A CLASSIFIER FOR SIMPLE SPACE CURVE SINGULARITIES

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Abstract. The classification of Bruce and Gaffney resp. Gibson and Hobbs for simple plane curve singularities resp. simple space curve singularities is characterized in terms of invariants. This a basis for the implementation of a classifier in the computer algebra system SINGULAR.

1. Introduction

The germ of a space curve is given by a germ of an analytic map $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$. Simple singularities of curves have been classified by Bruce and Gaffney in the case $n = 2$ and Gibson and Hobbs for the case $n = 3$. We will describe the implementation of a classifier in SINGULAR for simple curve singularities in case $n \leq 3$.
Instead of considering the germ $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ we may as well consider the corresponding $\mathbb{C}$-algebra morphism $f_* : \mathbb{C}[[x_1, .., x_n]] \to \mathbb{C}[[t]]$.
Let $A_n := \{f : \mathbb{C}[[x_1, .., x_n]] \to \mathbb{C}[[t]] \mid dim_{\mathbb{C}}\, \mathbb{C}[[t]] \,/\, Im(f) < \infty\}$. The group $\mathcal{A}_n := Aut_{\mathbb{C}}\mathbb{C}[[x_1, .., x_n]] \times Aut_{\mathbb{C}}\, \mathbb{C}[[t]]$ acts on A_n by $(\phi, \psi)(f) = \psi \circ f \circ \phi^{-1}$.

Definition 1.1. f is called $\mathcal{A}$-equivalent to g if f, g are in the same orbit of $\mathcal{A}_n$. Since a $\mathbb{C}$-algebra morphism $f : \mathbb{C}[[x_1, .., x_n]] \to \mathbb{C}[[t]]$ is determined by $f(x_i) := x_i(t)$ we may identify
$A_n = \{(x_1(t), x_2(t), .., x_n(t)) \in \mathbb{C}[[t]]^n | dim_{\mathbb{C}}\, \mathbb{C}[[t]] \,/\, \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]] < \infty\}$.
In this terminology $(x_1(t), x_2(t), .., x_n(t))$, $(y_1(t), y_2(t), .., y_n(t)) \in A_n$ are $\mathcal{A}$-equivalent iff there exist $H_1, H_2, .., H_n \in \langle Y_1, .., Y_n\rangle\, \mathbb{C}[[Y_1, ..Y_n]]$, $det\,(\frac{\partial H_i}{\partial Y_j}(0)) \neq 0$, and a unit $u \in \mathbb{C}[[t]]$ such that $x_i(t) = H_i(\, y_1(t\, u(t)), .., y_n(t\, u(t))\,)$ for all i. $A_n \subseteq \mathbb{C}[[t]]^n$ is equipped in a canonical way with a topology induced by the classical topology of the affine spaces $(\mathbb{C}[[t]] \,/\, t^N)^n$. It is the coarsest topology of $\mathbb{C}[[t]]^n$ such that the canonical maps $\mathbb{C}[[t]]^n \to (\mathbb{C}[[t]] \,/\, t^N)^n$ are continuous for all N.

Definition 1.2. $f \in A_n$ is called to be $\mathcal{A}$-simple, if there exist a neighbourhood $\mathcal{U} \subseteq A_n$ of f such that $\mathcal{U}$ contains only finitely many orbits of $\mathcal{A}_n$.

The following tables give the results of the classification of Gibson and Hobbs respectively Bruce and Gaffney.

Normal Form		Generators of the Semi-group
(t)		1
(t^2, t^{2k+1})		$2, 2k+1$
(t^3, t^{3k+1})		$3, 3k+1$
(t^3, t^{3k+2})		$3, 3k+2$
$(t^3, t^{3k+1}+t^{3p+2})$	$k \geq 2, k \leq p < 2k$	$3, 3k+1$
$(t^3, t^{3k+2}+t^{3p+1})$	$k \geq 2, k < p \leq 2k$	$3, 3k+2$
(t^4, t^5)		$4, 5$
(t^4, t^5+t^7)		$4, 5$
(t^4, t^6+t^{2k+1})	$k \geq 3$	$4, 6, 2k+7$
(t^4, t^7)		$4, 7$
(t^4, t^7+t^9)		$4, 7$
(t^4, t^7+t^{10})		$4, 7$

Normal Form		Generators of the Semi-group
$(t^3, t^{3k+1}, t^{3n+2})$	$k \leq n < 2k$	$3, 3k+1, 3n+2$
$(t^3, t^{3k+2}, t^{3n+1})$	$k < n \leq 2k$	$3, 3k+2, 3n+1$
$(t^3, t^{3k+1}+t^{3p+2}, t^{3n+2})$	$k \leq p < n < 2k$	$3, 3k+1, 3n+2$
$(t^3, t^{3k+2}+t^{3p+1}, t^{3n+1})$	$k < p < n \leq 2k$	$3, 3k+2, 3n+1$
(t^4, t^5, t^6)		$4, 5, 6$
(t^4, t^5, t^7)		$4, 5, 7$
(t^4, t^5, t^{11})		$4, 5, 11$
(t^4, t^5+t^7, t^{11})		$4, 5, 11$
(t^4, t^6, t^{2k+1})	$k \geq 3$	$4, 6, 2k+1$
$(t^4, t^6+t^{2k-1}, t^{2k+1})$	$k \geq 4$	$4, 6, 2k+1$
$(t^4, t^6+t^{2k-3}, t^{2k+1})$	$k \geq 5$	$4, 6, 2k+1, 2k+3$
$(t^4, t^6+t^{2k-7}, t^{2k+1})$	$k \geq 7$	$4, 6, 2k-1, 2k+1$
(t^4, t^7, t^9+t^{10})		$4, 7, 9$
(t^4, t^7, t^9)		$4, 7, 9$
(t^4, t^7, t^{10})		$4, 7, 10$
(t^4, t^7, t^{13})		$4, 7, 13$
(t^4, t^7, t^{17})		$4, 7, 17$
(t^4, t^7+t^9, t^{10})		$4, 7, 10$
(t^4, t^7+t^9, t^{13})		$4, 7, 13$
(t^4, t^7+t^9, t^{17})		$4, 7, 17$
$(t^4, t^7+t^{10}, t^{17})$		$4, 7, 17$

The aim of this paper is to describe the implementation of a classifier of simple space curve singularities for $n \leq 3$ in SINGULAR. The new investigation is that we do not compute the normal form of a given singularity (see tables above) because this would be very time consuming. We give a characterization of the different types of singularities in terms of certain invariants and use this characterization to identify the singularities.

2. Invariants: Semigroup of the curve and its differential module

In this section we will recall the invariants we need to set up an efficient classifier.

Definition 2.1. Let $(x_1(t), x_2(t), .., x_n(t)) \in A_n$.

(1) The δ-invariant of the corresponding algebra $\mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$ is $\delta := dim_{\mathbb{C}}\mathbb{C}[[t]] \,/\, \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$.

(2) Let $t^c\mathbb{C}[[t]] = Ann_{\mathbb{C}[[t]]}(\mathbb{C}[[t]] \,/\, \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]])$ then c is called the conductor of $\mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$. Note that $t^c\mathbb{C}[[t]]$ is also an ideal in $\mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$ the conductor ideal.

(3) $\Gamma := \{ord_t(f) | f \in \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]\}$ is called the semi-group of $\mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$.

The semi-group will play an important role in identifying the singularity. The semi-group can be computed using sagbi bases, this we will explain in the next section.

Definition 2.2. Let $A = \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]]$ be a subalgebra of $\mathbb{C}[[t]]$. The A-module of the Kähler differentials denoted by Ω is defined as the A-module generated by $\{\frac{da}{dt} | a \in A\}$. It is easy to see that $\Omega = \langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, .., \frac{dx_n}{dt} \rangle_A$.

Similarly, to the semi-group we define the semi-module of the differential module $\Omega \subset \mathbb{C}[[t]]$ as $\Gamma_\Omega = \{ord(\gamma) | \gamma \in \Omega\}$.

3. SAGBI BASES:THE SPECIAL CASE $\mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]] \subseteq \mathbb{C}[[t]]$

In this part we would like to recall the notion of a Sagbi basis for the subalgebra A of $\mathbb{C}[[t]]$ and A-modules $M \subseteq \mathbb{C}[[t]]$. Details can be found in the paper of Hefez and Hernandes [HH]. For a power series $f = \sum_{v \geq m} a_v t^v \in \mathbb{C}[[t]]$, $a_m \neq 0$, we will denote by $LT(f) = a_m t^m$, $LM(f) = t^m$ the leading term resp. the leading monomial.

Definition 3.1. Let $A \subseteq C[[t]]$ be a subalgebra. $G \subseteq A$ is called sagbi basis of A if for any $f \in A$, $f \neq 0$, there exist $g_1, g_2, .., g_s \in G$ and $H \in \mathbb{C}[Y_1, .., Y_s]$ such that $LM(f) = H(LM(g_1), LM(g_2), .., LM(g_s))$.

Definition 3.2. The Sagbi basis $G = \{g_1, g_2, \cdots, g_s\}$ is called to be reduced, if the coefficient of the leading terms is 1 and it has the following properties:

(1) $LM(g_i) \notin \mathbb{C}[LM(g_1), LM(g_2), .., LM(g_{i-1}), LM(g_{i+1}), .., LM(g_s)]$ for all i.
(2) Let $m \neq LM(g_i)$ be a monomial of g_i then $m \notin \mathbb{C}[LM(g_1), .., LM(g_s)]$.

Note that in case of $\dim_{\mathbb{C}} \mathbb{C}[[t]] \,/\, A < \infty$ there exist a reduced Sagbi basis.

Definition 3.3. let $A = \mathbb{C}[[x_1(t), x_2(t), .., x_n(t)]] \subseteq \mathbb{C}[[t]]$ be a subalgebra such that $\dim_{\mathbb{C}} \mathbb{C}[[t]] \,/\, A < \infty$. Let $G = \{g_1, g_2, , \cdots, g_s\}$ be a Sagbi basis of A. Let $M \subseteq \mathbb{C}[[t]]$ be an A-module. $H \subseteq M$ is called a G-standard basis, if for every $m \in M$, $m \neq 0$, there exist $h \in H$ and $Q \in \mathbb{C}[Y_1, .., Y_s]$ such that $LT(m) = LT(h).Q(LT(g_1), LT(g_2), .., LT(g_s))$.

Remark 3.4. Sagbi bases for subalgebras $A \subseteq \mathbb{C}[[t]]$ with $dim_{\mathbb{C}}\ \mathbb{C}[[t]]/\,A < \infty$ and standard bases of A-module M have been implemented in SINGULAR [DGPS].

Proposition 3.5. *Let $A \subseteq \mathbb{C}[[t]]$ be a subalgebra such that $dim_{\mathbb{C}}\ \mathbb{C}[[t]]/\,A < \infty$ and $M \subseteq \mathbb{C}[[t]]$ be an A-module. Let $G = \{g_1, g_2, .., g_s\}$ be a sagbi bases of A and $H = \{h_1, h_2, .., h_k\}$ be a G-standard basis of M. Then $\{ord_t g_1, ord_t g_2, .., ord_t g_s\}$ resp. $\{ord_t h_1, ord_t g_2, .., ord_t h_k\}$ generate the semi-group of A resp. the semi-module of M.*

For a proof of the proposition cf. [HH].

4. Classifying the Singularities Using the Invariants

Proposition 4.1. *The polynomials in the normal form of the space curves given in the lists above are already reduced sagbi bases of the subalgebra of* $\mathbb{C}[[t]]$ *generated by them except in the following cases where one additional element is needed:*

(1) $t^4, t^6 + t^{2k+1}$, $k \geq 3$, *we need additionally* t^{2k+7}
(2) $t^4, t^6 + t^{2k-3}, t^{2k+1}$, $k \geq 5$, *we need additionally* t^{2k+3}
(3) $t^4, t^6 + t^{2k-7}, t^{2k+1}$, $k \geq 7$, *we need additionally* t^{2k-1}

Proof. We will give the proof for the case $(t^3, t^{3k+2} + t^{3p+1})$, $k \geq 2, k < p \leq 2k$ The proof for the other non-exceptional cases is similar. Let $G = \{g_1, g_2\} = \{t^3, t^{3k+2} + t^{3p+1}\} \subseteq A = \mathbb{C}[[t^3, t^{3k+2} + t^{3p+1}]]$. We consider t^3 , t^{3k+2} being the leading terms of g_1, g_2. The leading exponent of the g_1 and g_2 have greatest common divisor equal to 1. We have to consider a polynomial H as $H(LT(g_1), LT(g_2)) = 0$. Obviously it is enough to consider $H(Y_1, Y_2) = Y_1^{3k+2} - Y_2^3$.

$$\begin{aligned} H(g_1, g_2) &= (t^3)^{3k+2} - (t^{3k+2} + t^{3p+1})^3 \\ &= -3t^{6k+3p+5} - 3t^{6p+3k+4} - t^{9p+3} \\ &= t^3[-3(t^3)^{k+p}[t^{3k+2} + t^{3p+1}] - (t^3)^{3p}] \\ &= g_1(-3(g_1)^{k+p} g_2 - (g_1)^{3p}). \end{aligned}$$

This implies that the normal form of $H(g_1, g_2)$ with respect to G is zero. There are no more relations to consider. Hence G is a required sagbi basis in this case. Now we will give the proof for one of the exceptional cases. Consider the algebra generated by $t^4, t^6 + t^{2k-3}, t^{2k+1}$.
We consider $H(Y_1, Y_2) = Y_1^3 - Y_2^2$ then $H(t^4, t^6 + t^{2k-3}) = -2t^{2k+3} + t^{4k-6}$. This leads to a new element for the sagbi basis. t^{4k-6} is the leading monomial of $(t^4)^{k-3}(t^6 + t^{2k-3})$. We can use these relation and similar once to cancel this term. We obtain $t^4, t^6 + t^{2k-3}, t^{2k+1}, t^{2k+3}$ as a candidate for the sagbi basis. As above we have to check that for any element $H(Y_1, .., Y_4)$ of a generating set of polynomials of the algebraic relations between $t^4, t^6, t^{2k+1}, t^{2k+3}$ $H(t^4, t^6 + t^{2k-3}, t^{2k+1}, t^{2k+3})$ can be reduced to zero. This can easily be checked. The other exceptional cases can be treated in a similar way. □

Proposition 4.2. *The following tables contain the G-standard basis of the module of Kähler differentials of the simple space curve singularities.*

Sagbi Basis of the Algebra	*Standard basis of Kähler Differentials*
(t)	(1)
(t^2, t^{2k+1})	(t, t^{2k})
(t^3, t^{3k+1})	(t^2, t^{3k})
(t^3, t^{3k+2})	(t^2, t^{3k+1})
$(t^3, t^{3k+1} + t^{3p+2})$ $k \geq 2, k \leq p < 2k$	$(t^3, t^{3k} + \frac{3p+2}{3k+1} t^{3p+1}, t^{3p+4})$
$(t^3, t^{3k+2} + t^{3p+1})$ $k \geq 2, k < p \leq 2k$	$(t^3, t^{3k+1} + \frac{3p+1}{3k+2} t^{3p}, t^{3p+3})$
(t^4, t^5)	(t^3, t^4)
$(t^4, t^5 + t^7)$	$(t^3, t^4 + \frac{7}{5} t^6, t^{10})$
$(t^4, t^6 + t^{2k+1}, t^{2k+7})$ $k \geq 3$	$k = 3, (t^3, t^5 + \frac{7}{6} t^6, t^{10}, t^{12} - \frac{15}{26} t^{14}, t^{14})$ $k > 3, (t^3, t^5 + \frac{2k+1}{6} t^{2k}, t^{2k+6}, t^{2k+4})$
(t^4, t^7)	(t^3, t^6)
$(t^4, t^7 + t^9)$	$(t^3, t^6 + \frac{9}{7} t^8, t^{12})$
$(t^4, t^7 + t^{10})$	$(t^3, t^6 + \frac{10}{7} t^9, t^{16})$

Sagbi basis of Algebra	*Standard basis of Kähler Differentials*
$(t^3, t^{3k+1}, t^{3n+2})$ $k \leq n < 2k$	(t^3, t^{3k}, t^{3n+1})
$(t^3, t^{3k+2}, t^{3n+1})$ $k < n \leq 2k$	(t^3, t^{3k+1}, t^{3n})
$(t^3, t^{3k+1} + t^{3p+2}, t^{3n+2})$ $k \leq p < n < 2k$	$n = p+1, (t^2, t^{3k} + \frac{3p+2}{3k+1} t^{3p+1}, t^{3n+1})$ $n \neq p+1, (t^2, t^{3k} + \frac{3p+2}{3k+1} t^{3p+1}, t^{3n+1}, t^{3p+4})$
$(t^3, t^{3k+2} + t^{3p+1}, t^{3n+1})$ $k < p < n \leq 2k$	$n = p+1, (t^2, t^{3k+1} + \frac{3p+1}{3k+2} t^{3p}, t^{3n})$ $n \neq p+1, (t^2, t^{3k+1} + \frac{3p+1}{3k+2} t^{3p}, t^{3n}, t^{3p+3})$
(t^4, t^5, t^6)	(t^3, t^4, t^5)
(t^4, t^5, t^7)	(t^3, t^4, t^6)
(t^4, t^5, t^{11})	(t^3, t^4, t^10)
$(t^4, t^5 + t^7, t^{11})$	$(t^3, t^4 + \frac{7}{5} t^6, t^{10})$
(t^4, t^6, t^{2k+1}) $k \geq 3$	(t^3, t^5, t^{2k})
$(t^4, t^6 + t^{2k-1}, t^{2k+1})$ $k \geq 4$	$(t^3, t^5 + \frac{2k-1}{6} t^{2k-2}, t^{2k}, t^{2k+2})$
$(t^4, t^6 + t^{2k-3}, t^{2k+1}, t^{2k+3})$ $k \geq 5$	$(t^3, t^5 + \frac{2k-3}{6} t^{2k-4}, t^{2k}, t^{2k+2})$
$(t^4, t^6 + t^{2k-7}, t^{2k+1}, t^{2k-1})$ $k \geq 7$	$(t^3, t^5 + \frac{2k-7}{6} t^{2k-8}, t^{2k}, t^{2k-2}, t^{2k-4})$
$(t^4, t^7, t^9 + t^{10})$	$(t^3, t^6, t^8 + \frac{10}{9} t^9, t^{13})$
(t^4, t^7, t^9)	(t^3, t^6, t^8)
(t^4, t^7, t^{10})	(t^3, t^6, t^9)
(t^4, t^7, t^{13})	(t^3, t^6, t^{12})
(t^4, t^7, t^{17})	(t^3, t^6, t^{16})
$(t^4, t^7 + t^9, t^{10})$	$(t^3, t^6 + \frac{9}{7} t^8, t^9, t^{12})$
$(t^4, t^7 + t^9, t^{13})$	$(t^3, t^6 + \frac{9}{7} t^8, t^{12})$
$(t^4, t^7 + t^9, t^{17})$	$(t^3, t^6 + \frac{9}{7} t^8, t^{12}, t^{16})$
$(t^4, t^7 + t^{10}, t^{17})$	$(t^3, t^6 + \frac{10}{7} t^9, t^{16})$

Proof. we will just prove one case. The other are similar.
Consider the curve defined by $(t^4, t^6 + t^{2k-7}, t^{2k+1})$, $k \geq 7$. The corresponding algebra has reduced sagbi basis $G = \{g_1, \cdots, g_4\} = \{t^4, t^6 + t^{2k-7}, t^{2k-1}, t^{2k+1}\}$. The module of the Kähler differentials is generated by $\{h_1, \cdots, h_4\} = \{t^3, t^5 + \frac{2k-7}{6} t^{2k-8}, t^{2k-2}, t^{2k}\}$. Now we consider all the combinations of g_i and h_j of the form $LT(h_i)H_1(LT(g_1), \cdots, LT(g_4)) - LT(h_j)H_2(LT(g_1), \cdots, LT(g_4)) = 0$.
Consider $g_1h_2 - g_2h_1$ the leading term cancels and we obtain $(\frac{2k-13}{6})t^{2k-4}$ since

the term t^{2k-4} is not of the form $LT(h_i)H(LT(g_1), LT(g_2), \cdots, LT(g_4))$. We add the new term into the G-standard basis of the module, $h_5 := t^{2k-4}$. We continue and obtain no new term.
Thus the set $H = \{h_1, h_2, \cdots, h_5\} = \{t^3, t^5 + \frac{2k-1}{6}t^{2k-8}, t^{2k-4}, t^{2k-2}, t^{2k}\}$ is the required G-standard basis of the Kähler differentials. The other cases can be treated similarly. □

Proposition 4.3. *Let $A := \mathbb{C}[[x_1(t), x_2(t), x_3(t)]] \subseteq \mathbb{C}[[t]]$ be a subalgebra, $dim\, \mathbb{C}[[t]] \,/\, \mathbb{C}[[x_1(t), x_2(t), x_3(t)]] < \infty$. Let Γ be the semi-group of A.*
If Γ is in the list of semi-groups of the singularities listed in the tables above then A is a simple singularity.

Proof. We give the proof for one case. The other cases are similar. Assume $\Gamma =< 4, 6, 2k+1 >$ is the semigroup of A. Let $L = \{t^4+H_1, t^6+H_2, t^{2k+1}+H_3\}$ be a Sagbi basis corresponding to this semigroup where $ord(H_1) > 4, ord(H_2) > 6, ord(H_3) > 2k+1$ and $k \geq 3$. Using the automorphism of $\mathbb{C}[[t]]$ mapping $t^4 + H_1$ to t^4 we may assume that $H_1 = 0$. Since the conductor of Γ is $2k + 4$ and $< 4, 6 >\subseteq \Gamma$ we may assume that $H_2 = \alpha_7 t^7 + \alpha_9 t^9 + \cdots + \alpha_{2k+1}t^{2k+1} + \alpha_{2k+3}t^{2k+3}$ and $H_3 = \beta t^{2k+3}$. Let α_i be minimal such that $\alpha_i \neq 0$. Since $(t^6 + H_2)^2 - (t^4)^3 = 2\alpha_i t^{6+i} + \cdots$ we obtain $6 + i \in \Gamma$, i.e. $i = 2k - 5$ or $i \geq 2k - 1$. Then above basis reduces to $\{t^4, t^6 + \gamma_0 t^{2k-5} + \gamma_1 t^{2k-3} + \gamma_2 t^{2k-1} + \gamma_3 t^{2k+3}, t^{2k+1} + \omega t^{2k+3}\}$, $\gamma_1 = \gamma_0\omega$.
If $\gamma_0 \neq 0$ then $A = \mathbb{C}[[t^4, t^6 + H_2]]$ and A is a simple plane curve singularity. If $\gamma_0 = 0$ then $\gamma_1 = 0$ and $L = \{t^4, t^6 + \gamma_2 t^{2k-1} + \gamma_3 t^{2k+3}, t^{2k+1} + \omega t^{2k+3}\}$. Using the transformation $t \to t - \frac{1}{2k+1}\omega t^3$ we may assume that $\omega = 0$. This transformation creates in t^4 resp. t^6 only additional terms of even degree which can be removed afterwords. Using the transformation $t \to t - \frac{1}{2k-1}\frac{\gamma_3}{\gamma_2}t^5$ (if $\gamma_2 \neq 0$). We may assume as before $\gamma_3 = 0$ This leads to the case $L = \{t^4, t^6 + t^{2k-1}, t^{2k+1}\}$. If $\gamma_2 = 0$ and $\gamma_3 = 0$ we are in the case $L = \{t^4, t^6, t^{2k+1}\}$. It remains to consider the case $\gamma_2 = 0$,$\gamma_3 \neq 0$. We may assume that $\gamma_3 = 1$,i.e. $L = \{t^4, t^6 + t^{2k+3}, t^{2k+1}\}$. The transformation $t \to t - \frac{1}{6}t^{2k-2}$ gives

$$\begin{array}{ccc} t^4 \to & t^4 - \frac{2}{3}t^{2k+1} & \mod t^{2k+4} \\ t^6 + t^{2k+3} \to & t^6 & \mod t^{2k+4} \\ t^{2k+1} \to & t^{2k+1} & \mod t^{2k+4} \end{array}$$

We obtain finally $L = \{t^4, t^6, t^{2k+1}\}$.

□

Proposition 4.4. *The type of the simple space curve singularities is completely characterized by the semi-group and the semi-module of its Kähler differentials except in the following cases:*

(1) $(t^3, t^{3k+1} + \lambda\, t^{3(n-1)+2}, t^{3n+2})$ $k < n < 2k,\ \lambda \in \{0, 1\}$
(2) $(t^3, t^{3k+2} + \lambda\, t^{3(n-1)+1}, t^{3n+1})$ $k < n \leq 2k,\ \lambda \in \{0, 1\}$

Proof. Tables given in section 1 shows that there are different singularities with same semi-group. An analysis of the tables given in Proposition 4.2 show that all of them but the two cases above can be distinguished by the semi-module of the Kähler differentials. □

Proposition 4.5. *Let $\Gamma = \langle 3, 3k+1, 3n+2\rangle$ resp. $\Gamma = \langle 3, 3k+2, 3n+1\rangle$ be the semi-group of $A := \mathbb{C}[[x_1(t), x_2(t), x_3(t)]] \subseteq \mathbb{C}[[t]]$. Let $\{t^3 + h_1, t^{3k+1} + h_2, t^{3n+2} + h_3\}$*

resp. $\{t^3+h_1, t^{3k+2}+h_2, t^{3n+1}+h_3\}$ *be the sagbi-basis of* A. *Let* $\mathbb{C}[[t^3, t^{3k+1}+\lambda\, t^{3(n-1)+2}]]$ *resp.* $\mathbb{C}[[t^3, t^{3k+2}+\lambda\, t^{3(n-1)+1}]]$ *be the normal form of* $\mathbb{C}[[t^3+h_1, t^{3k+1}+h_2]]$ *resp.* $\mathbb{C}[[t^3+h_1, t^{3k+2}+h_2]]$ $\lambda\in\{0,1\}$. *Then* $t^3, t^{3k+1}+\lambda\, t^{3(n-1)+2}, t^{3n+2}$ *resp.* $t^3, t^{3k+2}+\lambda\, t^{3(n-1)+1}, t^{3n+1}$ *is the normal form of* A.

Proof. We give the proof of one case and the other case is similar. By the assumptions above we have that $\mathbb{C}[[t^3+h_1, t^{3k+1}+h_2]] \simeq \mathbb{C}[[t^3, t^{3k+1}+\lambda t^{3(n-1)+2}]]$. Let $\Phi : \mathbb{C}[[t]] \to \mathbb{C}[[t]]$ be the automorphism such that
$\Phi(\mathbb{C}[[t^3+h_1, t^{3k+1}+h_2]]) = \mathbb{C}[[t^3, t^{3k+1}+\lambda t^{3(n-1)+2}]]$. Using this automorphism we may assume that $h_1 = 0$, $h_2 = \lambda t^{3(n-1)+2}$. Since the conductor of $\Gamma =< 3, 3k+1, 3n+2 >$ is 3n we may also assume that $h_3 = 0$. □

5. Description of the classifier

The following two algorithms classify the simple space curve singularities and the simple plane curve singularities.

Algorithm 1 Simple Space Curves (`spaceCur`)

Input: $x_1(t), x_2(t), x_3(t) \in \mathbb{C}[[t]]$ and $A = \mathbb{C}[[x_1(t), x_2(t), x_3(t)]]$.
Output: $y_1(t), y_2(t), y_3(t)$, the normal form or 0 if it is not simple.

Compute $G = \{g_1, g_2, \cdots, g_s\}$, a reduced Sagbi basis of A such that $LT(g_i) = t^{a_i}$, $a_1 < a_2 < \cdots < a_s$.
Compute M, a minimal G-standard basis for the Kähler differentials of A.
$\Gamma = \langle a_1, a_2, \cdots, a_s\rangle$, the semigroup of A and C the conductor of Γ.
If $s = 2$ return `planeCur(G)`.
Compute Γ_M the semi module of M and compute C_M the conductor of Γ_M.
if $a_1 = 3$ **then**
  if $a_2 = 3k+1$ **then**
    $H =$ `planeCur`(g_1, g_2)
    if $H = (t^3, t^{3k+1}+\lambda t^{3(n-1)+2})$ **then**
      return $(t^3, t^{3k+1}+\lambda t^{3(n-1)+2}, t^{3n+2})$;
  if $a_2 = 3k+2$ **then**
    $H =$ `planeCur`(g_1, g_2)
    if $H = (t^3, t^{3k+2}+\lambda t^{3(n-1)+1})$ **then**
      return $(t^3, t^{3k+2}+\lambda t^{3(n-1)+1}, t^{3n+1})$
if $a_1 = 4$ **then**
  if $a_2 = 5$ **then**
    if $a_3 = 6$ **then**
      return (t^4, t^5, t^6);
    if $a_3 = 7$ **then**
      return (t^4, t^5, t^7);
    if $a_3 = 11$ **then**
      Compute G', a reduced sagbi-basis of $\{g_1, g_2\}$ and, Compute M', a minimal G'-standard basis of the module of Kähler differentials of $\mathbb{C}[[g_1, g_2]]$.
      if $\sharp(M') = 2$ **then**
        return (t^4, t^5, t^{11});
      if $\sharp(M') = 3$ **then**
        return (t^4, t^5+t^7, t^{11});

Algorithm 1 Simple Space Curves (`spaceCur`)

if $a_1 = 4$ **then**
  if $a_2 = 6$ **then**
    if $s = 3$ **then**
      if $\sharp(M) = 4$ **then**
        return $(t^4, t^6 + t^{a_3-2}, t^{a_3})$.
      if $\sharp(M) = 3$ **then**
        return (t^4, t^6, t^{a_3}).
    if $s = 4$ **then**
      if $\sharp(M) = 4$ **then**
        return $(t^4, t^6 + t^{a_3-4}, t^{a_3})$;
      if $\sharp(M) = 5$ **then**
        return $(t^4, t^6 + t^{a_4-8}, t^{a_4})$.
  if $a_2 = 7$ **then**
    if $a_3 = 9$ **then**
      if $\Gamma_M = \langle 3, 6, 8, 13 \rangle$ **then**
        return $(t^4, t^7, t^9 + t^{10})$;
      if $\Gamma_M = \langle 3, 6, 8 \rangle$ **then**
        return (t^4, t^7, t^9)
    if $a_3 = 10$ **then**
      if $\Gamma_M = \langle 3, 6, 9, 12 \rangle$ **then**
        return $(t^4, t^7 + t^9, t^{10})$
      if $\Gamma_M = \langle 3, 6, 9 \rangle$ **then**
        return (t^4, t^7, t^{10})
    if $a_3 = 13$ **then**
      Compute G', a reduced sagbi-basis of $\{g_1, g_2\}$.
      Compute M', a minimal G'-standard basis of the module of Kähler differentials of $\mathbb{C}[[g_1, g_2]]$.
      if $\sharp(M') = 2$ **then**
        return (t^4, t^7, t^{13});
      if $\sharp(M') = 3$ **then**
        return $(t^4, t^7 + t^9, t^{13})$;
    if $a_3 = 17$ **then**
      if $\Gamma_M = \langle 3, 6, 12, 16 \rangle$ **then**
        return $(t^4, t^7 + t^9, t^{17})$;
      if $\Gamma_M = \langle 3, 6, 17 \rangle$ **then**
        Compute G', a reduced sagbi-basis of $\{g_1, g_2\}$.
        Compute M', a minimal G'-standard basis of the module of Kähler differentials of $\mathbb{C}[[g_1, g_2]]$.
        if $\sharp(M') = 2$ **then**
          return (t^4, t^7, t^{17});
        if $\sharp(M') = 3$ **then**
          return $(t^4, t^7 + t^{10}, t^{13})$;
return (0)

Algorithm 2 Simple Plane Curves (`planeCur`)

Input: $x_1(t), x_2(t) \in \mathbb{C}[[t]]$.
Output: $y_1(t), y_2(t)$, the normal form or 0 if it is not simple.

Compute G, a Sagbi basis of $A = \mathbb{C}[[x_1(t), x_2(t)]]$.
Compute M, a minimal G-standard basis for the module of Kähler differentials of A.
Compute $\Gamma = \langle a_1, a_2, \cdots, a_s \rangle$ the semigroup of A, $a_1 < a_2 < \cdots < a_s$.
Compute C the conductor of Γ.
Compute Γ_M the semi module of M.
Compute C_M the conductor of Γ_M
if $a_1 = 1$ **then**
  return (t);
if $a_1 = 2$ **then**
  return (t^2, t^{C+1});
if $a_1 = 3$ **then**
  if $a_2 = 3k + 1$ **then**
    if $\Gamma_M = \langle 2, 3k \rangle$ **then**
      return (t^3, t^{3k+1})
    if $\Gamma_M = \langle 2, 3k, C_M + 2 \rangle$ **then**
      return $(t^3, t^{3k+1} + t^{C_M})$
  if $a_2 = 3k + 2$ **then**
    if $\Gamma_M = \langle 2, 3k + 1 \rangle$ **then**
      return (t^3, t^{3k+2})
    if $\Gamma_M = \langle 2, 3k + 1, C_M + 2 \rangle$ **then**
      return $(t^3, t^{3k+2} + t^{C_M})$
if $a_1 = 4$ **then**
  if $a_2 = 5$ **then**
    if $\Gamma_M = \langle 3, 4 \rangle$ **then**
      return (t^4, t^5);
    if $\Gamma_M = \langle 3, 4, 10 \rangle$ **then**
      return $(t^4, t^5 + t^7)$;
  if $a_2 = 6$ **then**
    return $(t^4, t^6 + t^{C_M - 9})$;
  if $a_2 = 7$ **then**
    if $\Gamma_M = \langle 3, 6 \rangle$ **then**
      return (t^4, t^7);
    if $\Gamma_M = \langle 3, 6, 12 \rangle$ **then**
      return $(t^4, t^7 + t^9)$;
    if $\Gamma_M = \langle 3, 6, 16 \rangle$ **then**
      return $(t^4, t^7 + t^{10})$;
return (0);

Example 5.1.
```
ring r=0,t,Ds;
ideal I=t3+3t4+3t5+4t6+6t7+3t8+3t9+3t10+t12,t2+2t3+t4+2t5+2t6+t8;
planeCur(I);
>[1]=t2
```

```
>[2]=t3
```

Example 5.2. ring r=0,t,Ds;

```
ideal I=t3+3t4+3t5+t6,t13+14t14+92t15+377t16+1079t17+2288t18+3718t19+
4719t20+4719t21+3718t22+2288 t23+1079t24+377t25+92t26+14t27+t28,
t20+20t21+190t22+1140t23+4845t24+15504t25+38760t26+77520t27+
125970t28+16796 0t29+184756t30+167960t31+125970t32+77520t33+
38760t34+15504t35+4845t36+1140t37+19 0t38+20t39+t40;
spaceCur(I);
>[1]=t3
>[2]=t13+t14
>[3]=t20
```

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