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Unimodal Singularities of Parametrized Plane Curves

by
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Abstract

We use the results of Hefez and Hernandez [7] to give a different proof for the classification of simple singularities of mappings $(\mathbb{C},0) \longrightarrow (\mathbb{C}^2,0)$ with respect to $\mathcal{A}$ equivalence done by Bruce and Gaffney [2] and give the classification of unimodal singularities. The result are extended to a classification over the real numbers $\mathbb{R}$.

Key Words: parametrized plane curves, unimodal singularities.
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1 Introduction

Let K be a field of characteristic 0 contained in $\mathbb{C}$ the field of complex numbers. We are interested in classifying the $\mathcal{A}-$finitely determined map germs $(K,0) \longrightarrow (K^2,0)$ for $K=\mathbb{R}$ or $\mathbb{C}$ with respect to $\mathcal{A}-$equivalence or equivalently the finitely determined $K-$algebra automorphisms $K[[x,y]] \longrightarrow K[[t]]$ with respect to the action of the group $\mathcal{A} = Aut_K(K[[t]]) \times Aut_K(K[[x,y]])$ acting on

$$A(1,2) = \{(x(t),y(t)) | x(t),y(t) \in <t> K[[t]], dim_K[[t]]/K[[x(t),y(t)]] < \infty\}$$

as follows:

$$\mathcal{A} \times A(1,2) \longrightarrow A(1,2)$$

$$((\Phi,\Psi),f) \mapsto \Phi^{-1} \circ f \circ \Psi$$

If we identify $Aut_K(K[[t]]) = tK[[t]]^*$, $Aut_K(K[[x,y]]) = \{(\Psi_1,\Psi_2) | \Psi_i \in <x,y> K[[x,y]],$ $det(\frac{\partial(\Psi_1,\Psi_2)}{\partial(x,y)}(0)) \neq 0\}$ then the action of it can be explicitely written as

$$\Psi \circ (x(t),y(t)) \circ \Phi^{-1} = (\Psi_1(x(\overline{\Phi}(t)),y(\overline{\Phi}(t))), \Psi_2(x(\overline{\Phi}(t)),y(\overline{\Phi}(t)))) \text{ with } \overline{\Phi}(\Phi(t)) = \mathrm{t}.$$

If $f,g \in A(1,2)$ are in the same orbit under the action of $\mathcal{A}$ we call f $\mathcal{A}-$ equivalent to g and write $f \sim_{\mathcal{A}} g$.

Bruce and Gaffney used Arnold's [1] classification of hypersurface singularities to obtain (with Arnold's notation) the $\mathcal{A}$- simple germs $(\mathbb{C},0) \longrightarrow (\mathbb{C}^2,0)$.

A_{2k}	(t^2, t^{2k+1})
E_{6k}	$(t^3, t^{3k+1} + t^{3p+2})$, $k \leq$ p $< 2k$ (t^3, t^{3k+1})
E_{6k+2}	$(t^3, t^{3k+2} + t^{3p+1}))$, $k <$ p $\leq 2k$ (t^3, t^{3k+2})
W_{12}	$(t^4, t^5 + t^7)$ (t^4, t^5)
W_{18}	$(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ (t^4, t^7)
$W^{\#}_{1,2k-5}$	$(t^4, t^6 + t^{2k+1})$, $k \geq 3$

We use the following theorem of Hefez and Hernandez [7] to obtain this result and refine it for a classification over $\mathbb{R}$. With the same idea we obtain the classification $\mathcal{A}$ unimodal germs. To give a formulation of the theorem we recall the definition of the Zariski number. Let $R := K[[x(t), y(t)]] \subseteq K[[t]]$ such that $\delta_R = dim_K K[[t]]/R < \infty$. Let $\Gamma_R = \{ord_t(f) | f \in R\}$, and $\beta = (\beta_0, \cdots, \beta_s)$, $\beta_0 < \beta_1 < \cdots < \beta_s$ the minimal generators of Γ_R, especially let $n := \beta_0$ and $m := \beta_1$. Note that $c_R = 2\delta_R$ is the conductor of R, i.e. $t^{c_R} K[[t]] \subseteq R$. Let $\Omega_R =< \frac{dx(t)}{dt}, \frac{dy(t)}{dt} >_R$ be the $R-$module of differentials. Let $\Gamma(\Omega_R) = \{ord_t(w) | w \in \Omega_R\}$ be the semi-module of values and $\Lambda = (\Gamma(\Omega_R) + 1) \cup \{0\}$. Then $\Gamma_R \subseteq \Lambda_R$. The Zariski number is

$$\lambda := \begin{cases} \infty : & \text{if } \Gamma_R = \Lambda_R \\ a : & \text{where a=min } \{s | s \in \Lambda_R - \Gamma_R\} - n \end{cases}$$

The pair (β, λ) is an $\mathcal{A}-$invariant.

Theorem 1. *[7] Let $R = K[[x(t), y(t)]] \subseteq K[[t]]$ such that $\delta_R < \infty$.*

1. *$\Lambda_R = \Gamma_R$ iff $(x(t), y(t)) \sim_{\mathcal{A}} (t^n, t^m)$.*

2. *If $\lambda < \infty$ then there exist $\overline{y}(t) = t^m + \overline{a}_\lambda t^\lambda + \sum_{\substack{i+n \notin \Lambda \\ i > \lambda}} \overline{a}_i t^i$, $\overline{a}_i \in K$, $\overline{a}_\lambda \neq 0$ such that $(x(t), y(t)) \sim_{\mathcal{A}} (t^n, \overline{y}(t))$. If $K = \mathbb{C}$ (resp. $K = \mathbb{R}$) we can choose $\overline{a}_\lambda = 1$ (resp. $\overline{a}_\lambda = \pm 1$).*

3. *$(t^n, t^m + b_\lambda t^\lambda + \sum_{\substack{i+n \notin \Lambda \\ i > \lambda}} b_i t^i) \sim_{\mathcal{A}} (t^n, t^m + \overline{b}_\lambda t^\lambda + \sum_{\substack{i+n \notin \Lambda \\ i > \lambda}} \overline{b}_i t^i)$, $b_\lambda, \overline{b}_\lambda \neq 0$, iff $b_i^{\lambda - m} = (\frac{b_\lambda}{\overline{b}_\lambda})^{i-m} \overline{b}_i^{\lambda - m}$. If $K = \mathbb{C}$ (resp. $K = \mathbb{R}$) and $b_\lambda = \overline{b}_\lambda = 1$ (resp. $b_\lambda = \pm 1, \overline{b}_\lambda = \pm 1$) then $b_i = r^{i-m} \overline{b}_i$ (resp.$b_i = \pm r^{i-m} \overline{b}_i$), $r \in \mathbb{C}$, $r^{\lambda - m} = 1$.*

We obtain the following corollary:

Corollary 1. *The modality [1] of $f = (x(t), y(t))$ is greater or equal to the cardinality of the set $\{i \in \mathbb{Z} | i > \lambda, i + n \notin \Lambda\}$.*

Definition 1. *The integers $i > \lambda$ with $i + n \notin \Lambda$ are called the moduli of Λ.*

[1] Recall, that the modality of f is the least number l such that a small neighbourhood of f in $A(1, 2)$ can be covered by a finite number of $l-$parameter families of orbits.

2 Classification of Simple and Unimodal Germs

The classification is done fixing the $\mathcal{A}-$invariant (β, λ).

Lemma 1. *The invariant (β, λ) of the map germ $(x(t), y(t)) \in A(1,2)$ is semicontinuous with respect to the lexicographical ordering* [2].

Proof. We may assume $x(t) = t^n$ and $y(t) = t^m + \sum_{i>m} a_i t^i$ with $\beta_0 = n$ and $\beta_1 = m$. Let $k_0 = \beta_0, k_1 = \beta_1$ and $k_v = min\{i | a_i \neq 0, gcd(i, k_0, \cdots, k_{v-1}) < gcd(k_0, \cdots, k_{v-1})\}$ define the sequence of characteristic exponents and
$k = (k_0, \cdots, k_s)$ then [9]

$$\beta_i = k_i + \sum_{j=1}^{i-1} (\frac{gcd(k_0, \cdots, k_{j-1})}{gcd(k_0, \cdots, k_{i-1})} - \frac{gcd(k_0, \cdots, k_j)}{gcd(k_0, \cdots, k_{i-1})}) k_j$$

and

$$k_i = \beta_i + \sum_{j=1}^{i-1} (\frac{gcd(\beta_0, \cdots, \beta_{j-1})}{gcd(\beta_0, \cdots, \beta_j)} - 1)) \beta_j.$$

It is equivalent to prove that (k, λ) is semicontinuous. Let $(X(t,z), Y(t,z))$ be a deformation of $(x(t), y(t))$. If $ord_t X(t,z) < \beta_0$ or $ord_t Y(t,z) < \beta_0$ we are done. Assume now $ord_t X(t,z) \geq \beta_0$, and $ord_t Y(t,z) \geq \beta_0$. Since $X(t,0) = t^{\beta_0}$, we have $X(t,z) = t^{\beta_0}(1 + h(t,z))$ with $h(t,0) = 0$. Then $(X(t,z), Y(t,z)) \sim_{\mathcal{A}} (t^{\beta_0}, \overline{Y}(t,z))$ with $(t^{\beta_0}, Y(t,0)) \sim_{\mathcal{A}} (t^{\beta_0}, y(t))$. We may assume $\overline{Y}(t,0) = y(t)$ and $ord_t \overline{Y}(t,z) \geq \beta_0$, because $\mathcal{A}-$equivalence does not change the characteristic exponents. Let $\overline{Y}(t,z) = \sum_{i \geq \beta_0} \overline{a}_i(z) t^i$, we may assume $\overline{a}_i(z) = 0$ if $\beta_0 \mid i$. For a fixed z with $|z|$ small, let $\overline{k}_1(z), \cdots, \overline{k}_s(z)$ be the sequence of characteristic exponents of $\overline{Y}(t,z)$. Assume $\overline{k}_1(z) = k_1, \cdots, \overline{k}_l(z) = k_l$ for some $l \geq 0$. If $l < r$ consider $\overline{k}_{l+1}(z)$. We have $k_{l+1} = min\{i \mid a_i \neq 0, gcd(i, k_0, \cdots, k_l) < gcd(k_0, \cdots, k_l)\}$ and $\overline{k}_{l+1}(z) = min\{i \mid \overline{a}_i(z) \neq 0, gcd(i, k_0, \cdots, k_l) < gcd(k_0, \cdots, k_l)\}$. Since $\overline{a}_i(0) = a_i$ and z is small we have $\overline{a}_{k_{l+1}}(z) \neq 0$. This implies $\overline{k}_{l+1}(z) \leq k_{l+1}$. If $\overline{k}_{l+1}(z) = k_{l+1}$ we continue like that and may assume that $\overline{k}_i(z) = k_i$ for all i. This implies that we have a deformation with constant semi group and we have to prove that in this situation the Zariski number does not increase in a deformation.
If $\lambda = \infty$, we are done. Assume that $\lambda < \infty$. We may assume that $x(t) = t^n$, $y(t) = t^m + \sum_{i \geq \lambda} a_i t^i$, $a_\lambda \neq 0$ and $X(t,z) = t^n$, $Y(t,z) = t^m + \sum_{i \geq k > m} \overline{a}_i(t,z) t^i$ such that $\overline{a}_i(t,0) = a_i$ for $i \geq \lambda$ and $\overline{a}_i(t,0) = 0$ for $i < \lambda$. If $k \geq \lambda$ we are done. Assume that $k < \lambda$. If $k + n \notin \Gamma$ we are done since in this case $k < \lambda$ is the Zariski number of $(X(t,z), Y(t,z))$ for $|z|$ small, $z \neq 0$. We will show that $k + n \in \Gamma$ implies $k \in \Gamma$ or $k + n - m$ in Γ. If $\Gamma = < n, m >$ this is clear. If $\Gamma = < n, m, \beta_2, \cdots >$ we obtain that $\lambda \leq k_2$, the second characteristic exponent and $\beta_2 = k_2 + (\frac{n}{gcd(n,m)} - 1)m$. In this case $\lambda + n \leq k_2 + n < k_2 + m \leq \beta_2$ and therefore $k + n < \beta_2$. This implies $k + n \in < n, m >$ and as in the first case $k \in \Gamma$ or $k + n - m \in \Gamma$. For the case $k \in \Gamma$ or $k + n - m \in \Gamma$ Zariski gave an explicite $\mathcal{A}-$equivalence for the family $(t^n, Y(t,z))$ to $(t^n, \overline{Y}(t,z))$ such that $\overline{Y}(t,z) = t^m + \sum_{i>k} \overline{b}_i(t,z) t^i$ and $\overline{b}_i(t,0) = a_i$ for $i \geq \lambda$, $\overline{b}_i(t,0) = 0$ for $i < \lambda$. Now we can continue as above . The process will stop either

[2]This means that in a deformation $(X(t,z), Y(t,z))$ the corresponding invariant (β_z, λ_z) is lexicographically smaller or equal to (β, λ).

if $k=\lambda$ (a deformation with constant Zariski number) or $k<\lambda$ and $k+n\notin\Gamma$ (k is the Zariski number of $(t^n,Y(t,z))$ for $|z|$ small, $z\neq 0$). ∎

Lemma 2. *The following list of semigroups Γ of plane curve singularities has only semimodules Λ with no moduli.*

β	λ	Λ	$Normal form$
1	∞	Γ	$(t,0)$
$(2,2k+1)$	∞	Γ	(t^2,t^{2k+1})
$(3,3k+1)$	$3p+2$ $k\leq p<2k$ ∞	$<\Gamma,3p+5>_\Gamma$ Γ	$(t^3,t^{3k+1}\pm t^{3p+2})$ " $-$ " *over* $\mathbb{R}$ *if* $2\,\|p$ *and* $2\nmid k$ (t^3,t^{3k+2})
$(3,3k+2)$	$3p+1$ $k<p\leq 2k$ ∞	$<\Gamma,3p+4>_\Gamma$ Γ	$(t^3,t^{3k+2}\pm t^{3p+1})$ " $-$ " *over* $\mathbb{R}$ *if* $2\nmid p$ *and* $2\nmid k$ (t^3,t^{3k+2})
$(4,5)$	7 ∞	$<\Gamma,11>_\Gamma$ Γ	(t^4,t^5+t^7) (t^4,t^5)
$(4,6,2k+7)$	$2k+1$ $k\geq 3$	$<\Gamma,2k+5>_\Gamma$	(t^4,t^6+t^{2k+1})
$(4,7)$	9 13 ∞	$<\Gamma,13>_\Gamma$ $<\Gamma,17>_\Gamma$ Γ	(t^4,t^7+t^9) (t^4,t^7+t^{13}) (t^4,t^7)

Proof. The proof can be done checking case by case. We compute Λ for each case and check that $i>\lambda$ implies $i+n\notin\Lambda$. ∎

Lemma 3.
1. *If* $\Gamma=<4,9>_{\mathbb{N}}$ *and* $\lambda=10$ *then* $\Lambda=<\Gamma,14,19>_\Gamma$ *or* $\Lambda=<\Gamma,14>_\Gamma$ *has one modulus* 11 (*i.e.* $11+4=15\notin\Lambda$) *resp.* 15.
   $(t^4,t^9+t^{10}+\sum_{i>10}a_it^i)\sim_{\mathcal{A}}(t^4,t^9+t^{10}+at^{11})$ *for suitable* $a\in K$, *resp.*
   $(t^4,t^9+t^{10}+\sum_{i>10}a_it^i)\sim_{\mathcal{A}}(t^4,t^9+t^{10}+\frac{19}{18}t^{11}+at^{15})$ *for suitable* $a\in K$.

2. *If* $\Gamma=<5,6>_{\mathbb{N}}$ *and* $\lambda=8$ *then* $\Lambda=<\Gamma,13>_\Gamma$ *has one modulus* 9.
   $(t^5,t^6+t^8+\sum_{i>8}a_it^i)\sim_{\mathcal{A}}(t^5,t^6+t^8+at^9)$ *for suitable* $a\in K$.
   $(t^5,t^6+t^7+\sum_{i>7}b_it^i)\sim_{\mathcal{A}}(t^5,t^6+\sum_{i>7}\overline{b}_it^i)$ *for suitable* $\overline{b}_i\in K$.

3. *If* $\Gamma=<4,13>_{\mathbb{N}}$ *and* $\lambda=14$ *then* $\Lambda=<\Gamma,18,27>_\Gamma$ *has two moduli* $15,19$
   $(t^4,t^{13}+t^{14}+\sum_{i>14}a_it^i)\sim_{\mathcal{A}}(t^4,t^{13}+t^{14}+at^{15}+bt^{19})$ *for suitable* $a,b\in K$.

4. *If* $\Gamma=<5,9>_{\mathbb{N}}$ *and* $\lambda=11$ *then* $\Lambda=<\Gamma,16>_\Gamma$ *has two moduli* $12,17$
   $(t^5,t^9+t^{11}+\sum_{i>11}a_it^i)\sim_{\mathcal{A}}(t^5,t^9+t^{11}+at^{12}+bt^{17})$ *for suitable* $a,b\in K$.

5. *If* $\Gamma=<6,7>_{\mathbb{N}}$ *and* $\lambda=9$ *then* $\Lambda=<\Gamma,15,23>_\Gamma$ *has two moduli* $10,11$
   $(t^6,t^7+t^8+\sum_{i>8}a_it^i)\sim_{\mathcal{A}}(t^6,t^7+\sum_{i>9}\overline{a}_it^i)$ *for suitable* $\overline{a}_i\in K$ *and*
   $(t^6,t^7+t^9+\sum_{i>9}a_it^i)\sim_{\mathcal{A}}(t^6,t^7+t^9+at^{10}+bt^{11})$ *for suitable* $a,b\in K$.

Proof. We compute Λ for every case and the integer i such that $i+n \notin \Lambda$. We implemented a SINGULAR procedure for this purpose, see the next example. □

Example 1. *The* SINGULAR *library classify-aeq.lib contains a procedure normalForm. The procedure computes a list, the normal form over* $\mathbb{Q}$ *the semi group* Γ*, the semi module* Λ*, the Zariski number* λ *and the moduli.*

```
 >LIB"classify_aeq.lib";
 >ring r=0,t,ds;
 >ideal I=t6,t7+t9;
 >normalForm(I);
[1]:
   _[1]=t6
   _[2]=t7+t9
[2]:
   0,6,7,12,13,14,18,19,20,21,24,25,26,27,28,30
[3]:
   0,6,7,12,13,14,15,18
[4]:
   9
[5]:
   10,11
> I=t6,t7+t8;
>normalForm(I);
[1]:
   _[1]=t6
   _[2]=t7-15/14t9+184/147t10-391/2744t11
[2]:
   0,6,7,12,13,14,18,19,20,21,24,25,26,27,28,30
[3]:
   0,6,7,12,13,14,15,18
[4]:
   9
[5]:
   10,11
> I=t5,t6+t7+13/12t8+133/108t9;
>normalForm(I);
[1]:
   _[1]=t5
   _[2]=t6-5225/559872t14
[2]:
   0,5,6,10,11,12,15,16,17,18,20
[3]:
   0,5,6,10,11,12,15
[4]:
   14
[5]:
   0
```

In the last example we obtain as normal form over $\mathbb{R}$ $(t^5, t^6 - t^{14})$ *and no modulus but the modality is 1. In first two examples we obtain* $\lambda = 9$ *and 2 moduli* $10, 11$. *This implies especially that the modality is at least* 2.

Lemma 4. *The following list of semi groups* Γ *of plane curve singularities has only semi modules* Λ *with one modulus or no moduli.*

β	λ	Λ	*modulus*	*Normalform*
$(4,9)$	10	$<\Gamma, 14, 19>_\Gamma$	11	$(t^4, t^9 + t^{10} + at^{11})$
	10	$<\Gamma, 14>_\Gamma$	15	$(t^4, t^9 + t^{10} + \frac{19}{18}t^{11} + at^{15})$
	11	$<\Gamma, 15>_\Gamma$	$-$	$(t^4, t^9 + t^{11})$
	15	$<\Gamma, 19>_\Gamma$	$-$	$(t^4, t^9 + t^{15})$
	19	$<\Gamma, 23>_\Gamma$	$-$	$(t^4, t^9 + t^{19})$
	∞	Γ	$-$	(t^4, t^9)
$(4,10,2k+11)$	$2k+1$	$<\Gamma, 2k+5>_\Gamma$ [3]	$2k+3$	$(t^4, t^{10} + t^{2k+1} + at^{2k+3})$
$(4,11)$	13	$<\Gamma, 17>_\Gamma$	14	$(t^4, t^{11} + t^{13} + at^{14})$
	14	$<\Gamma, 18, 25>_\Gamma$	17	$(t^4, t^{11} + t^{14} + at^{17})$
	14	$<\Gamma, 18>_\Gamma$	21	$(t^4, t^{11} + t^{14} + \frac{25}{22}t^{17} + at^{21})$
	17	$<\Gamma, 21>_\Gamma$	$-$	$(t^4, t^{11} + t^{17})$
	21	$<\Gamma, 25>_\Gamma$	$-$	$(t^4, t^{11} + t^{21})$
	25	$<\Gamma, 29>_\Gamma$	$-$	$(t^4, t^{11} + t^{25})$
	∞	Γ	$-$	(t^4, t^{11})
$(5,6)$	8	$<\Gamma, 13>_\Gamma$	9	$(t^5, t^6 \pm t^8 + at^9)$
	9	$<\Gamma, 14>_\Gamma$	$-$	$(t^5, t^6 + t^9)$
	14	$<\Gamma, 19>_\Gamma$	$-$	$(t^5, t^6 \pm t^{14})$
	∞	Γ	$-$	(t^5, t^6)
$(5,7)$	8	$<\Gamma, 13>_\Gamma$	11	$(t^5, t^7 + t^8 + at^{11})$
	11	$<\Gamma, 16>_\Gamma$	13	$(t^5, t^7 + t^{11} + at^{13})$
	13	$<\Gamma, 18>_\Gamma$	$-$	$(t^5, t^7 + t^{13})$
	18	$<\Gamma, 23>_\Gamma$	$-$	$(t^5, t^7 + t^{18})$
	∞	Γ	$-$	(t^5, t^7)
$(5,8)$	9	$<\Gamma, 14>_\Gamma$	12	$(t^5, t^8 + t^9 + at^{12})$
	12	$<\Gamma, 17>_\Gamma$	14	$(t^5, t^8 \pm t^{12} + at^{14})$
	14	$<\Gamma, 19>_\Gamma$	17	$(t^5, t^8 \pm t^{14} + at^{17})$
	17	$<\Gamma, 22>_\Gamma$	$-$	$(t^5, t^8 + t^{17})$
	22	$<\Gamma, 27>_\Gamma$	$-$	$(t^5, t^8 \pm t^{22})$
	∞	Γ	$-$	(t^5, t^8)

[3] Note that in the cases $(4,10,21)$ resp. $(4,10,23)$ we need additionally 21 resp. 29 to generate Λ.

Proof. We compute Λ for every case and the integers $i > \lambda$ such that $i + n \notin \Lambda$. $\square$

Theorem 2. *The second table contains the simple singularities of perameterized plane curves and the third table contains the unimodal singularities.*

Proof. We have to prove that in a deformation of a map germ of the second table (resp. the third table) are only finitely many (resp. one parameter families) of different types of map germs, that any map germ of the third table deforms into one with exactly one modulus and any map germ with (β, λ) lexicographically greater then a map germ in first two tables deforms in a family with two moduli. This is an immediate consequence of lemma 1 (the semi continuity of (β, λ)) and lemma 3 since the tables are ordered by increasing (β, λ). □

Remark 1. *The singularities with* $\beta = (4, 9)$ *resp.*$(4, 11)$ *in the third table correspond to* W_{24} *resp.* W_{30}. *The singularities with* $\beta = (4, 10, 2k + 1)$ *correspond to* $W^{\#}_{2,2k-9}$ *in Arnolds classification [1].*

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