[\mathbf{x}]$, π: $\mathbb{A}^3 = \mathrm{Spec}\, K[\mathbf{x}] \to \mathbb{A}^2 = \mathrm{Spec}\, K[x_1, x_1x_3 - \frac{1}{2}x_2^2]$, $(x_1, x_2, x_3) \to (x_1, x_1x_3 - \frac{1}{2}x_2^2)$. Then

$$\pi^{-1}(x, y) = \{(x, \alpha, x^{-1}(y + \tfrac{1}{2}\alpha^2)) \mid \alpha \in K\} = \mathrm{orbit}(x, 0, y/x) \quad \text{if } x \neq 0,$$

$$\pi^{-1}(0, y) = \{(0, x, \alpha) \mid x^2 = -2y,\ \alpha \in K\} = \mathrm{orbit}(0, x, 0) \cup \mathrm{orbit}(0, -x, 0).$$

(3) The restriction of π to the open subset $D(x_1)$ is a geometric quotient

$$\pi\colon \mathrm{Spec}\, K[x_1, x_2, x_3]_{x_1} \to \mathrm{Spec}(K[x_1, x_2, x_3]_{x_1})^L.$$

(4) If $A = K[x_1, x_2, x_3]_{x_1x_3 - \frac{1}{2}x_2^2}$, then L acts freely on $\mathrm{Spec}\, A$ but the restriction to $D(x_1x_3 - \frac{1}{2}x_2^2)$, π: $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^L$ is not a geometric quotient since some fibres of π are unions of two orbits. Notice that the assumption of Lemma 3.1 is not satisfied, which would imply that there is an $a \in A$ such that $\delta(a)$ is a unit.

(5) The restriction $\pi|D(x_1x_3 - \frac{1}{2}x_2^2)$ is also not a quotient in the analytic category since $(\operatorname{Spec} A)/L$ is not Hausdorff with respect to the analytic topology (see (2)).

(6) Although $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is not a geometric quotient, it becomes one after an étale covering. Let $F = X_1Z^2 + 2X_2Z + 2X_3$; then

$$A \to B = A[Z]/F$$

is étale since $[\partial F/\partial Z]^2 \equiv -\frac{1}{2}(x_1x_3 - \frac{1}{2}x_2^2) \bmod F$. The action of L can be lifted to B by $\delta Z = -1$. This implies $B^L[Z] = B$, and $\operatorname{Spec} B \to \operatorname{Spec} B^L$ is a geometric quotient (cf. 3.1).

(7) The restriction of π to the closed subset $V(x_1) \subseteq \operatorname{Spec} A$ is a geometric quotient since, on this set, x_2 is invariant and a unit, that is, $(A/x_1)^L[x_3] = A/x_1$ (cf. 3.1).

Now (3) and (7) imply that $\mathbb{A}^3$ is stratified canonically by

$$V_1 = \{(x_1, x_2, x_3)\colon\ x_1 \neq 0\}, \quad V_2 = \{(x_1, x_2, x_3)\colon\ x_1 = 0,\ x_2 \neq 0\}$$

and

$$V_3 = \{(x_1, x_2, x_3)\colon\ x_1 = x_2 = 0\}.$$

Because of (4), this is the best result we can get.

Example 2.3 (nowhere existence of geometric quotient). Let K be algebraically closed, $A = K[\varepsilon, x]$, $\varepsilon^2 = 0$, and $L = K\delta$ with $\delta = \varepsilon x(\partial/\partial x)$. Then we have

(1) $\delta(A) \subseteq \varepsilon A$ and $\delta^2 = 0$,

(2) $A^L = K + \varepsilon A$, that is, $\operatorname{Spec} A^L$ is a fat point,

(3) $(\exp t\delta)(x + a) = x(1 + t\varepsilon) + a$, for $a \in A^L$, that is all points of $\operatorname{Spec} A$ are fixed under the action of L.

Now (2) and (3) imply that there is no open subset $U \subseteq \operatorname{Spec} A$ such that $U \to U/L$ is a geometric quotient. This is also a simple example where $A^L = K[\varepsilon x, \varepsilon x^2, \ldots]$ is not of finite type over K.

Example 2.4 (stratification with respect to central series is not optimal). Let

$$A = k[x_1, x_2, x_3, x_4, x_5]_{2x_1x_3 - x_2^2} \quad \text{and} \quad L = \sum_{i=1}^{3} K\delta_i$$

with

$$\delta_1 = x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + (2x_1x_3 - x_2^2)\frac{\partial}{\partial x_4},$$

$$\delta_2 = x_2\frac{\partial}{\partial x_5},$$

$$\delta_3 = x_1\frac{\partial}{\partial x_5}.$$

The centre of L is $Z = K\delta_3$, the lower central series is just given by $L \supset Z$. It is not difficult to see (because $\operatorname{Spec} A = D(x_1) \cup D(x_2)$ and $\delta_1(x_4)$ is a unit in A) that $A^L[x_4, x_5] = A$ and $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is a geometric quotient. But $\operatorname{Spec} A \to \operatorname{Spec} A^Z$ is not a geometric quotient. The maximal open set where the geometric quotient by Z exists is $D(x_1)$ which is the set of stable points of $\operatorname{Spec} A$ with respect to Z and A.

EXAMPLE 2.5 (Winkelmann [17]; geometric quotient of affine space need not be affine). Let

$$\delta := x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} + (1 + x_1x_4 - x_2x_3) \frac{\partial}{\partial x_5} \in \mathrm{Der}_K^{\mathrm{nil}} K[x_1, \ldots, x_5]$$

and $L = K\delta$. The following hold:

(1) $\operatorname{Spec} K[x_1, \ldots, x_5] = D(x_1) \cup D(x_2) \cup D(1 + x_1x_4 - x_2x_3)$, that is, the action of L is free;

(2) $x_1, x_2, 1 + x_1x_4 - x_2x_3 \in K[x_1, \ldots, x_5]^L$;

(3) the canonical map

$$\pi\colon \operatorname{Spec} K[x_1, \ldots, x_5] \to \operatorname{Spec} K[x_1, \ldots, x_5]^L$$

is not surjective; the open subset $U = D(x_1, x_2, 1 + x_1x_4 - x_2x_3) \subseteq \operatorname{Spec} K[x_1, \ldots, x_5]^L$ is a proper subset and $\pi(\operatorname{Spec} K[x_1, \ldots, x_5]) = U$; U is not affine;

(4) by Remark 3.4 (cf. also Theorem 3.10) we have

$$H^1(L, K[x_1, \ldots, x_5]) \neq 0,$$

but

$$\begin{aligned} H^1(L, K[x_1, \ldots, x_5]_{x_1}) &= H^1(L, K[x_1, \ldots, x_5]_{x_2}) \\ &= H^1(L, K[x_1, \ldots, x_5]_{1+x_1x_4-x_2x_3}) = 0; \end{aligned}$$

(5) $\pi\colon \operatorname{Spec} K[x_1, \ldots, x_5] \to U$ is a geometric quotient by (4), whence each point of $\operatorname{Spec} K[x_1, \ldots, x_5]$ is stable.

EXAMPLE 2.6 (Dixmier and Raynaud [5]; non-existence of a maximal open subset for which the geometric quotient exists). Let

$$\delta := x_1 \frac{\partial}{\partial x_3} + (2x_2x_3 - 1) \frac{\partial}{\partial x_4} \in \mathrm{Der}_K^{\mathrm{nil}} K[x_1, \ldots, x_4].$$

Let $X \subseteq \operatorname{Spec} K[x_1, \ldots, x_4]$ be the closed subset defined by $x_1x_4 - x_3(x_2x_3 - 1) = 0$ and $A = K[x_1, \ldots, x_4]/x_1x_4 - x_3(x_2x_3 - 1)$. Let $L = K\delta$. The following hold:

(1) $\delta(x_1x_4 - x_3(x_2x_3 - 1)) = 0$, that is, $\delta \in \mathrm{Der}_K^{\mathrm{nil}} A$;

(2) $A^L = K[x_1, x_2]$ and L acts freely on X;

(3) $X = D(x_1) \cup V(x_1, x_3) \cup V(x_1, x_2x_3 - 1)$ and $D(x_1)$, $V(x_1, x_3)$, and $V(x_1, x_2x_3 - 1)$ are L-invariant under the action of L on X; if we let $(x_1, x_2, x_3, x_4) \in X$, then

$$\exp(t\delta)(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 + tx_1, x_4 + t(2x_2x_3 - 1) + t^2x_1x_2);$$

(4) if we let $X_1 = X - V(x_1, x_2x_3 - 1)$ and $X_2 = (X - V(x_1, x_3)) \cap D(x_2)$, then $X = X_1 \cup X_2$; X_1 and X_2 are affine open subsets of X, namely

$$X_1 \cong \operatorname{Spec} K[x_1, x_2, g],$$

with

$$g = \begin{cases} x_3/x_1 & \text{on } D(x_1), \\ x_4/(x_2x_3 - 1) & \text{on } D(x_2x_3 - 1), \end{cases}$$

and $\delta(g) = 1$, and

$$X_2 \cong \operatorname{Spec} K[x_1, x_2, h]_{x_2},$$

with

$$h = \begin{cases} x_4/x_2x_3 & \text{on } D(x_3) \cap D(x_2), \\ (x_2x_3 - 1)/x_1x_2 & \text{on } D(x_1) \cap D(x_2), \end{cases}$$

and $\delta(h) = 1$; this implies that $X_1 \to \operatorname{Spec} K[x_1, x_2]$ and $X_2 \to \operatorname{Spec} K[x_1, x_2]_{x_2}$ are geometric quotients;

(5) π: $X = \operatorname{Spec} A \to \operatorname{Spec} A^L$ is not a geometric quotient since the fibre of the points $(0, x_2) \in \operatorname{Spec} A^L = K^2$, with $x_2 \neq 0$, is the union of the two orbits $\{(0, x_2, 0, t)\colon t \in K\}$ and $\{(0, x_2, 1/x_2, t)\colon t \in K\}$;

(6) by (3), (4) and (5) we obtain $X^s = D(x_1) \subset X_1 = X_1^s$ (with $D(x_1) \neq X_1$) $X_2 = X_2^s$, $X_1^s(A) = X_2^s(A) = X^s$.

3. *Free actions and a relation to the Jacobian Umkehrproblem*

In this chapter A denotes an arbitrary commutative K-algebra, with K a field of characteristic 0. We study free actions of a nilpotent Lie algebra L on the affine K-scheme $\operatorname{Spec} A$ and derive necessary and sufficient conditions for $\operatorname{Spec} A \to \operatorname{Spec} A^L$ to be a geometric quotient. The following simple lemma is the starting point of all that follows (cf. also [**7**, **15**, **10**]).

LEMMA 3.1. *Let $\delta \in \operatorname{Der}_K^{\mathrm{nil}}(A)$, $x \in A$ and $\delta(x) \in A^\delta$ be a unit. Then $A^\delta[x] = A$ and x is transcendental over A^δ.*

Proof. We may replace x by $x/\delta(x)$ and hence assume $\delta(x) = 1$. We only need to show that $A \subset A^\delta[x]$. So, let $a \in A$ and n be such that $\delta^{n+1}(a) = 0$. Assume by induction that $\{b \in A \mid \delta^n(b) = 0\} \subset A^\delta[x]$ and consider $b := a - (1/n!)\, \delta^n(a)x^n$. Then $\delta^n(b) = 0$, and hence $b \in A^\delta[x]$. On the other hand, $\delta^n(a) \in A^\delta$, so that $a \in A^\delta[x]$. Assume there exists a non-trivial polynomial $p \in A^\delta[X]$ of minimal degree such that $p(x) = 0$. Then $(\delta p)(x) = \delta(p(x)) = 0$ and δp is a non-trivial polynomial of lower degree vanishing in x, which is a contradiction.

REMARK 3.2. If δ and x are as in Lemma 3.1, we easily obtain invariant functions by putting

$$i(y) = \sum_{\nu \geq 0} (1/\nu!)(-1)^\nu \delta^\nu(y)x^\nu, \quad \text{for } y \in A.$$

Then $i(y) \in A^\delta$ and $i(y) = \delta^0(y) = y$ if $y \in A^\delta$.

REMARK 3.3. If $a \in A$ is a unit and if A is reduced, then $\delta(a) = 0$ for each $\delta \in \operatorname{Der}_K^{\mathrm{nil}}(A)$.

Proof. Let $ab = 1$. Since $\exp t\delta$ is an algebraic automorphism, we get $\exp t\delta(a) \exp t\delta(b) = 1$ in $A[t]$. This implies $\delta(a) = 0$ in A/p for each $p \in \operatorname{Spec} A$ since A/p is an integral domain (consider the maximal n for which $\delta^n(a) \neq 0$). Since A is reduced, the intersection of all $p \in \operatorname{Spec} A$ is zero. This implies $\delta(a) = 0$.

Notice that without the assumption of A being reduced the remark is not true: let $A = K[\varepsilon, x]$, $\varepsilon^2 = 0$, and $a = 1 + \varepsilon x$. Then a is a unit and $\partial(a)/\partial x = \varepsilon$.

To prepare the main theorem of this chapter we will give sufficient conditions for A to be a polynomial ring over A^L which are extremely useful in the applications as well as for further theoretical results. They say that the derivatives of certain subminors of the matrix $(\delta_i(x_j))$ are linear combinations of 'earlier' columns. We call this the column-minor criterion. In Theorem 3.8 we give a kind of dual criterion for rows.

At an intermediate stage we need the *Lie algebra cohomology* (cf. [3]). Recall the definition of H^1: if $\delta_1, \ldots, \delta_n$ is a basis of L and $[\delta_i, \delta_j] = \sum_k c_{ijk}\delta_k$, then $H^1(L, A) = \ker d_1 / \operatorname{Im} d_0$, where

$$d_0\colon A \to A^n, \qquad \text{with } d_0(a) = (\delta_1(a), \ldots, \delta_n(a)),$$

$$d_1\colon A^n \to \Lambda^2 A^n, \quad \text{with } d_1(a) = \left(\delta_i(a_j) - \delta_j(a_i) - \sum_k c_{ijk} a_k\right)_{i<j}.$$

REMARK 3.4. If L is abelian, then $H^1(L, A) = 0$ if and only if there are $x_1, \ldots, x_n \in A$ such that $\delta_i(x_j) = \delta_i^j$. Moreover, in this case $A = A^L[x_1, \ldots, x_n]$ and $x_1, \ldots, x_n$ are algebraically independent over A^L.

Proof. Note that $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \ker d_1$ since L is abelian. Hence, if $H^1(L, A) = 0$, there are $x_i \in A$ such that $d_0(x_i) = e_i$, for $i = 1, \ldots, n$.

Conversely, assume $\delta_i(x_j) = \delta_i^j$ and let $(a_1, \ldots, a_n) \in \ker d_1$, that is, $\delta_i(a_j) = \delta_j(a_i)$. Applying Lemma 3.1 n times, we obtain that $A = A^L[x_1, \ldots, x_n]$, with $x_1, \ldots, x_n$ algebraically independent over A^L. Therefore, we can write

$$a_i = \sum a_\nu^{(i)} x^\nu, \quad \text{where } a_\nu^{(i)} \in A^L,$$

and get $\nu_i a^{(j)}_{\nu_1, \ldots, \nu_i - 1, \ldots, \nu_n} = \nu_j a^{(i)}_{\nu_1, \ldots, \nu_j - 1, \ldots, \nu_n}$. Put

$$b = \sum b_\nu x^\nu, \quad b_{\nu_1, \ldots, \nu_n} = a^{(i)}_{\nu_1, \ldots, \nu_i - 1, \ldots, \nu_n} \nu_j;$$

then $d_0(b) = (a_1, \ldots, a_n)$.

THEOREM 3.5 (*column-minor criterion*). *Let* $\delta_1, \ldots, \delta_n \in \operatorname{Der}_K^{\mathrm{nil}}(A)$ *and* $x_1, \ldots, x_n \in A$, *satisfy the following properties:*

(1) $[\delta_i, \delta_j] \in \sum_{\nu=1}^n A\delta_\nu$,

(2) $\det(\delta_i(x_j))$ *is a unit in* A,

(3) *for any* $k = 1, \ldots, n$ *and any* k*-minor* M *of the first* k *columns of* $(\delta_i(x_j))$ *we have*

$$\boldsymbol{\delta}(M) \in \sum_{\nu<k} A\boldsymbol{\delta}(x_\nu),$$

with the conventions $x_0 = 0$ *and* $\boldsymbol{\delta} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}$.

Let $L \subseteq \sum_{\nu=1}^n A\delta_\nu$ *be any* K*-Lie algebra such that* $\delta_1, \ldots, \delta_n \in L$. *Then* $A^L[x_1, \ldots, x_n] = A$ *and* $x_1, \ldots, x_n$ *are algebraically independent over* A^L. *In particular,* $(\operatorname{Spec}(A))^s = \operatorname{Spec} A$.

REMARK 3.6. Condition (3) is implied by

$$(3') \qquad \boldsymbol{\delta}\delta_j(x_k) \in \sum_{\nu<k} A\boldsymbol{\delta}(x_\nu), \quad \text{for } k=1, \dots, n,$$

that is, the derivative-vector of each element of the matrix $(\delta_i(x_j))$ is an A-linear combination of earlier columns.

Proof. Let M be a k-minor of the first k columns of $(\delta_i(x_j))$. If $k=1$, then condition (3) is the same as (3'). Assume that $k>1$ and (3) is true for all $(k-1)$-minors of the first $k-1$ columns of $(\delta_i(x_j))$.

We have $M = \sum (-1)^{k+\nu}\delta_{l\nu}(x_k)M_\nu$, where $\delta_{l_1}, \dots, \delta_{l_k}$ are the vector fields defined by the rows of M and $M_1, \dots, M_k$ are the corresponding $(k-1)$-minors. Now

$$\boldsymbol{\delta}M = \sum (-1)^{k+\nu}(\boldsymbol{\delta}(\delta_{l_\nu}(x_k))M_\nu + \delta_{l_\nu}(x_k)\boldsymbol{\delta}M_\nu)$$

and (3) follows from (3') and the induction hypothesis.

REMARK 3.7. If A is reduced, then (3) in Theorem 3.5 can be replaced by the following weaker condition:

(3'') For any $k=1, \dots, n$ there is a k-minor M_k of the first k columns such that M_k is not a zero divisor and is obtained by deleting a row and the $(k+1)$th column in M_{k+1} and satisfies

$$\boldsymbol{\delta}M_k \in \sum_{\nu<k} A\boldsymbol{\delta}(x_\nu).$$

This will become clear during the proof of Theorem 3.5.

Proof of Theorem 3.5. We prove the theorem by induction on n. The case $n=1$ is Lemma 3.1.

Let $n>1$ and define $\bar{\delta}_j = \sum_k b_{jk}\delta_j$, $(b_{ij}) = (\delta_i(x_j))^{-1}$. We claim that $\bar{\delta}_j \in \mathrm{Der}_K^{\mathrm{nil}}(A)$ and $[\bar{\delta}_i, \bar{\delta}_j]=0$. Then, $\bar{\delta}_i(x_j) = \delta_j^i$ and, by Remark 3.4, we obtain $A^{\bar{L}}[x_1, \dots, x_n] = A$, where $\bar{L} = \sum_{i=1}^n K\bar{\delta}_i$. Let $L \subseteq \sum_{\nu=1}^n A\delta_\nu$ be a K-Lie algebra and $\delta_1, \dots, \delta_n \in L$. The theorem follows since $A^{\bar{L}} = A^L$.

In order to prove the claim, let (for any k such that $\delta_k(x_1) \neq 0$ and is not nilpotent; by (2) such a k exists)

$$\delta_j^{(k)} := \delta_j - \frac{\delta_j(x_1)}{\delta_k(x_1)}\delta_k.$$

Then, by definition, $\delta_j^{(k)}(x_1) = 0$. Since $\delta_k(x_1) \in A^L$, we obtain $A^{\delta_k}_{\delta_k(x_1)}[x_1] = A_{\delta_k(x_1)}$ (Lemma 3.1) and $\delta_j^{(k)} \in \mathrm{Der}_K^{\mathrm{nil}} A_{\delta_k(x_1)}$ (use the fact that $\delta_j^{(k)}(a) = \delta_j(a)$ for $a \in A^{\delta_k}_{\delta_k(x_1)}$ and $\delta_j^{(k)}(x_1) = 0$). We will prove that $\delta_1^{(k)}, \dots, \delta_{k-1}^{(k)}, \delta_{k+1}^{(k)}, \dots, \delta_n^{(k)}$ and $x_2, \dots, x_n$ satisfy the conditions (1), (2) and (3) of the theorem.

Assuming this for a moment and using the induction hypothesis for $x_2, \dots, x_n$ and $\delta_1^{(k)}, \dots, \delta_{k-1}^{(k)}, \delta_{k+1}^{(k)}, \dots, \delta_n^{(k)}$ and any Lie-algebra $L^{(k)} \subseteq \sum_{\nu\neq k} A_{\delta_k(x_1)}\delta_\nu^{(k)}$ such that $\delta_\nu^{(k)} \in L^{(k)}$ we get

$$A^{L^{(k)}}_{\delta_k(x_1)}[x_2, \dots, x_n] = A_{\delta_k(x_1)}.$$

Now let $a \in A^{L^{(k)}}_{\delta_k(x_1)}$, that is, $\delta^{(k)}_\nu(a) = 0$, which implies that

$$\delta_\nu(a) = \frac{\delta_\nu(x_1)}{\delta_k(x_1)}\,\delta_k(a). \qquad (*)$$

Consider

$$\begin{aligned}
\delta^{(k)}_\nu \delta_k(a) &= [\delta^{(k)}_\nu, \delta_k](a) \\
&= [\delta_\nu, \delta_k](a) \\
&= \sum_\mu c_{\nu,k,\mu}\delta_\mu(a) \\
&= \sum_\mu c_{\nu,k,\mu}\frac{\delta_\mu(x_1)}{\delta_k(x_1)}\,\delta_k(a) \quad \text{(because of } (*)) \\
&= \frac{\delta_k(a)}{\delta_k(x_1)}[\delta_\nu, \delta_k](x_1) = 0
\end{aligned}$$

(because $\delta_\nu(x_1), \delta_k(x_1) \in A^L$). This implies that $\delta_k \in \operatorname{Der}^{\mathrm{nil}}_K A^{L^{(k)}}_{\delta_k(x_1)}$. Furthermore, $x_1 \in A^{L^{(k)}}_{\delta_k(x_1)}$. This implies (Lemma 3.1) that $A^L_{\delta_k(x_1)}[x_1] = A^{L^{(k)}}_{\delta_k(x_1)}$ and consequently

$$A^L_{\delta_k(x_1)}[x_1, \ldots, x_n] = A_{\delta_k(x_1)}.$$

Therefore, $\bar\delta_j \in \operatorname{Der}^{\mathrm{nil}}_K A_{\delta_k(x_1)}$ and $[\bar\delta_i, \bar\delta_j](a) = 0$ for all $a \in A_{\delta_k(x_1)}$. This holds for all k with $\delta_k(x_1) \neq 0$ (if $\delta_k(x_1)$ is nilpotent, it is trivial). Assumption (2) implies that

$$\operatorname{Spec} A = \bigcup D(\delta_k(x_1)).$$

Hence, $\bar\delta_j \in \operatorname{Der}^{\mathrm{nil}}_K A$ and $[\bar\delta_i, \bar\delta_j] = 0$. It remains to prove that $\delta^{(k)}_1, \ldots, \delta^{(k)}_{k-1}, \delta^{(k)}_{k+1}, \ldots, \delta^{(k)}_n$ and $x_2, \ldots, x_n$ satisfy (1), (2), (3). Let $[\delta_i, \delta_j] = \sum_\nu c_{ij\nu}\delta_\nu$. Then

$$[\delta^{(k)}_i, \delta^{(k)}_j] = \sum_\nu \left(c_{ij\nu} - \frac{\delta_i(x_1)}{\delta_k(x_1)}c_{kj\nu} - \frac{\delta_j(x_1)}{\delta_k(x_1)}c_{ik\nu}\right)\delta^{(k)}_\nu,$$

that is, (1) is satisfied. Part (2) follows from

$$\det(\delta_i(x_j)) = (-1)^{k+1}\delta_k(x_1)\det(\delta^{(k)}_i(x_j))_{i\neq k, j\geq 2}.$$

To prove (3) let $M^{(k)}$ be an l-minor of the first l columns of $(\delta^{(k)}_i(x_j))_{j\geq 2, i\neq k}$. Let M be the $(l+1)$-minor of $(\delta_i(x_i))$ defined by the first $l+1$ columns and by the rows corresponding to the rows of $M^{(k)}$ and the kth row of $(\delta_i(x_j))$. Then

$$M = \pm\delta_k(x_1)M^{(k)}$$

and

$$\boldsymbol{\delta} M = \pm\delta_k(x_1)\boldsymbol{\delta} M^{(k)} = \sum_{\nu<l+1} c_\nu \boldsymbol{\delta}(x_\nu),$$

by assumption. This implies in particular that $\delta_j M = \sum_{\nu<l+1} c_\nu\delta_j(x_\nu)$ for suitable $c_\nu \in A$ and all j. We obtain

$$\delta^{(k)}_j M = \sum_{\nu<l+1} c_\nu \delta^{(k)}_j(x_\nu),$$

$$\boldsymbol{\delta}^{(k)} M^{(k)} = \pm\frac{1}{\delta_k(x_1)}\boldsymbol{\delta}^{(k)} M \in \sum_{\nu<l+1} A_{\delta_k(x_1)}\boldsymbol{\delta}^{(k)}(x_\nu),$$

and (3) is proved.

THEOREM 3.8 (*row-minor criterion*). *Let* $\delta_1, \ldots, \delta_n \in \mathrm{Der}_K^{\mathrm{nil}}(A)$ *and* $x_1, \ldots, x_n \in A$ *satisfy the following properties:*

(1) $[\delta_i, \delta_j] \in \sum_{v>\max\{i,j\}} A\delta_v$,

(2) $\det(\delta_i(x_j))$ *is a unit in* A,

(3) *for any* $k = 1, \ldots, n$ *and any* $n-k+1$ *minor* M *of the last* $n-k+1$ *rows of* $(\delta_i(x_j))$, *we have* $\delta_l M = 0$ *for* $l = 1, \ldots, k$.

Let $L \subseteq \sum_{v=1}^n A\delta_v$ *be any* K*-Lie algebra such that* $\delta_1, \ldots, \delta_n \in L$. *Then* $A^L[x_1, \ldots, x_n] = A$ *and* $x_1, \ldots, x_n$ *are algebraically independent over* A^L.

REMARK 3.9. Condition (3) follows from

$$(3') \qquad \delta_i\delta_l(\mathbf{x}) \in \sum_{v>l} A\delta_v(\mathbf{x}) \quad \text{for all } i,l$$

(with the convention that $\delta_{n+1} = 0$ and $\mathbf{x} = (x_1, \ldots, x_n)$).

Proof. Let M be an $n-k+1$-minor of the last $n-k+1$ rows of $(\delta_i(x_i))$. If $k = n$, then (3) is the same as (3′) for $l = n$. Assume (3) holds for all $(n-k)$-minors of the last $n-k$ rows and write

$$M = \sum (-1)^v \delta_k(x_{i_v}) M_v.$$

Then, for $l \leq k$,

$$\begin{aligned}
\delta_l M &= \sum (-1)^v (\delta_l\delta_k(x_{i_v})M_v + \delta_k(x_{i_v})\delta_l M_v) \\
&= \sum_{\substack{v,\mu \\ \mu>k}} (-1)^v c_{lk\mu}\delta_\mu(x_{i_v})M_v \quad (\text{because } \delta_l M_v = 0) \\
&= \sum_{\mu>k} c_{lk\mu} \sum_v (-1)^v \delta_\mu(x_{i_v}) \\
&= 0.
\end{aligned}$$

Proof of Theorem 3.8. Again we prove the theorem by induction on n. The case $n = 1$ is considered in Lemma 3.1.

Let $n > 1$ and define $\bar{\delta}_j = \sum_k b_{jk}\delta_j$, $(b_{ij}) = (\delta_i(x_j))^{-1}$. We claim that $\bar{\delta}_i \in \mathrm{Der}_K^{\mathrm{nil}}(A)$ and $[\bar{\delta}_i, \bar{\delta}_j] = 0$. This implies, as in the proof of Theorem 3.5, that $A^L[x_1, \ldots, x_n] = A$. In order to prove the claim we define, for any k such that $\delta_n(x_k) \neq 0$,

$$x_j^{(k)} := x_j - \frac{\delta_n(x_j)}{\delta_n(x_k)} x_k.$$

Then $x_j^{(k)} \in A^{\delta_n}_{\delta_n(x_k)}$ since $\delta_n(x_j) \in A^L$ by (3). Using Lemma 3.1 we obtain $A^{\delta_n}_{\delta_n(x_k)}[x_k] = A_{\delta_n(x_k)}$. Now, by assumption (1), $\delta_1, \ldots, \delta_{n-1} \in \mathrm{Der}_K^{\mathrm{nil}} A^{\delta_n}_{\delta_n(x_k)}$. On the other hand,

$$\det(\delta_i(x_j)) = (-1)^{n+k}\delta_n(x_k)\det(\delta_i(x_j^{(k)}))_{i<n, j\neq k},$$

that is, $\det(\delta_i(x_j^{(k)}))_{i<n,j\neq k}$ is a unit in $A^{\delta_n}_{\delta_n(x_k)}$. Let $M^{(k)}$ be an $(n-k)$-minor of the last $n-k$ rows of $(\delta_i(x_j^{(k)}))_{i<n,j\neq k}$. If M denotes the $(n-k+1)$-minor of $(\delta_i(x_j))$ defined by the last $n-k+1$ rows and the columns defining $M^{(k)}$ and the kth

column, then $M = \pm\delta_k(x_k)M^{(k)}$. Now $\delta_l M = 0$ for $l \leq k$ implies $\delta_l M^{(k)} = 0$ in $A_{\delta_k(x_k)}$. We have proved that the conditions (1), (2), (3) are satisfied for $\delta_1, \ldots, \delta_{n-1}$ and $x_1^{(k)}, \ldots, x_{k-1}^{(k)}, x_{k+1}^{(k)}, \ldots, x_n^{(k)}$. Using the induction hypothesis we obtain

$$A^L_{\delta_n(x_k)}[x_1^{(k)}, \ldots, x_{k-1}^{(k)}, x_{k+1}^{(k)}, \ldots, x_n^{(k)}] = A^{\delta_n}_{\delta_k(x_k)}$$

and, finally,

$$A^L_{\delta_n(x_k)}[x_1, \ldots, x_k] = A_{\delta_n(x_k)}.$$

As in the proof of Theorem 3.5 we can deduce that $\bar{\delta}_i \in \mathrm{Der}_K^{\mathrm{nil}}(A)$ and $[\bar{\delta}_i, \bar{\delta}_j] = 0$.

Now we are prepared to prove the main theorem of this chapter.

THEOREM 3.10. *Let $L \subseteq \mathrm{Der}_K^{\mathrm{nil}}(A)$ be a finite-dimensional nilpotent Lie algebra and $r = \dim_K L$. The following conditions are equivalent:*

(1) $H^1(L, A) = 0$;
(2) $H^n(L, A) = 0$ *for* $n \geq 1$;
(3) *there are* $x_1, \ldots, x_r \in A$ *and* $\delta_1, \ldots, \delta_r \in L$ *such that*
  (3.1) $\delta_i(x_i) = 1$,
  (3.2) $\delta_i(x_j) = 0$ *if* $j < i$,
  (3.3) $\delta_k\delta_i(x_j) = 0$ *if* $k \geq j$;
(4) *there are* $x_1, \ldots, x_r \in A$ *and* $\delta_1, \ldots, \delta_r \in L$ *such that*
  (4.1) $\det(\delta_i(x_j))$ *is a unit,*
  (4.2) *for any k-minor M of the first k columns of* $(\delta_i(x_j))$, *with* $k = 1, \ldots, r$, *we have*

$$\boldsymbol{\delta}(M) \in \sum_{v<k} A\boldsymbol{\delta}(x_v);$$

(5) *there are* $x_1, \ldots, x_r \in A$ *and* $\delta_1, \ldots, \delta_r \in L$ *such that*
  (5.1) $[\delta_i, \delta_j] \in \sum_{v > \max\{i,j\}} K\delta_v$,
  (5.2) $\det(\delta_i(x_j))$ *is a unit,*
  (5.3) *for any* $r - k + 1$*-minor M of the last* $r - k + 1$ *rows of* $(\delta_i(x_j))$, *with* $k = 1, \ldots, r$, *we have* $\delta_l M = 0$ *for* $l = 1, \ldots, k$;
(6) *there are* $x_1, \ldots, x_r \in A$ *algebraically independent over* A^L *and* $\delta_1, \ldots, \delta_r \in L$ *such that* $A^L[x_1, \ldots, x_r] = A$, *and* $\det(\delta_i(x_j))$ *is a unit* (Spec $A \to$ Spec A^L *is a trivial geometric quotient with fibre isomorphic to L*).

Proof. To prove that (2) is equivalent to (1) we use the Hochschild–Serre spectral sequence (cf. [**3**, p. 351]) for a sub-Lie algebra Z contained in the centre of L:

$$E_2^{pq} = H^p(L/Z, H^q(Z, A)) \Rightarrow H^{p+q}(L, A).$$

Theorem 5.11 in [**3**, p. 328] for the case of $\dim Z = 1$ (especially $H^q(Z, A) = 0$ if $q \neq 0, 1$) gives rise to the exact sequence

$$\ldots \to H^n(L/Z, A^Z) \to H^n(L, A) \to H^{n-1}(L/Z, H^1(Z, A)) \to H^{n+1}(L/Z, A^Z) \to \ldots.$$

Now we can use induction on the dimension of L. The case $n = 1$ is obvious since always $H^i(L, A) = 0$ if $i > \dim L$.

Assume $H^1(L, A) = 0$ and let $Z = K\delta$ for some $\delta \neq 0$ in the centre of L. From the exact sequence we obtain $H^1(L/Z, A^Z) = 0$. By the induction hypothesis this implies that $H^n(L/Z, A^Z) = 0$ for $n \geq 1$ and $H^n(L, A) = H^{n-1}(L/Z, H^1(Z, A))$. In particular, $0 = H^0(L/Z, H^1(Z, A)) = H^1(Z, A)^{L/Z}$. By definition of H^1 there is an $x \in A$ such that $\delta(x) = 1$. By Lemma 3.1 we have $A^Z[x] = A$. This implies that $H^1(Z, A) = 0$ (Remark 3.4) and, consequently, $H^n(L, A) = 0$.

To prove that (1) implies (3) we again use induction on the dimension of L. If $\dim L = 1$, then the result is a consequence of Remark 3.4. Now let δ_r be a non-trivial element from the centre of L. As before, $H^1(L, A) = 0$ implies that $H^1(Z, A) = 0$ and $H^1(L/Z, A^Z) = 0$. Thus, there is an $x_r \in A$ such that $\delta_r(x_r) = 1$.

By the induction hypothesis there are $x_1, \ldots, x_{r-1} \in A^Z$ and $\bar{\delta}_1, \ldots, \bar{\delta}_{r-1} \in L/Z$ such that

$$\bar{\delta}_i(x_i) = 1,$$

$$\bar{\delta}_i(x_j) = 0 \quad \text{if } j < i,$$

$$\bar{\delta}_e\bar{\delta}_i(x_j) = 0 \quad \text{if } e \geq j.$$

Let $\delta_1, \ldots, \delta_{r-1} \in L$ represent $\bar{\delta}_1, \ldots, \bar{\delta}_{r-1}$. Then $\delta_1, \ldots, \delta_r$, $x_1, \ldots, x_r$ satisfy (3.1), (3.2) and (3.3). The implication (3)$\Rightarrow$(4) is obvious because a k-minor of the first k rows of $(\delta_i(x_j))$ is either 1 or 0. Moreover, the assumptions of (3) imply that $[\delta_i, \delta_j] \in \sum_{v > \max\{i,j\}} K\delta_v$. This can be proved using induction on r.

The conditions of (3) imply that $\delta_1, \ldots, \delta_r$ is a basis of L, δ_r is in the centre of $L, x_1, \ldots, x_{r-1} \in A^{\delta_r}$, and the classes $\bar{\delta}_1, \ldots, \bar{\delta}_{r-1}$ of $\delta_1, \ldots, \delta_{r-1}$ and $x_1, \ldots, x_{r-1} \in A^{\delta_r}$ satisfy (3) too. On the other hand, any $(r-k+1)$-minor of the last $r-k+1$ rows of $(\delta_i(x_j))$ is either 1 or 0. This shows that (3) also implies (5). Now, using Theorem 3.5 and Theorem 3.8, we obtain that (4), respectively (5), implies (6).

The implication (6)$\Rightarrow$(1) is proved in the following supplement.

Supplement. Let $B \subseteq A$ be a subalgebra and $L \subseteq \mathrm{Der}_B^{\mathrm{nil}} A$ be a nilpotent Lie algebra with the following properties:

(i) L is a free B-module of rank r,

(ii) $L = Z_0(L) \supseteq Z_1(L) \supseteq \ldots \supseteq Z_l(L) \supseteq Z_{l+1}(L) = 0$ is filtered by sub Lie-algebras $Z_j(L)$ such that $[L, Z_j(L)] \subseteq Z_{j+1}(L)$ for all j, and $Z_i(L)/Z_j(L)$ are free B-modules of finite rank for $i = 0, \ldots, l, j = 1, \ldots, l+1$.

Then the conditions (1), …, (6) of Theorem 3.10 are equivalent.

Proof. Choosing a suitable base of the free B-module L, we find that the proof for the implications (1) to (6) works as well as for Theorem 3.10. It remains to prove that (6) implies (1). Let $(a_{ij}) = (\delta_i(x_j))^{-1}$; then $\sum_v \delta_i(x_v)a_{vj} = \delta_i^j$ implies that

$$\sum_v \delta_l\delta_i(x_v)a_{vj} + \sum_v \delta_i(x_v)\delta_l(a_{vj}) = 0.$$

Using this equality we obtain

$$\sum_v [\delta_i, \delta_l](x_v)a_{vj} = \sum_v \delta_i(x_v)\delta_l(a_{vj}) - \sum_v \delta_l(x_v)\delta_i(a_{vj}).$$

Let $[\delta_i, \delta_l] = \sum_\mu c_{il\mu}\delta_\mu$; then

$$c_{ilj} = \sum_{\nu,\mu} c_{il\mu}\delta_\mu(x_\nu)a_{\nu j} = \sum_{\nu,\mu} \delta_i(x_\nu)\delta_l(x_\mu)\left(\frac{\partial a_{\nu j}}{\partial x_\mu} - \frac{\partial a_{\mu j}}{\partial x_\nu}\right).$$

Now let $\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \in A^r$ such that $\delta_i(a_k) - \delta_k(a_i) = \sum_\nu c_{ik\nu}a_\nu$. We have to prove that there is a $z \in A$ such that $\delta_j(z) = a_j$. We use the equality obtained for the c_{ilj} and obtain

$$\sum_\nu \delta_i(x_\nu)\frac{\partial a_l}{\partial x_\nu} - \sum_\nu \delta_l(x_\nu)\frac{\partial a_i}{\partial x_\nu} = \sum_{\nu,\mu,j} \delta_i(x_\nu)\delta_l(x_\mu)\left(\frac{\partial a_{\nu j}}{\partial x_\mu} - \frac{\partial a_{\mu j}}{\partial x_\nu}\right)a_j$$

for all i,l. This implies that

$$\sum_j a_{\nu j}\frac{\partial a_j}{\partial x_k} - \sum_j a_{kj}\frac{\partial a_j}{\partial x_\nu} = \sum_j \left(\frac{\partial a_{kj}}{\partial x_\nu} - \frac{\partial a_{\nu j}}{\partial x_k}\right)a_j,$$

that is,

$$\frac{\partial}{\partial x_k}\left(\sum_j a_{\nu j}a_j\right) = \frac{\partial}{\partial x_\nu}\left(\sum_j a_{kj}a_j\right)$$

for all ν,k. Consequently, we obtain $z \in A$ such that $\partial z/\partial x_k = \sum_j a_{kj}a_j$. This implies that $\delta_k(z) = a_k$.

We should like to thank Hanspeter Kraft for the following remarks:

REMARKS 3.11. 1. The equivalent conditions of Theorem 3.10 imply that the geometric quotient $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is a principal fibre bundle with group $\exp(L)$ in the sense of Mumford [**13**, 0.3, Definition 0.10].

We have to show that the following morphism is an isomorphism:

$$(\sigma, p_2)\colon \exp L \times \operatorname{Spec} A \xrightarrow{\cong} \operatorname{Spec}(A \otimes_{A^L} A),$$

where σ denotes the action and p_2 the projection. Since the coordinate ring of $\exp L$ is isomorphic to $K[y_1, \ldots, y_r]$, we have to show the isomorphism

$$A \otimes_{A^L} A \xrightarrow{\cong} A \otimes_K K[y_1, \ldots, y_r], \quad a \otimes b \mapsto (a \otimes 1)\sigma^*(b).$$

We use condition (3) and induction on r. For $r = 1$ we have the morphism

$$A^\delta[x] \otimes_{A^\delta} A^\delta[y] \to A^\delta[x] \otimes_K K[y],$$

$x \mapsto x$ and $y \mapsto \sigma^*(1 \otimes y) = \exp(y\delta)(x) = x + y$, which is certainly an isomorphism. By the induction hypothesis, we have

$$A^{\delta_r} \otimes_{A^L} A^{\delta_r} \xrightarrow{\cong} A^{\delta_r} \otimes_K K[y_1, \ldots, y_{r-1}].$$

Applying $A^{\delta_r}[x_r] \otimes_{A^{\delta_r}} - = A \otimes_{A^{\delta_r}}$ from the left and $- \otimes_{A^{\delta_r}} A = - \otimes_{A^{\delta_r}} A^{\delta_r}[y_r]$ from the right we see that the above isomorphism composes with $x_r \mapsto x_r$ and $y_r \mapsto \exp(y_r\delta_r)(x_r)$ to the desired isomorphism.

2. If we use Lemma 7.4.1 of [**9**] we also get the converse, that is, the conditions of Theorem 3.10 are equivalent to $\operatorname{Spec} A \to \operatorname{Spec} A^L$ being a geometric quotient and principal fibre bundle with group $\exp(L)$.

3. Of course, the equivalent conditions of 3.10 imply that the action of L is free (either in our sense or in the strong sense of Mumford [**13**]) and $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is a geometric quotient. We do not know whether the converse is true. This would follow from [**9**], but according to Kraft the statement 'If the action is free we have clearly ...' in [**9**, p. 115] is not justified.

4. Note, however, that if A is of finite type over K and if the action of L on A is free in the sense of Mumford, then a geometric quotient $\operatorname{Spec} A \to \operatorname{Spec} A^L$ is a principal fibre bundle with group $\exp L$ by [**13**, 0.3, Proposition 0.9].

Theorem 3.10 suggests the following conjecture.

CONJECTURE. Let $L \subseteq \operatorname{Der}_K^{\mathrm{nil}}(A)$ be a nilpotent Lie algebra and $r = \dim L$. Let $\delta_1, \ldots, \delta_r \in L$, $x_1, \ldots, x_r \in A$ such that $\det(\delta_i(x_j))$ is a unit. Then there exist $y_1, \ldots, y_r \in A$ such that $A = A^L[y_1, \ldots, y_r]$ (equivalently, $H^1(L, A) = 0$). For the moment we can show this only up to an étale covering:

REMARK 3.12. Let $L \subseteq \operatorname{Der}_K^{\mathrm{nil}}(A)$ be abelian and $r = \dim L$. Let $\delta_1, \ldots, \delta_r \in L$, $x_1, \ldots, x_r \in A$ such that $\det(\delta_i(x_j))$ is a unit. Then there is a $B \supseteq A$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale, the action of L lifts to B and $H^1(L, B) = 0$. If K is algebraically closed and A is of finite type over K then B can be chosen such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. Let $F_i(Z_1, \ldots, Z_r) := (\exp \sum Z_j\delta_j)(x_i)$ and $B := A[Z_1, \ldots, Z_r]/(F_1, \ldots, F_r)$. Then the action of L on A lifts to B by $\delta_i(Z_j) = -\delta_j^i$. This implies that $H^1(L, B) = 0$ (Remark 3.4). On the other hand, applying the automorphism $\exp \sum Z_l\delta_l$ yields that

$$\left(\exp \sum Z_l\delta_l\right)(\det(\delta_i(x_j))) = \det\left(\left(\exp \sum Z_l\delta_l\right)(\delta_i(x_j))\right) = \det\left[\frac{\partial F_i}{\partial Z_j}\right]$$

is a unit in $A[Z_1, \ldots, Z_r]$. This implies that $B \supseteq A$ is étale.

Replacing x_i by $x_i + \alpha_i$, where $\alpha_i \in K$, we obtain, with the construction above for every closed point of $\operatorname{Spec} A$ defined by the prime ideal p such that $x_1 + \alpha_1, \ldots, x_r + \alpha_r \in p$, an étale neighbourhood $\operatorname{Spec} B$ of p such that $H^1(L, B) = 0$. This proves the rest of the remark.

The following stronger version of the conjecture is equivalent to the *Jacobian Umkehrproblem*. Let $L \supseteq \operatorname{Der}_K^{\mathrm{nil}}(A)$ be an abelian Lie algebra and $r = \dim L$. Let $\delta_1, \ldots, \delta_r \in L$, $x_1, \ldots, x_r \in A$ be such that $\det(\delta_i(x_j))$ is a unit. Then $A^L[x_1, \ldots, x_r] = A$. (For $A = K[Y_1, \ldots, Y_r]$, $\delta_i = \partial/\partial Y_i$, we obtain the Jacobian Umkehrproblem.)

It is not difficult to see that a solution of the Jacobian Umkehrproblem also solves the conjecture if A is reduced. Theorem 3.10 now provides a solution of the Jacobian Umkehrproblem under additional conditions:

COROLLARY 3.13. *Let* $x_1, \ldots, x_n \in K[z_1, \ldots, z_n]$ *such that*

(1) $\det(\partial x_i/\partial x_j) = 1$,

and assume that the following condition is satisfied:

(2) *there is a sequence of non-vanishing* k*-minors* M_k *of the first* k *columns,* $k = 1, \ldots, n$*, with the following properties:* M_k *is obtained by deleting a row and the* $(k+1)$*th column in* M_{k+1} *and satisfies*

$$\frac{\partial}{\partial z} M_k \in \sum_{v<k} K[z_1, \ldots, z_n] \frac{\partial x_v}{\partial z}$$

(*with the convention that* $x_0 = 0$ *and* $\partial/\partial z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$).

Then $K[x_1, \ldots, x_n] = K[z_1, \ldots, z_n]$.

4. *General actions and algorithmic stratification*

In this chapter we will give conditions for the existence of a geometric quotient for the case where the action of L is not necessarily free. Again, A denotes a commutative K-algebra, $\mathrm{char}(K) = 0$. The differential

$$d\colon A \to \mathrm{Hom}_K(L, A), \quad da(\delta) = \delta(a),$$

will play an essential role in the construction of the stratifications. Moreover, we consider the exterior derivation $A \to \Omega_{A/A^L}$ into the module of Kähler differentials. Note that d factors

$$d\colon A \to \Omega_{A/A^L} \to \mathrm{Hom}_K(\mathrm{Der}_{A^L}(A), A) \to \mathrm{Hom}_K(L, A).$$

The exterior derivation will also be denoted by d. For any subset $M \subset A$, AdM denotes the submodule generated by dM either in $\mathrm{Hom}_K(L, A)$ or in Ω_{A/A^L}.

The next theorem is a generalization of Theorem 3.10 to not-necessarily-free actions of an abelian Lie algebra.

THEOREM 4.1. *Let* A *be reduced and noetherian and* $L \subseteq \mathrm{Der}_K^{\mathrm{nil}} A$ *be a finite-dimensional abelian Lie algebra. The following conditions are equivalent.*

(1) *There exists an open subset* U *of* $\mathrm{Spec}\, A^L$ *such that* $\mathrm{Spec}\, A \to U \subseteq \mathrm{Spec}\, A^L$ *is a geometric quotient and locally trivial.*

(2) *The orbit dimension under the action of* L *is locally constant and* $\Omega_{A/A^L} = Ad \int A^L$ *where* $\int A^L := \{a \in A \mid \delta(a) \in A^L \text{ for all } \delta \in L\}$.

(3) *There are* $x_1, \ldots, x_n \in A$ *and* $\delta_1, \ldots, \delta_m \in L$ *such that*

(3.1) $\delta_i(x_j) \in A^L$ *for all* i, j,

(3.2) $\mathrm{rank}(\delta_i(x_j))$ *is locally constant on* $\mathrm{Spec}\, A$ *and equal to the orbit dimension.*

(3′) *Let* $d\colon A \to \mathrm{Hom}_K(L, A)$ *be the differential defined as above. Then* AdA *is locally free and* $AdA = Ad \int A^L$.

(4) *There is a covering* $\bigcup_{f \in I} D(f) = \mathrm{Spec}\, A$, $I \subseteq A^L$, *and for* $f \in I$ *there exists a sub-Lie algebra* $L^{(f)} \subseteq L$ *such that*

(4.1) $A_f \otimes_K L^{(f)} = A_f \otimes_K L$,

(4.2) $H^1(L^{(f)}, A_f) = 0$.

Proof. First, we prove that Condition (1) implies (2), (3′) and (4). The conditions of (2) are local in the sense that it is sufficient to prove them on an invariant affine covering of $\operatorname{Spec} A$. So we may assume that $A = A^L[x_1, \ldots, x_n]$ and $\operatorname{rank}(\delta_i(x_j)(t)) = n$ for all $t \in \operatorname{Spec} A$, where $L = \sum_{i=1}^m K\delta_i$.

Now let M be an n-minor of the matrix $(\delta_i(x_j))$ not vanishing identically. We will see that M is invariant. Since A is noetherian and reduced, it is enough to check it on the components of $\operatorname{Spec} A$, that is, we may assume that A is an integral domain. Let L_0 be the sub-Lie algebra generated by the vector fields corresponding to the rows of the matrix defining the minor M, $L_0 = \sum_{\nu=1}^n K\delta_{i_\nu}$. Since $M \neq 0$, we obtain $A^L = A^{L_0}$. Using the same method as in the proof of Proposition 1.6, we obtain $A_f = A_f^{L_0}[y_1, \ldots, y_n]$ for a suitable $f \in A^{L_0}$ and $\delta_{i_\nu}(y_\nu) = 1$, $\delta_{i_\nu}(y_i) = 0$ if $\nu > j$. On the other hand, $A_f = A_f^{L_0}[x_1, \ldots, x_n]$ implies that

$$(\delta_{i_\nu}(y_j)) = (\delta_{i_\nu}(x_k))\left(\frac{\partial y_i}{\partial x_k}\right),$$

that is, $1 = M \det(\partial y_i/\partial x_k)$. This implies that $M \in A_f^{L_0}$ (Remark 3.3) and, since A is an integral domain, $M \in A^{L_0}$. Since M is invariant under the action of L, we obtain

$$A_M = A_M^L[x_1, \ldots, x_n] = A_M^{L_0}[x_1, \ldots, x_n].$$

Since L_0 acts freely, we can use Theorem 3.10 to obtain $H^1(L_0, A_M) = 0$. This implies that there are $z_1, \ldots, z_n \in A_M$ such that $\delta_{i_\nu}(z_j) = \delta_j^i$. Remark 3.4 implies that $A_M^{L_0}[z_1, \ldots, z_n] = A_M$. If $\delta \in L$ is any vector field, then $\delta = \sum_{\nu=1}^n h_\nu \delta_{i_\nu}$, with $h_\nu \in A_M$, since the other rows of $(\delta_i(x_j))$ are linearly dependent on the rows corresponding to M. Now L is abelian and $[\delta, \delta_{i_\nu}] = 0$ for all ν implies that $h_\nu \in A_M^L$. This implies that $\delta(z_i) \in A_M^L = A_M^{L_0}$ for all $\delta \in L$, that is, $z_1, \ldots, z_n \in \int A_M^L$. Hence, $A_M dA_M = A_M d \int A_M$ and $A_M dA_M$ is locally free and (3′) follows.

On the other hand, $\Omega_{A_M/A_M^L} = \sum_{\nu=1}^N A_M dz_\nu$, that is, $A_M d\int A_M^L = \Omega_{A_M/A_M^L}$, and we have proved that (2) is true on the open set $D(M)$. Now $\operatorname{Spec} A$ is covered by the open sets defined by all n-minors of $(\delta_i(x_j))$ and this proves (2) and (4). Condition (4) implies (1) because of Theorem 3.10 (take $U = \bigcup_{f \in I} D(f) \subset \operatorname{Spec} A^L$).

Next, we prove that (2) implies (3). Choose $x_1, \ldots, x_n \in \int A^L$ such that $\Omega_{A|A^L} = \sum_{i=1}^n A dx_i$. By definition of $\int A^L$, we have $\delta(x_i) \in A^L$ for all $\delta \in L$. Let $\delta_1, \ldots, \delta_m$ be a basis of L. We have to prove that $\operatorname{rank}(\delta_i(x_j)(t))$ is equal to the dimension of the orbit of t for all points $t \in \operatorname{Spec} A$. Let us consider the exact sequence

$$\Omega_{A^L} \otimes_{A^L} A \to \Omega_A \to \Omega_{A/A^L} \to 0.$$

Locally at t we may assume that Ω_A is generated by $dx_1, \ldots, dx_n, dy_1, \ldots, dy_s$, $dy_i \in \Omega_{A^L}$ such that $x_1, \ldots, x_n, y_1, \ldots, y_s$ generate the maximal ideal of the local ring of t. Now $\operatorname{rank}(\delta_i(x_j), \delta_i(y_j))(t)$ is equal to the dimension of the orbit of t (cf. Lemma 4.2). However, $dy_i \in \Omega_{A^L}$ implies $d_j(y_i) = 0$, which proves the claim.

To prove that (3) implies (1) let $M = \det(\delta_{i_\nu}(x_{j_\mu}))_{\nu,\mu \leq r}$ be an r-minor not vanishing identically and r the orbit dimension on $D(M) \subseteq \operatorname{Spec} A$. Let L_M be the Lie algebra generated by $\delta_{i_1}, \ldots, \delta_{i_r}$. Using Theorem 3.5, we obtain

$$A_M^{L_M}[x_{j_1}, \ldots, x_{j_r}] = A_M.$$

If $A_M^L \nsubseteq A_M^{L_M}$, then there is an $f \in A^L$ such that $A_{Mf}^L[y_1, \ldots, y_t] = A_{Mf}^{L_M}$ for suitable $y_1, \ldots, y_t$ (Proposition 1.6). This implies that the orbit dimension is not r and this is a contradiction to the assumption. We obtain $A_M^L[x_{j_1}, \ldots, x_{j_r}] = A_M$. Let $U \subseteq \operatorname{Spec} A^L$ be the open set defined by all minors of size equal to the orbit dimension of $(\delta_i(x_j))$; then $\operatorname{Spec} A \to U$ is a geometric quotient and locally trivial.

Finally, we show that (3′) implies (3). Choose $x_1, \ldots, x_n \in \int A^L$ such that $AdA = \sum_{\nu=1}^n Adx_\nu$. If $\delta_1, \ldots, \delta_m$ is a basis of L, then $\delta_i(x_j) \in A^L$. The second condition of (3) is implied by the following lemma.

Lemma 4.2. *Let A be noetherian and $L = \sum_{i=1}^m K\delta_i \subseteq \operatorname{Der}_K^{\mathrm{nil}} A$ be a nilpotent Lie algebra. Let d: $A \to \operatorname{Hom}_K(L, A)$ be the differential defined by $da(\delta) = \delta(a)$ for $a \in A$, $\delta \in L$, and let t be a point of* $\operatorname{Spec} A$. *Then the following hold:*

(i) *if $x_1, \ldots, x_n$ generate the maximal ideal of the local ring of t then* $\operatorname{rk}(\delta_i(x_j)(t))$ *is equal to the dimension of the L-orbit of t;*

(ii) *if $AdA = \sum_{j=1}^n Adx_j$ then*

$$\dim_{\kappa(t)} AdA \otimes_A \kappa(t) = \operatorname{rk}(\delta_i(x_j)(t))$$

and is equal to the dimension of the L-orbit of t.

Proof. (i) Let $\delta_i|_t$ be defined by $\delta_i|_t(a) = \delta_i(a)(t)$ for $a \in A$. Then $\delta_i|_t$ generate the tangent space to the L-orbit of t. Certainly, the dimension of this tangent space is equal to $\operatorname{rk}(\delta_i(x_j)(t))$. Since $\operatorname{char}(K) = 0$, this is equal to the orbit-dimension at t [**1**, II, 6.7].

(ii) Let Φ: $\operatorname{Hom}_K(L, A) \xrightarrow{\sim} A^m$ be the isomorphism defined by $\Phi(\varphi) = (\varphi(\delta_1), \ldots, \varphi(\delta_m))$, where $\delta_1, \ldots, \delta_m$ form a K-basis of L. Then $\Phi(AdA)$ is the submodule generated by $\Phi(dx_i) = (\delta_1(x_i), \ldots, \delta_m(x_i))$. Certainly Φ does commute with localization. Hence $AdA \otimes_A \kappa(t) = \operatorname{rk}(\delta_i(x_j)(t))$, which is equal to the orbit dimension by (i).

Remark 4.3. The condition $AdA = Ad \int A^L$ implies that L is abelian. More generally, let $L \subset \operatorname{Der}_K A$ be any Lie-algebra and $Z_j = [L, Z_{j-1}]$, $Z_0 = L$, be the lower central series. Define $F^0(A) = A^L$ and $F^i(A) := \int F^{i-1}(A) := \{a \in A \mid \delta(a) \in F^{i-1}(A)$ for all $\delta \in L\}$, for $i \geqslant 1$. Then $AdA = AdF^k(A)$ implies that $Z_k = 0$, in particular, L is nilpotent. To see this, let $x \in A$ and $\delta \in Z_j$. Then $\delta(x) \in F^{k-j-1}(A)$, and hence $\delta(x) = 0$ if $\delta \in Z_k$ and $x \in F^k(A)$. For arbitrary $x \in A$ we have $dx = \sum \xi_i dx_i$, with $x_i \in F^k(A)$ by assumption. This shows that $\delta(x) = dx(\delta) = 0$, that is, $Z_k = 0$.

Remark 4.4. If L is nilpotent but not abelian, we could use the lower (or upper) central series

$$L = Z_0 \supset Z_1 \supset \ldots \supset Z_l \supset \{0\},$$

and derive a criterion for the existence of a locally trivial quotient by applying Theorem 4.1 successively to the abelian Lie algebras Z_i/Z_{i+1}. We do not formulate this for two reasons. Firstly, it is of minor practical use since it would require knowledge about the invariant functions A^{Z_i} ($i = 1, \ldots, l$), and secondly, it is too strong, as Example 1.13 shows: there, the quotient with respect to L exists but not with respect to the centre.

Instead we prefer to prove a criterion which uses a filtration of L and of A (with properties like Z_j and F^i in Remark 4.3) and which does not require any knowledge about invariant functions. Moreover, if the quotient does not exist on all of $\operatorname{Spec} A$, it provides a stratification of $\operatorname{Spec} A$, into locally closed invariant subspaces admitting a locally trivial quotient. The construction of the strata is completely explicit in terms of the given coordinates and vector fields.

Algorithmic stratification

Let A be a noetherian K-algebra and $L \subseteq \operatorname{Der}_K^{\mathrm{nil}} A$ be a finite-dimensional nilpotent Lie algebra. Suppose that $A = \bigcup_{i \in \mathbb{Z}} F^i(A)$ has a filtration

$$F^\bullet\colon\ 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$$

by sub-vector spaces $F^i(A)$ such that

$$\text{(F)} \qquad \delta F^i(A) \subseteq F^{i-1}(A) \quad \text{for all } i \in \mathbb{Z} \text{ and all } \delta \in L.$$

Assume, furthermore, that

$$Z_\bullet\colon\ L = Z_0(L) \supseteq Z_1(L) \supseteq \dots \supseteq Z_l(L) \supseteq Z_{l+1}(L) = 0$$

is filtered by sub-Lie algebras $Z_j(L)$ such that

$$\text{(Z)} \qquad [L, Z_j(L)] \subseteq Z_{j+1}(L) \quad \text{for all } j \in \mathbb{Z}.$$

Let $d\colon A \to \operatorname{Hom}_K(L, A)$ be the differential defined by $da(\delta) = \delta(a)$ for $a \in A$ and $\delta \in L$. The filtration $F^\bullet$ of A induces a filtration of AdA, the A-module generated by the image of d:

$$0 = AdF^0(A) \subseteq AdF^1(A) \subseteq \dots \subseteq AdF^i(A) \subseteq \dots \subseteq AdF^k(A) = AdA.$$

(Since A is noetherian and AdA a submodule of the free module $\operatorname{Hom}_K(L, A)$, there is a k such that $AdF^k(A) = AdA$.) The filtration $Z_\bullet$ of L induces projections

$$\pi_j\colon \operatorname{Hom}_K(L, A) \to \operatorname{Hom}_K(Z_j(L), A)$$

and via π_j the module AdA defines submodules $\pi_j(AdA) \subseteq \operatorname{Hom}_K(Z_j(L), A)$. We will study now the flattening stratification of $\operatorname{Spec} A$ with respect to the modules $\operatorname{Hom}_K(L, A)/AdF^i(A)$ and $\operatorname{Hom}_K(Z_j(L), A)/\pi_j(AdA)$.

We use the following notation. For any A-module M of finite presentation

$$A^q \xrightarrow{\varphi} A^p \longrightarrow M \longrightarrow 0$$

let $I_j(M)$ be the ideal of A generated by the j-minors of φ with the convention that $I_0(M) = A$ and $I_j(M) = 0$ if $j > \min\{p, q\}$. Then $I_j(M)$ is the $(p-j)$th Fitting ideal and, consequently, independent of the presentation of M, a fact which is used below several times. For $\mathbf{r} = (r_1, \dots, r_k)$ and $\mathbf{s} = (s_1, \dots, s_l)$, with r_i and s_i non-negative integers, we define ideals

$$\mathfrak{a}_{\mathbf{r},\mathbf{s}} := \sum_{i=1}^{k} I_{r_i+1}(\operatorname{Hom}_K(L, A)/AdF^i(A)) + \sum_{j=1}^{l} I_{s_j+1}(\operatorname{Hom}_K(Z_j(L), A)/\pi_j(AdA)),$$

$$\mathfrak{b}_{\mathbf{r},\mathbf{s}} := \prod_{i=1}^{k} I_{r_i}(\operatorname{Hom}_K(L, A)/AdF^i(A)) \prod_{j=1}^{l} I_{s_j}(\operatorname{Hom}_K(Z_j(L), A)/\pi_j(AdA)).$$

We define

$$U_{\mathbf{r},\mathbf{s}} := U_{\mathbf{r},\mathbf{s}}(F^\bullet, Z_\bullet) := V(\mathfrak{a}_{\mathbf{r},\mathbf{s}}) \cap D(\mathfrak{b}_{\mathbf{r},\mathbf{s}}) \subseteq \operatorname{Spec} A$$

to be the quasi-affine subscheme defined by the intersection of the open set $D(\mathfrak{b}_{\mathbf{r},\mathbf{s}})$ and the closed subset $V(\mathfrak{a}_{\mathbf{r},\mathbf{s}}) = \operatorname{Spec} A/\mathfrak{a}_{\mathbf{r},\mathbf{s}}$.

REMARK 4.5. We give an explicit description of the strata $U_{\mathbf{r},\mathbf{s}}$, using coordinates and vector fields. Let $x_1, \dots, x_n \in A$ and $\delta_1, \dots, \delta_m \in L$ satisfy the following properties:

(1) there are $\nu_1, \dots, \nu_k$, with $0 \leqslant \nu_1 \leqslant \dots \leqslant \nu_k = n$, such that $dx_1, \dots, dx_{\nu_i}$ generate, over A, the A-module $AdF^i(A)$;

(2) there are $\mu_0, \dots, \mu_l$, with $1 = \mu_0 \leqslant \mu_1 \leqslant \dots \leqslant \mu_l$, such that $\delta_{\mu_j}, \dots, \delta_m$ generate, over K, the K-vector space $Z_j(L)$.

Then $I_h(\mathrm{Hom}_K(L, A)/AdF^i(A))$ is the ideal generated by the h-minors of the matrix $(\delta_\alpha(x_\beta))_{\beta \leqslant \nu_i}$, and $I_h(\mathrm{Hom}_K(Z_j(L), A)/\pi_j(AdA))$ is the ideal generated by the h-minors of the matrix $(\delta_\alpha(x_\beta))_{\alpha \geqslant \mu_j}$.

Hence, for any point $t \in \mathrm{Spec}\, A$ we have

$$t \in U_{\mathbf{r},\mathbf{s}} \quad \Leftrightarrow \quad \begin{cases} \mathrm{rank}(\delta_\alpha(x_\beta)(t))_{\beta \leqslant \nu_i} = r_i & \text{for } i = 1, \dots, k, \\ \mathrm{rank}(\delta_\alpha(x_\beta)(t))_{\alpha \geqslant \mu_j} = s_j & \text{for } j = 1, \dots, l. \end{cases}$$

REMARK 4.6. Assume that $U_{\mathbf{r},\mathbf{s}} \neq \varnothing$, let $t \in U_{\mathbf{r},\mathbf{s}}$ and $\kappa(t)$ be its residue field. Then Lemma 4.2 implies that

$$\begin{aligned} r_i &= \dim_{\kappa(t)} AdF^i(A) \otimes_A \kappa(t), \quad \text{for } i = 1, \dots, k, \\ s_j &= \dim_{\kappa(t)} \pi_j(AdA) \otimes_A \kappa(t), \\ &= \text{orbit dimension of } Z_j(L) \text{ at } t, \quad \text{for } j = 1, \dots, l, \\ r_k &= \text{orbit dimension of } L \text{ at } t. \end{aligned}$$

This implies, in particular, that $0 \leqslant r_1 \leqslant r_2 \leqslant \dots \leqslant r_k$, $0 \leqslant s_l \leqslant s_{l-1} \leqslant \dots \leqslant s_1 \leqslant r_k$ and that the set $\{(\mathbf{r}, \mathbf{s}) \in \mathbb{Z}^k \times \mathbb{Z}^l \colon U_{\mathbf{r},\mathbf{s}} \neq \varnothing\}$ is finite.

THEOREM 4.7. *Let A be a noetherian K-algebra. Assume either that A is reduced or, with the above notation, that* Spec $A = \bigcup_{\mathbf{r},\mathbf{s}} U_{\mathbf{r},\mathbf{s}}$. *Then*

(1) *$\{U_{\mathbf{r},\mathbf{s}}\}$ is the flattening stratification of* Spec A *with respect to the modules* $\mathrm{Hom}_K(L, A)/AdF^i(A)$, *for $i = 1, \dots, k$, and* $\mathrm{Hom}_K(Z_j(L), A)/\pi_j(AdA)$ *for $j = 1, \dots, l$,*

(2) *$U_{\mathbf{r},\mathbf{s}}$ is invariant under the action of L,*

(3) *$U_{\mathbf{r},\mathbf{s}}$ admits a locally trivial geometric quotient with respect to the action of L.*

We call $U_{\mathbf{r},\mathbf{s}}$ the algorithmic stratification *of* Spec A *(with respect to the filtrations $F^\bullet$, $Z_\bullet$ and the action of L satisfying* (F) *and* (Z)).

Proof. (1) For fixed i, the locally closed subspaces

$$V(I_{r_i+1}(\mathrm{Hom}_K(L, A)/AdF^i(A))) \cap D(I_{r_i}(\mathrm{Hom}_K(L, A)/AdF^i(A))),$$

where $0 \leqslant r_i \leqslant r_k$, define the flattening stratification of Spec A with respect to $\mathrm{Hom}_K(L, A)/AdF^i(A)$, and a similar result holds for $\mathrm{Hom}(Z_j(L), A)/\pi_j(AdA)$ (cf. [**12**, Lecture 8]. Then $\{U_{\mathbf{r},\mathbf{s}}\}$ is just the intersection of these stratifications.

(2) It is sufficient to prove that the ideals $I_h(\mathrm{Hom}_K(L, A)/AdF^i(A))$ and $I_h(\mathrm{Hom}_K(Z_j(L), A)/\pi_j(AdA))$ are stable under the action of L. For arbitrary $\delta_1, \dots, \delta_r \in L$ and $x_1, \dots, x_r \in A$ let

$$\det(\delta_1, \dots, \delta_r, x_1, \dots, x_r) := \det(\delta_i(x_j)).$$

We need the following lemma.

LEMMA 4.8. *Let* $\delta, \delta_1, \dots, \delta_r \in L$, $x_1, \dots, x_r \in A$. *Then*

$$\begin{aligned}\delta(\det(\delta_1, \dots, \delta_r, x_1, \dots, x_r)) &= \sum_{\nu=1}^{r} \det(\delta_1, \dots, \delta_r, x_1, \dots, x_{\nu-1}, \delta(x_\nu), x_{\nu+1}, \dots, x_r) \\ &+ \sum_{\nu=1}^{r} \det(\delta_1, \dots, [\delta, \delta_\nu], \delta_{\nu+1}, \dots, \delta_r, x_1, \dots, x_r).\end{aligned}$$

Proof. We have

$$\begin{aligned}\delta(\det(\delta_1, \dots, \delta_r, x_1, \dots, x_r)) &= \sum_{\nu=1}^{r} \begin{vmatrix} \delta_1(x_1) & \dots & \delta\delta_1(x_\nu) & \dots & \delta_1(x_r) \\ \vdots & & \vdots & & \vdots \\ \delta_r(x_1) & & \delta\delta_r(x_\nu) & \dots & \delta_r(x_r) \end{vmatrix} \\ &= \sum_{\nu=1}^{r} \begin{vmatrix} \delta_1(x_1) & \dots & \delta_1\delta(x_\nu) + [\delta, \delta_1](x_\nu) & \dots & \delta_1(x_r) \\ \vdots & & \vdots & & \vdots \\ \delta_r(x_1) & \dots & \delta_r\delta(x_\nu) + [\delta, \delta_r](x_\nu) & \dots & \delta_r(x_r) \end{vmatrix} \\ &= \sum_{\nu=1}^{r} \det(\delta_1, \dots, \delta_r, x_1, \dots, \delta(x_\nu), \dots, x_r) \\ &+ \sum_{\nu=1}^{r} \begin{vmatrix} \delta_1(x_1) & \dots & [\delta, \delta_1](x_\nu) & \dots & \delta_1(x_r) \\ \vdots & & \vdots & & \vdots \\ \delta_r(x_1) & \dots & [\delta, \delta_r](x_\nu) & \dots & \delta_r(x_r) \end{vmatrix}.\end{aligned}$$

The latter sum can be developed into

$$\begin{aligned}&\sum_{\nu} \sum_{\mu} (-1)^{\nu+\mu}[\delta, \delta_\mu](x_\nu)\det(\delta_1, \dots, \hat{\delta}_\mu, \dots, \delta_r, x_1, \dots, \hat{x}_\nu, \dots, x_r) \\ &\qquad = \sum_{\mu} \det(\delta_1, \dots, [\delta, \delta_\mu], \dots, \delta_r, x_1, \dots, x_r),\end{aligned}$$

which yields the lemma.

Now let $\delta_1, \dots, \delta_h \in L$ (respectively $\delta_1, \dots, \delta_h \in Z_j(L)$) and $x_1, \dots, x_h \in F^i(A)$ (respectively $x_1, \dots, x_h \in A$). Then

$$\det(\delta_1, \dots, \delta_h, x_1, \dots, x_h) \in I_h(\mathrm{Hom}_K(L, A)/AdF^i(A))$$

(respectively $\det(\delta_1, \dots, \delta_h, x_1, \dots, x_h) \in I_h(\mathrm{Hom}_K(Z_j(L), A)/\pi_j(AdA)))$ and the ideal is generated by all such determinants. Note that $x_\nu \in F^i(A)$ implies $\delta(x_\nu) \in F^i(A)$ and $\delta_\nu \in Z_j(L)$ implies $[\delta, \delta_\nu] \in Z_j(L)$ by the properties of the filtrations.

Using the lemma we obtain the required invariance of the ideals above. To prove Theorem 4.7(3) we need the following notation.

DEFINITION 4.9. Let $\mathbf{r} = (r_1, \dots, r_k)$ and $\mathbf{s} = (s_1, \dots, s_l)$ be sequences of integers such that $0 \leqslant r_1 \leqslant \dots \leqslant r_k$, $0 \leqslant s_l \leqslant s_{l-1} \leqslant \dots \leqslant s_1 \leqslant r_k$, and let $\delta_1, \dots, \delta_{r_k} \in L$ and $x_1, \dots, x_{r_k} \in A$. The matrix $(\delta_i(x_j))$ is called $(\mathbf{r},\mathbf{s})$-*nested* (with respect to the filtrations $F^\bullet$ and $Z_\bullet$) if

(1) $x_1, \dots, x_{r_i} \in F^i(A)$ for $i = 1, \dots, k$,
(2) $\delta_{r_k - s_j + 1}, \dots, \delta_{r_k} \in Z_j(L)$ for $j = 1, \dots, l$.

We set

$$\mathcal{N}_{\mathbf{r},\mathbf{s}} := \{d \in A \mid d = \det(\delta_1, \dots, \delta_{r_k}, x_1, \dots, x_{r_k}), x_1, \dots, x_{r_k} \in A,\ \delta_1, \dots, \delta_{r_k} \in L \text{ and } (\delta_i(x_j)) \text{ is } (\mathbf{r}, \mathbf{s})\text{-nested}\}.$$

LEMMA 4.10. (i) *Let* $d \in \mathcal{N}_{\mathbf{r},\mathbf{s}}$. *Then* $\delta(d) \in \mathfrak{a}_{\mathbf{r},\mathbf{s}}$ *for all* $\delta \in L$.
(ii) *We have* $U_{\mathbf{r},\mathbf{s}} = V(\mathfrak{a}_{\mathbf{r},\mathbf{s}}) \cap (\bigcup_{d \in \mathcal{N}_{\mathbf{r},\mathbf{s}}} D(d))$.

Proof. To prove (i) let $d = \det(\delta_1, \dots, \delta_{r_k}, x_1, \dots, x_{r_k}) \in \mathcal{N}_{\mathbf{r},\mathbf{s}}$. Using Lemma 4.8 we obtain

$$\delta(d) = \sum_{\nu} \det(\delta_1, \dots, \delta_{r_k}, x_1, \dots, \delta(x_\nu), \dots, x_{r_k})$$
$$+ \sum_{\nu} \det(\delta_1, \dots, [\delta, \delta_\nu], \dots, \delta_{r_k}, x_1, \dots, x_{r_k}).$$

Let ν be fixed and choose i minimal such that $\nu \leqslant r_i$ and j maximal such that $r_k - s_{j-1} + 1 \leqslant \nu$. Using $d \in \mathcal{N}_{\mathbf{r},\mathbf{s}}$ and the properties of the filtrations we obtain $x_1, \dots, x_{r_{i-1}}, \delta(x_\nu) \in F^{i-1}(A)$ and $\delta_{r_k - s_j + 1}, \dots, \delta_{r_k}, [\delta, \delta_\nu] \in Z_j(L)$. This implies (by definition of $\mathfrak{a}_{\mathbf{r},\mathbf{s}}$) that

$$\det([\delta, \delta_\nu], \delta_{r_k - s_j + 1}, \dots, \delta_{r_k}, \bar{x}_1, \dots, \bar{x}_{s_j+1}) \in \mathfrak{a}_{\mathbf{r},\mathbf{s}}$$

for all $\bar{x}_1, \dots, \bar{x}_{s_j+1} \in A$ and

$$\det(\bar{\delta}_1, \dots, \bar{\delta}_{r_{i-1}+1}, x_1, \dots, x_{r_{i-1}}, \delta(x_\nu)) \in \mathfrak{a}_{\mathbf{r},\mathbf{s}}$$

for all $\bar{\delta}_1, \dots, \bar{\delta}_{r_{i-1}+1} \in L$. Consequently,

$$\det(\delta_1, \dots, \delta_{r_k}, x_1, \dots, \delta(x_\nu), \dots, x_{r_k}) \in \mathfrak{a}_{\mathbf{r},\mathbf{s}}$$

and

$$\det(\delta_1, \dots, [\delta, \delta_\nu] \dots, \delta_{r_k}, x_1, \dots, x_{r_k}) \in \mathfrak{a}_{\mathbf{r},\mathbf{s}}$$

for all ν which proves (i).

To prove (ii) we choose $x_1, \dots, x_n \in A$ and $\delta_1, \dots, \delta_m \in L$ as in Remark 4.5 and such that $x_1, \dots, x_{r_i} \in F^i(A)$ for $i = 1, \dots, k$. Let $t \in U_{\mathbf{r},\mathbf{s}}$; then $\operatorname{rank}(\delta_\alpha(x_\beta)(t))_{\beta \leqslant \nu_i} = r_i$ and $\operatorname{rank}(\delta_\alpha(x_\beta)(t))_{\alpha \geqslant \mu_j} = s_j$. This implies that there is a quadratic submatrix M of $(\delta_\alpha(x_\beta))$ which is $(\mathbf{r}, \mathbf{s})$-nested and $\det M(t) \neq 0$. This proves (ii) of the lemma.

To prove (3) of the theorem we choose any $d = \det(\delta_1, \dots, \delta_{r_k}, x_1, \dots, x_{r_k}) \in \mathcal{N}_{\mathbf{r},\mathbf{s}}$ such that $V(\mathfrak{a}_{\mathbf{r},\mathbf{s}}) \cap D(d) \neq \varnothing$. We want to apply Theorem 3.5 and Remark 3.6. Let $\bar{A} := (A/\mathfrak{a}_{\mathbf{r},\mathbf{s}})_{\bar{d}}$ and $\bar{x}_1, \dots, \bar{x}_{r_k} \in \bar{A}$ be the images of $x_1, \dots, x_{r_k}$. Let $\bar{L}$ be the image of L in $\operatorname{Der}_K \bar{A}$ under the induced representation which exists since $\mathfrak{a}_{\mathbf{r},\mathbf{s}}$ is L-invariant by the proof of Theorem 4.7(2). Since d is $\bar{L}$-invariant by Lemma 4.10, $\bar{L} \subseteq \operatorname{Der}_K^{\text{nil}} \bar{A}$ is nilpotent (and also finite-dimensional since L is so). Let $\bar{\delta}_1, \dots, \bar{\delta}_{r_k}$ be the images of $\delta_1, \dots, \delta_{r_k}$ in $\bar{L}$. We have to show that $\bar{L} \subset \sum_{\nu=1}^{r_k} \bar{A}\bar{d}_\nu$. Let $\bar{\delta} \in \bar{L}$; then, since $\det(\bar{\delta}_1, \dots, \bar{\delta}_{r_k}, \bar{x}_1, \dots, \bar{x}_{r_k})$ is a unit in $\bar{A}$, there are $\xi_i \in \bar{A}$ such that, for $\tilde{\delta} = \bar{\delta} - \sum_{i=1}^{r_k} \xi_i \bar{\delta}_i$, we have $\tilde{\delta}(\bar{x}_j) = 0$ for $j = 1, \dots, r_k$. Since $\det(\bar{\delta}_1, \dots, \bar{\delta}_{r_k}, \bar{\delta}, x_1, \dots, x_{r_k}, \bar{y}) = 0$ for all $\bar{\delta} \in \bar{L}$, $\bar{y} \in \bar{A}$ (definition of $\mathfrak{a}_{\mathbf{r},\mathbf{s}}$), we obtain $\tilde{\delta}(\bar{y}) = 0$ for all $\bar{y} \in \bar{A}$, that is, $\bar{\delta} = \sum_{i=1}^{r_i} \xi_i \bar{\delta}_i$.

Property (3′) of Remark 3.6,

$$\bar{\boldsymbol{\delta}}\bar{\delta}_j(\bar{x}_s) \in \sum_{\nu<s} \bar{A}\bar{\boldsymbol{\delta}}(\bar{x}_\nu), \quad \text{for } j,\ s = 1, \ldots, r_k,$$

means just (where $d\colon A \to \mathrm{Hom}_K(L, A)$ is the differential)

$$d\bar{\delta}_j(\bar{x}_s) \in \sum_{\nu<s} \bar{A}\, d\bar{x}_\nu.$$

This holds because of the fact that, for all $i = 1, \ldots, k$,

$$d\bar{x}_1, \ldots, d\bar{x}_{r_i} \text{ generate } AdF^i(A) \otimes_A \bar{A}$$

and if i is minimal such that $r_i \geq s$, then $\bar{\delta}_j(\bar{x}_s) \in \overline{F^{i-1}(A)}$, the image of $F^{i-1}(A)$ in $\bar{A}$. The fact that $d\bar{x}_1, \ldots, d\bar{x}_{r_i}$ generate $AdF^i(A) \otimes_A \bar{A}$ is a consequence of $\det(\bar{\delta}_1, \ldots, \bar{\delta}_{r_k}, \bar{x}_1, \ldots, \bar{x}_{r_k})$ being a unit in $\bar{A}$: let $a \in F^iA$ and $\bar{a}$ be the corresponding element in $\bar{A}$. By definition of $\mathfrak{a}_{\mathbf{r},s}$ and Remark 4.5, the $(r_i + 1)$-minors of the matrix

$$\begin{pmatrix} \bar{\delta}_1(\bar{x}_1) & \ldots & \bar{\delta}_1(\bar{x}_{r_i}) & \bar{\delta}_1(\bar{a}) \\ \vdots & & & \\ \bar{\delta}_{r_k}(\bar{x}_1) & \ldots & \bar{\delta}_{r_k}(\bar{x}_{r_i}) & \bar{\delta}_{r_k}(\bar{a}) \end{pmatrix}$$

vanish. This implies that, in any open set $D(M)$ of $\mathrm{Spec}\,\bar{A}$, defined by an r_i-minor M of $(\bar{\delta}_\alpha(\bar{x}_\beta))_{\beta \leq r_i}$, the last column of the matrix above is a linear combination of the first r_i columns, that is, $d\bar{a} \in \sum_{\nu=1}^{r_i} \bar{A}_M d\bar{x}_\nu$. The r_i-minors of the matrix $(\bar{\delta}_\alpha(\bar{x}_\beta))_{\beta \leq r_i}$ define an open covering of $\mathrm{Spec}\,\bar{A}$, and consequently $\bar{A}dF^i(A) = \sum_{\nu=1}^{r_i} \bar{A}d\bar{x}_\nu$. Now we may apply Theorem 3.5 and obtain the result that $\bar{A}^L[\bar{x}_1, \ldots, \bar{x}_{r_k}] = \bar{A}$, which proves the theorem.

Improvement of the stratification for abelian L

In the case of L being abelian we use the trivial filtration of L given by $L = Z_0(L) \supseteq Z_1(L) = 0$ and the notation $U_{\mathbf{r}}(F^\bullet)$ for the algorithmic stratification. If $L^{(0)} \oplus L^{(1)}$ is abelian and $L^{(0)}$ admits an algorithmic stratification with respect to some filtration $F^\bullet$, and if $L^{(1)}$ acts freely and satisfies the hypothesis of Theorem 3.10, we can combine Theorem 4.7 and Theorem 3.10 and obtain a geometric quotient by L on the strata of the algorithmic stratification with respect to $F^\bullet$ and $L^{(0)}$. Even if, in the situation above, L admits an algorithmic stratification, it can be useful to split $L = L^{(0)} \oplus L^{(1)}$ because the stratification with respect to $L^{(0)}$ may have bigger strata.

We will now analyse this situation in the graded case. Let $A = \bigoplus_{\nu \geq 0} A_\nu$, with $A_0 = K$, be a reduced noetherian graded K-algebra. Let $F^i(A) = \bigoplus_{\nu \leq n_i} A_\nu$, with $0 \leq n_0 < n_1 < \ldots$, for $i = 0, 1, \ldots$, be a filtration of A. Let $L = L^{(0)} \oplus L^{(1)} \subseteq \mathrm{Der}_K^{\mathrm{nil}}(A)$ be a finite-dimensional abelian Lie algebra satisfying the following properties:

(1) $\delta F^i(A) \subseteq F^{i-1}(A)$ for all $\delta \in L^{(0)}$,

(2) $H^1(L^{(1)}, A^{L^{(0)}}) = 0$,

(3) $L^{(i)} := \bigoplus_{\nu<0} L_\nu^{(i)}$ is graded $(i = 0, 1)$ and $\delta \in L_\nu^{(i)}$, $a \in A_\mu$ imply that $\delta(a) \in A_{\mu+\nu}$.

Let $\{U_{\mathbf{r}}(F^\bullet(A))\}$ be the algorithmic stratification of $\mathrm{Spec}\,A$ with respect to the action of $L^{(0)}$ and the filtration $F^\bullet(A)$, and let $\{U_{\mathbf{r}}(F^\bullet(A) \cap A^{L^{(1)}})\}$ be the algorithmic stratification of $\operatorname{Spec} A^{L^{(1)}}$ with respect to the induced action of $L^{(0)}$ on $A^{L^{(1)}}$ and the induced filtration $F^{\bullet}(A) \cap A^{L^{(1)}}$.

PROPOSITION 4.11. *Let π: $\operatorname{Spec} A \to \operatorname{Spec} A^{L^{(1)}}$ be the canonical morphism. Then $\pi^{-1}(U_r(F^{\bullet}(A) \cap A^{L^{(1)}})) = U_r(F^{\bullet}(A))$ and $U_r(F^{\bullet}(A))$ is invariant under the action of L.*

Proof. Let $\delta_1, \ldots, \delta_r$ be a homogeneous basis of $L^{(1)}$. By Remark 3.4, there are $x_1, \ldots, x_r \in A^{L^{(0)}}$ such that $\delta_i(x_j) = \delta^j_i$, especially $A^{L^{(1)}}[x_1, \ldots, x_r] = A$. The elements $x_1, \ldots, x_r$ can be chosen to be homogeneous. This implies that $F^i(A)$ is the K-vector space generated by all elements of the form $g x_1^{\nu_1} \cdot \ldots \cdot x_r^{\nu_r}$, with $g \in A^{L^{(1)}}$ homogeneous, and $\deg(g) + \sum_{j=1}^{r} \nu_j \deg x_j \leq n_i$. Let d: $A \to \operatorname{Hom}_K(L^{(0)}, A)$ be the differential. Then $AdF^i(A)$ is generated by $\{dg: g \in A^{L^{(1)}} \cap F^i(A)\}$, that is, $AdF^i(A) = Ad(F^i(A) \cap A^{L^{(1)}})$. This holds because $x_i \in A^{L^{(0)}}$ implies $dx_i = 0$. Now $[L^{(0)}, L^{(1)}] = 0$ implies that, for $g \in A^{L^{(1)}}$, $\operatorname{Im}(dg) \subseteq A^{L^{(1)}}$, that is, $A^{L^{(1)}} d(F^i(A) \cap A^{L^{(1)}}) \subseteq \operatorname{Hom}_K(L^{(0)}, A^{L^{(1)}})$. Now A is a faithfully flat $A^{L^{(1)}}$-algebra, that is, the flattening stratification of $\operatorname{Hom}_K(L^{(0)}, A)/Ad(F^i(A) \cap A^{L^{(1)}})$ is the flattening stratification induced via π: $\operatorname{Spec} A \to \operatorname{Spec} A^{L^{(1)}}$ by the flattening stratification of $\operatorname{Hom}_K(L^{(0)}, A^{L^{(1)}})/A^{L^{(1)}} d(F^i(A) \cap A^{L^{(1)}})$. This completes the proof by definition of $U_r(F^{\bullet}(A) \cap A^{L^{(1)}})$ and $U_r(F^{\bullet}(A))$.

COROLLARY 4.12. *The geometric quotient $U_r(F^{\bullet}(A))/L$* exists.

Proof. The morphism π: $\operatorname{Spec} A \to \operatorname{Spec} A^{L^{(1)}}$ is a geometric quotient of $\operatorname{Spec} A$ under the action of $L^{(1)}$ by Theorem 3.10. The Lie algebra $L^{(0)} \cong L/L^{(1)}$ acts on $\operatorname{Spec} A^{L^{(1)}}$ and, by Theorem 4.7, the geometric quotient $U_r(F^{\bullet}(A) \cap A^{L^{(1)}})/L^{(0)}$ exists.

The usefulness of Proposition 4.11 and Corollary 4.12 lies in the fact that this stratification obtained from $L^{(0)}$ leads to bigger strata on which the quotient by $L = L^{(0)} \oplus L^{(1)}$ exists than the algorithmic stratification obtained from L. We do not need any knowledge of $A^{L^{(1)}}$ since the strata are computed from the same matrix $(\delta_i(x_j))$ just by larger subminors (cf. Remark 4.5).

REMARKS 4.13. (1) Let $Z_i(L)$ be either the lower or the upper central series of L and $F^i_m(A)$ defined by

$$F^0_m(A) := A^L, \quad F^i_m(A) := \int F^{i-1}_m$$

(as in Remark 4.3). These are canonical filtrations which satisfy the properties (F) and (Z). In particular, algorithmic stratifications do exist, but the strata are in general smaller than for the canonical stratification introduced in § 1 (cf. the following Example (5.3)). On the other hand, for a given classification problem there might be a filtration which leads to strata that are more natural with respect to the objects to be classified although the quotient exists on bigger strata. This is, for example, the case for the classification of modules in **[8]**. Note also that we need only a little information about the $F^{\bullet}(A)$ (respectively $AdF^i(A)$) in order to compute the strata (see examples below).

(2) Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a noetherian graded K-algebra and $L = \bigoplus_{\nu \leq -a} L_\nu$ a finite-dimensional graded Lie algebra, and let $a > 0$ be such that

$$[L_\nu, L_\mu] \subseteq L_{\nu+\mu} \quad \text{and} \quad \delta(A_\mu) \subseteq A_{\mu+\nu} \quad \text{for } \delta \in L_\nu.$$

Fix some integer b, with $1 \leq b \leq a$. Then the graded filtrations defined by

$$F_g^i(A) = \bigoplus_{\nu \leq (i+1)a-b} A_\nu, \quad Z_j(L) = \bigoplus_{\nu \leq -(j+1)a} L_\nu,$$

have the properties (F) and (Z).

(2′) Let A and L be as in (2) and assume that L is abelian. Fix $1 \leq b \leq a$. Then the graded filtrations defined by

$$F_g^i(A) = \bigoplus_{\nu \leq (i+1)a-b} A_\nu, \quad L = Z_0(L) \supset Z_1(L) = 0,$$

have the properties (F) and (Z).

(3) Let L be abelian and $L = Z_0(L) \supset Z_1(L) = 0$ and $F_m^i(A)$ be defined as in (1). Theorem 1.30 implies that if $U_{\mathbf{r},\mathbf{s}} =: U_\mathbf{r}$ $(l = 0)$ is open in $\operatorname{Spec} A$, then $r_1 = r_2 = \ldots = r_k$, since we can cover $U_\mathbf{r}$ by affine open subsets such that $AdA = Ad \int A^L$.

(4) Let A be a reduced and noetherian K-algebra and $B \subseteq A$ a subalgebra. Let $L \subseteq \operatorname{Der}_B^{\mathrm{nil}} A$ be a nilpotent Lie algebra. Suppose that A and L have filtrations $F^\bullet$ and $Z_\bullet$ satisfying (F) and (Z) and that $Z_i(L)/Z_j(L)$ are free B-modules of finite rank, with $i = 0, \ldots, l$ and $j = 1, \ldots, l+1$. Then Theorem 4.7 also holds for this situation.

Definition 4.14. The filtration $\{F_m^i(A)\}$ from (1) is called the *maximal filtration* of A (since it is maximal with respect to property (1)), and $\{F_g^i(A)\}$ from (2) is called the *graded filtration*.

Remark 4.15. It follows from Remark 4.5 (and will also show up in the following examples) that the filtration $AdF^\bullet A$ of AdA (and not so much $F^\bullet A$) is essential for the construction of the algorithmic stratification. Different F^iA may lead to the same AdF^iA, in particular, $AdF^\bullet A$ is a finite filtration while $F^\bullet A$ is usually not. Therefore, it might be useful to give a different interpretation of AdA. The differential

$$d\colon A \to \operatorname{Hom}_K(L, A), \quad da(\delta) = \delta(a),$$

factorizes as follows:

$$A \xrightarrow{d_0} \Omega^1_{A/K} \longrightarrow (\Omega^1_{A/K})^{**} = (\operatorname{Der}_K(A))^* \longrightarrow (A \otimes_K L)^* = \operatorname{Hom}_K(L, A),$$

where $\Omega^1_{A/K}$ are the Kähler differentials, $-^* = \operatorname{Hom}_A(-, A)$ and the maps are the canonical ones. Since all maps except d_0 are A-linear, AdA is just the image of $\Omega^1_{A/K}$ in the free A-module $\operatorname{Hom}_K(L, A)$. The condition (F) then reads as $dF^iA \subset \operatorname{Hom}_K(L, F^{i-1}A)$.

Remark 4.16. If $A = K[X_1, \ldots, X_N]/I = K[x_1, \ldots, x_n]$ and if $\delta_1, \ldots, \delta_m$ is a basis of L, then $AdA \subset \operatorname{Hom}_K(L, A) = A^m$ is generated by the columns of the matrix $(\delta_\alpha(x_\beta)_{\alpha=1,\ldots,m,\beta=1,\ldots,n})$. Hence, $dx_1, \ldots, dx_{\nu_i}$ generate $AdF^i(A)$ if and only if the image of $AdF^i(A)$ in A^m is generated by the columns of $(\delta_\alpha(x_\beta))$ with index

$\beta \leq \nu_i$. For $t \in X = \operatorname{Spec} A$, let $L \to T_t X$, $\delta_i \mapsto \sum_j \delta_i(x_j)(t)\partial/\partial X_i|_t$, be the differential of the orbit map $G \times_K \operatorname{Spec} \kappa(t) \to T_t X$, where $G = \exp(L)$. Hence, if we consider $T_t X$ as a subvector space of $\kappa(t)^n$, the rows of $(\delta_\alpha(x_\beta)(t))$ generate the tangent space to the G-orbit of t (cf. also Lemma 4.2). Moreover, the rows with index $\alpha \geq \mu_j$ generate the tangent space to the G_j-orbit of t if $\delta_{\mu_j}, \ldots, \delta_m$ form a basis of $Z_j(L)$ and $G_j = \exp Z_j(L)$.

5. *Examples (continuation)*

Finally let us compute the stratifications for some examples.

5.1. In Example 2.1, $A = K[x_1, \ldots, x_{2r}]$ is graded by $\deg x_i = 1$ if $i \leq r$, and by $\deg x_i = 2$ if $i > r$. Then all vector fields of L are homogeneous of degree -1, that is, $L = L_{-1}$. We get $F_g^0(A) = K$, $x_1, \ldots, x_r \in F_g^1(A)$ and $x_{r+1}, \ldots, x_{2r} \in F_g^2(A)$. This implies that $dF_g^0(A) = dF_g^1(A) = 0$ and $AdA = AdF_g^2(A) = \sum_{v=1}^r Adx_{r+v}$. Therefore, we have only the strata $U_{\mathbf{r}}$, where

$$\mathbf{r} = (0, 0, r_2), \quad r_2 = \operatorname{rk}(\delta_i(x_j)\,|_{j=r+1,\ldots,2r}^{i=4,\ldots,r});$$

that is, the algorithmic stratification with respect to $F_g^\bullet$ is the stratification by orbit dimension. This is also the canonical stratification. If we consider the maximal filtration, we obtain $F_m^0(A) = A^L$, $x_{r+1}, \ldots, x_{2r} \in F_m^1(A)$. This implies that $F_m^i(A) = F_g^{i+1}(A)$ and we obtain the same stratification.

5.2. In Example 2.2, $A = K[x_1, x_2, x_3]$ is graded by $\deg x_i = i$ and $\deg \delta = -1$. For the graded filtration we obtain $x_1 \in F_g^1(A)$, $x_2 \in F_g^2(A)$, $x_3 \in F_g^3(A)$ and $dF_g^1(A) = 0$, $AdF_g^2(A) = Adx_2$ and $AdF_g^3(A) = AdA = Adx_2 + Adx_3$. This implies that if $U_{\mathbf{r},\mathbf{s}} = U_{\mathbf{r}} \neq \varnothing$, $\mathbf{r} = (r_1, r_2, r_3)$, then $\mathbf{r} \in \{(0, 1, 1), (0, 0, 1), (0, 0, 0)\}$. Now

$$\begin{aligned} U_{(0,1,1)} &= D(x_1), \\ U_{(0,0,1)} &= V(x_1) \cap D(x_2), \\ U_{(0,0,0)} &= V(x_1, x_2). \end{aligned}$$

This is exactly the canonical stratification we have already constructed in Example 2.2. If we consider the maximal filtration, we obtain $F_m^0(A) = A^L$, $x_2 \in F_m^1(A)$ and $x_3 \in F_m^2(A)$; hence $AdA = AdF^2(A) = Adx_2 + Adx_3$. This implies that if $U_{\mathbf{r}} \neq \varnothing$ for some $r = (r_1, r_2)$, then $\mathbf{r} \in \{(1, 1), (0, 1), (0, 0)\}$, which leads to the same stratification as before.

5.3. We give an example which shows that the graded and the maximal filtration define different stratifications. Let

$$A = K[x_1, x_2, x_3, x_4] \quad \text{and} \quad L = K\delta$$

with

$$\delta = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + (2x_1x_3 - x_2^2)\frac{\partial}{\partial x_4}.$$

Then A is graded by $\deg x_i = i$ for $i \leq 3$, and $\deg x_4 = 5$. Hence $L = L_{-1}$, $F_g^0(A) = K$, $x_i \in F_g^i(A)$ for $i \leq 3$, and $x_4 \in F_g^5(A)$. This implies that $dF_g^i(A) = 0$ for $i = 0, 1$, $AdF_g^2(A) = Adx_2$, $AdF_g^3(A) = Adx_2 + Adx_3$, $AdF_g^4(A) = AdF_g^3(A)$ and $AdA = AdF_g^5(A) = \sum_{i=2}^4 Adx_i$. Hence, $U_{\mathbf{r}}(F_g^\bullet) \neq \varnothing$ implies that

$$\mathbf{r} = (r_1, \ldots, r_5) \in \{(0, 1, 1, 1, 1), (0, 0, 1, 1, 1), (0, 0, 0, 0, 1), (0, 0, 0, 0, 0)\}.$$

Because $\delta(x_4)=2x_1x_3-x_2^2$, we have $U_{(0,0,0,0,1)}=\varnothing$. Therefore, the graded stratification is given by

$$U_{(0,1,1,1,1)}=D(x_1),\quad U_{(0,0,1,1,1)}=V(x_1)\cap D(x_2),\quad U_{(0,0,0,0,0)}=V(x_1,x_2).$$

If we consider the maximal filtration, we obtain $x_1\in F_m^0(A)=A^L$, x_2, $x_4\in F_m^1(A)$, $x_3\in F_m^2(A)$. This implies that $AdA=AdF_m^2(A)$, and $dx_2, dx_4\in AdF_m^1(A)$ and $U_{\mathbf{r}}(F_m^\bullet)\neq\varnothing$ implies that $\mathbf{r}=(r_1,r_2)\in\{(1,1),(0,1),(0,0)\}$. Then $U_{(1,1)}=D(x_1,2x_1x_3-x_2^2)=D(x_1,x_2)$, which is the set of all points with orbit dimension 1, $U_{(0,1)}=\varnothing$, and $U_{(0,0)}=V(x_1,x_2)$. This is also the canonical stratification.

5.4. In Example 2.4, $A=K[x_1,\dots,x_5]$ is graded by $\deg x_i=i$ for $i\leqslant 3$, $\deg x_4=\deg x_5=5$. Then $L_{-1}=K\delta_1$, $L_{-2}=0$, $L_{-3}=K\delta_2$ and $L_{-4}=K\delta_3$. For the graded filtration we obtain $Z_0(L)=L$, $Z_1(L)=Z_2(L)=K\delta_2+K\delta_3$, $Z_3(L)=K\delta_3$, $Z_4(L)=0$. Also $F_g^0(A)=K$, $x_i\in F_g^i(A)$ for $i\leqslant 3$, x_4, $x_5\in F_g^5(A)$. This implies $AdA=AdF_g^5(A)$, $dF_g^i(A)=0$ for $i=0$, 1, $AdF_g^4(A)=AdF_g^3(A)=Adx_2+Adx_3$, $AdF_g^2(A)=Adx_2$.

If $U_{\mathbf{r},\mathbf{s}}(F_g^\bullet, Z_\bullet)\neq\varnothing$, then

$$\mathbf{r}=(r_1,\dots,r_5)\in\{(0,1,1,1,2),(0,0,1,1,2),(0,0,0,0,0)\}$$

and

$$\mathbf{s}=(s_1,s_2,s_3)\in\{(1,1,1),(0,0,0)\}.$$

We obtain for $U_{\mathbf{r},\mathbf{s}}\neq\varnothing$:

$$U_{(0,1,1,1,2),(1,1,1)}=D(x_1),\quad U_{(0,0,1,1,2)(1,1,1)}=V(x_1)\cap D(x_2),\quad U_{(\mathbf{0},\mathbf{0})}=V(x_1,x_2).$$

If we consider the maximal filtration, we obtain $Z_0(L)=L$, $Z_1(L)=K\delta_3$, $Z_2(L)=0$ and $F_m^0(A)=A^L$, x_1, x_2, $x_4\in F_m^1(A)$ and x_3, $x_5\in F_m^2(A)$, that is, $AdA=AdF_m^2(A)$. If $U_{\mathbf{r},\mathbf{s}}(F^\bullet,Z_\bullet)\neq\varnothing$, then $\mathbf{r}=(r_1,r_2)\in\{(1,2),(0,0)\}$, $\mathbf{s}=s_1\in\{0,1\}$. We obtain $U_{(1,2),1}=D(x_1)$, $U_{(1,2),0}=V(x_1)\cap D(x_2)$, $U_{(0,0),0}=V(x_1,x_2)$. This is the same stratification as before.

Example 2.4 shows that the stratification is not optimal.

5.5. If we take, in Example 2.5, the maximal filtration, we obtain for $A=K[x_1,\dots,x_5]$ that $AdA=AdF_m^1(A)=\sum_{\nu=3}^5 Adx_\nu$ is locally free of rank 1, that is, $\mathbf{r}=r_1=1$ is the only possibility and, consequently, $U_1=\operatorname{Spec}A$.

5.6. If we take, in Example 2.6, the maximal filtration, we obtain x_1, $x_2\in F_m^0(A)=A^L$, $x_3\in F_m^1(A)$, $x_4\in F_m^2(A)$, that is, $AdA=AdF_m^2(A)\supseteq AdF_m^1(A)\ni dx_3$. Since L has constant orbit-dimension on $\operatorname{Spec}A$, this implies that the stratification is given by $U_{(1,1)}$ and $U_{(0,1)}$, $U_{(1,1)}=D(x_1)$ and $U_{(0,1)}=V(x_1)$, which is also the canonical stratification.

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