SIMPLE SINGULARITIES OF PARAMETRIZED PLANE CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let K be an algebraically closed field of characteristic $p > 0$. The aim of the article is to give a classification of simple parametrized plane curve singularities over K. The idea is to give explicitly a class of families of singularities which are not simple such that almost all singularities deform to one of those and show that remaining singularities are simple.

Key words : parametrized plane curves, simple singularities, characteristic p.
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1. INTRODUCTION

The study and the classification of singularities have a long history. Very important contributions go back to Zariski (cf. [13]) and Arnold (cf. [1]). Most of the results were obtained over the complex numbers. Greuel and his students started a classification for hypersurface singularities in characteristic p (cf. [2],[3],[7], [8]). Bruce and Gaffney (cf. [4]) classified the simple parametrized plane curve singularities over the complex numbers (for space curves cf. [6]). The aim of this paper is to give a similar classification in positive characteristic.
Let K be an algebraically closed field of characteristic p. A parametrization of a germ of a plane curve singularity is given by a pair $(x(t), y(t))$ of power series, $x(t), y(t) \in K[[t]]$. Two parametrizations $(x(t), y(t))$ and $(\overline{x}(t), \overline{y}(t))$ are $\mathcal{A}-$equivalent (we write $(x(t), y(t)) \sim_{\mathcal{A}} (\overline{x}(t), \overline{y}(t))$) if there exist automorphisms

$$\psi : K[[t]] \to K[[t]] \text{ and } \varphi = (\varphi_1, \varphi_2) : K[[x, y]] \to K[[x, y]]$$

such that

$$(x(\psi(t)), y(\psi(t))) = (\varphi_1(\overline{x}(t), \overline{y}(t)), \varphi_2(\overline{x}(t), \overline{y}(t))).$$

We will always assume that a given parametrization $(x(t), y(t))$ is primitive, i.e. $\dim_K K[[t]]/K[[x(t), y(t)]] =: \delta < \infty$. Furthermore we may assume that $\mathrm{ord}_t x(t) =: n < ord_t y(t) =: m$ and $n \nmid m$.
Given a parametrization $f = (x(t), y(t))$ we denote by Γ_f (or just Γ if f is fixed) the semigroup

$$\Gamma_f = \{ord_t(h) | h \in K[[x(t), y(t)]]\}.$$

Generators of the semigroup can be computed using a sagbi basis (cf. [10]) for $K[[x(t), y(t)]]$.

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As in characteristic 0 it is easy to see ([11], [13]) that the following hold.

Theorem 1. (Zariski)

(1) *In case* $p \nmid n$ *(resp.* $p \nmid m$*) the parametrization is* $\mathcal{A}-$*equivalent to*

$$(t^n, \overline{y}(t)) \text{ (resp. } (\overline{x}(t), t^m)) \text{ for suitable } \overline{y}(t) \text{ (resp. } \overline{x}(t))$$

with $ord_t\overline{y}(t) = m$ *(resp.* $ord_t\overline{x}(t) = n$*).*

(2) *Assume that* $k \in \Gamma$ *then*

$$(t^n, t^m + \textstyle\sum_{i>m} a_i t^i) \sim_{\mathcal{A}} (t^n, t^m + \sum_{i>m} a'_i t^i)$$

with $a_i = a'_i$ *if* $i < k$ *and* $a'_k = 0$*.*

(3) *Assume that* $p \nmid m$ *and* $k + n - m \in \Gamma$ *then*

$$(t^n, t^m + \textstyle\sum_{i>m} a_i t^i) \sim_{\mathcal{A}} (t^n, t^m + \sum_{i>m} a'_i t^i)$$

with $a_i = a'_i$ *if* $i < k$ *and* $a'_k = 0$*.*

Definition 2. *A parametrization* $(x(t), y(t))$ *is called simple if there are only finitely many* $\mathcal{A}-$ *equivalent classes in a deformation of* $(x(t), y(t))$*.*

Remark 1. *As an immediate consequence we obtain:*
Given two parametrizations $(x(t), y(t))$ *(resp.*$(\overline{x}(t), \overline{y}(t))$ *and* $(x(t), y(t))$ *not simple. If* $(x(t), y(t))$ *is* $\mathcal{A}-$*equivalent to a parametrization in a deformation of* $(\overline{x}(t), \overline{y}(t))$ *then* $(\overline{x}(t), \overline{y}(t))$ *is not simple.*

The classification is based on the following idea:

(1) Find special classes of non-simple parametrizations such that all non-simple parametrizations have one of them represented in a suitable deformation.
(2) Especially parametrizations with $n \geq 5$ or $n = 4$ and $m \geq 9$ are not simple.
(3) Find normal forms depending on the semigroup for the remaining cases.
(4) The candidates for simple parametrizations have semigroups generated by at most 3 elements. These semigroups behave semicontinously [1] in a deformation.

In characteristic 0 we obtain the following list of simple parametrizations (cf.[4], [12]):

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[1] Let $\Gamma =< \beta_0, \ldots, \beta_l >$ and $\overline{\Gamma} =< \overline{\beta}_0, \ldots, \overline{\beta}_k >$ two semigroups given by the minimal set of generators. We define $\Gamma \leq \overline{\Gamma}$ iff $\Gamma = \overline{\Gamma}$ or there exists $i \leq \min(l, k)$ such that $\beta_0 = \overline{\beta}_0, \cdots, \beta_{i-1} = \overline{\beta}_{i-1}$ and $\beta_i < \overline{\beta}_i$.
In a deformation of a parametrization with semigroup Γ the semigroup is smaller or equal to Γ.

Characteristic $p = 0$	
Γ	Normal Form
$< 1 >$	$(t, 0)$
$< 2, 2k + 1 >$	(t^2, t^{2k+1})
$< 3, 3k + 1 >$	$(t^3, t^{3k+1} + t^{3l+2}),\ 1 \leq k \leq l < 2k - 1$ (t^3, t^{3k+1})
$< 3, 3k + 2 >$	$(t^3, t^{3k+2} + t^{3l+1}),\ 1 \leq k < l \leq 2k - 1$ (t^3, t^{3k+2})
$< 4, 5 >$	$(t^4, t^5 + t^7)$ (t^4, t^5)
$< 4, 6, 2k + 7 >$	$(t^4, t^6 + t^{2k+1}),\ k \geq 3$
$< 4, 7 >$	$(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ (t^4, t^7)

The main result of this paper is the following Theorem:

Theorem 3. *If the characteristic of K is greater than 0, we obtain in the following two tables the classification of simple parametrizations.*

Characteristic $p = 2$	
Γ	Normal Form
$< 1 >$	$(t, 0)$
$< 2, 2k + 1 >$	(t^2, t^{2k+1}) $(t^2 + t^{2m+1}, t^{2k+1})$, $0< m < k$
$< 3, 4 >$	(t^3, t^4) $(t^3, t^4 + t^5)$
$< 3, 5 >$	(t^3, t^5)
$< 3, 7 >$	(t^3, t^7) $(t^3, t^7 + t^8)$

Characteristic $p \geq 3$	
Γ	Normal Form
$< 1 >$	$(t, 0)$
$< 2, 2k+1 >$	(t^2, t^{2k+1})
$< 3, 3k+1 >$	$(t^3, t^{3k+1} + t^{3l+2}),\ 1 \leq k \leq l < 2k-1$ (t^3, t^{3k+1}) $k \leq \frac{cp+5}{3}$ with $c = p \mod 3$, $0 \leq c \leq 2$
$< 3, 3k+2 >$	$(t^3, t^{3k+2} + t^{3l+1}),\ 1 \leq k < l \leq 2k-1$ (t^3, t^{3k+2}) $k \leq \frac{cp+4}{3}$ with $c = 2p \mod 3$, $0 \leq c \leq 2$ additionally $(t^3 + t^4, t^5)$ if $p = 3$
$< 4, 5 >$	$(t^4, t^5 + t^7)$ (t^4, t^5) additionally $(t^4, t^5 + t^6)$ if $p = 5$
$< 4, 6, 2k+7 >$	$(t^4, t^6 + t^{2k+1}), k \geq 3$ and $p \neq 13$ $k = 3$ if $p = 3$ or $p = 5$ $k \leq 6$ if $p = 7$ $k \leq 12$ if $p = 11$ $k \leq \frac{p-9}{2}$ if $p \geq 17$
$< 4, 7 >$	$p \neq 7$ $(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ (t^4, t^7)

2. Some Classes of non Simple Curves

In this section we will prove that in any characteristic parametrizations with $n \geq 5$ or $n = 4$ and $m \geq 9$ are not simple. We will also see that in characteristic 3 parametrizations with semigroup $< 3, 7 >$ and in characteristic 7 parametrizations with semigroup $< 4, 7 >$ are not simple.

Lemma 1.

(1) $(t^4, t^9 + t^{10} + \sum_{i \geq 11} a_i t^i) \sim_{\mathcal{A}} (t^4, t^9 + t^{10} + \sum_{i \geq 11} b_i t^i)$ *implies* $a_{11} = b_{11}$.

(2) $(t^5, t^6 + t^8 + \sum_{i > 8} a_i t^i) \sim_{\mathcal{A}} (t^5, t^6 + t^8 + \sum_{i > 8} b_i t^i)$ *implies* $a_9 = b_9$.

Proof. Assume that $\varphi : K[[t]] \to K[[t]]$ is an automorphism defined by $\varphi(t) = t + \sum_{i \geq 2} c_i t^i$, $H, L \in K[[x, y]]$ with $H = H_1 x + H_2 y + H_3 x^2 + H_4 xy + \ldots$ $L = L_1 x + L_2 y + L_3 x^2 + L_4 xy + \ldots$ such that

(i) $\varphi(t)^4 = H(t^4, t^9 + t^{10} + \sum_{i>10} a_i t^i)$
(ii) $\varphi(t)^9 + \varphi(t)^{10} + \sum_{i>10} b_i \varphi(t)^i = L(t^4, t^9 + t^{10} + \sum_{i>10} a_i t^i)$.

From (i) obtain $H_1 = 1$ and $c_2 = c_3 = 0$ comparing the coefficients of t^4, t^5 and t^6. From (ii) we obtain $a_{11} = b_{11}$ looking at the coefficient of t^{11}. This proves (1). Similarly (2) can be proved. □

Lemma 2. *Let K be a field of characteristic 3 then the parametrizations with semigroup $< 3, 7 >$ and $< 5, 6 >$ are not simple.*

Proof. It is not difficult to see that

(1) $(t^3 + t^4 + \sum_{i\geq 5} a_i t^i, t^7) \sim_{\mathcal{A}} (t^3 + t^4 + \sum_{i\geq 5} b_i t^i, t^7)$ implies $a_5 = b_5$
(2) $(t^5, t^6 + \sum_{i\geq 8} a_i t^i) \sim_{\mathcal{A}} (t^5, t^6 + \sum_{i\geq 8} b_i t^i)$ and $a_9 = b_9 = 1$ implies $a_8 = b_8$.

The proof is similar to the proof of the previous lemma. □

Corollary 1. *Parametrizations with semigroup $< 4, 9 >$ and $< 5, 6 >$ are not simple.*

Proof. The corollary is a consequence of the lemma 1 and 2, since $(t^4, t^9 + \sum_{i\geq 10} a_i t^i) \sim_{\mathcal{A}} (t^4, t^9 + t^{10} + \frac{a_{11}}{a_{10}} t^{11} + ...)$ if $a_{10} \neq 0$ and $(t^5, t^6 + \sum_{i\geq 7} a_i t^i) \sim_{\mathcal{A}} (t^5, t^6 + t^8 + \frac{35a_7^3 - 63a_7a_8 + 27a_9}{27(12a_8 + 13a_7^2)^2} t^9 + ...)$ if $12a_8 + 13a_7^2 \neq 0$ and $p \neq 3$.
If $p = 3$ we use lemma 2. □

Lemma 3. *Let K be a field of characteristic 7 then the parametrizations with semigroup $< 4, 7 >$ are not simple.*

Proof. It is not difficult to see that $(t^4, t^7 + t^9 + \sum_{i\geq 10} a_i t^i) \sim_{\mathcal{A}} (t^4, t^7 + t^9 + \sum_{i\geq 10} b_i t^i)$ implies $a_{10} = b_{10}$. □

3. Curves with Semigroup $< 3, 3k + 1 >$ or $< 3, 3k + 2 >$

In this section we assume that the characteristic $p > 3$. The aim of this section is to find depending on the characteristic p a minimal k_0 such that $(t^3, t^{3k+1} + \sum_{i\geq 3k+2} a_i t^i)$ (with semigroup $< 3, 3k + 1 >$) resp. $(t^3, t^{3k+2} + \sum_{i\geq 3k+3} a_i t^i)$ (with semigroup $< 3, 3k + 2 >$) is not simple for all $k \geq k_0$. We will see that in case of semigroup $< 3, 3k + 1 >$ we obtain $k_0 = \frac{p+8}{3}$ if $p \equiv 1 \bmod 3$ and $k_0 = \frac{2p+8}{3}$ if $p \equiv 2 \bmod 3$. If the semigroup is $< 3, 3k + 2 >$ we obtain the minimal k_0 such that $(t^3, t^{3k+2} + \sum_{i\geq 3k+3} a_i t^i)$ is not simple for all $k \geq k_0$ as $k_0 = \frac{2p+7}{3}$ if $p \equiv 1 \bmod 3$ and $k_0 = \frac{p+7}{3}$ if $p \equiv 2 \bmod 3$.
We first consider parametrizations with semigroup $< 3, 3k + 1 >$. Let $(x(t), y(t))$ be a primitive parametrization with $ord_t x(t) = 3, ord_t y(t) = 3k + 1, k > 0$. As in characteristic 0 we obtain

$$(x(t), y(t)) \sim_{\mathcal{A}} (t^3, t^{3k+1}) \text{ or}$$
$$(x(t), y(t)) \sim_{\mathcal{A}} (t^3, t^{3k+1} + t^{3l+2} + \sum_{i>3l+2} a_i t^{3(l+i)+2}), k \leq l < 2k - 1.$$

We can compute the semigroup Γ of $K[[x(t), y(t)]]$ and obtain

$\Gamma =< 3, 3k+1 >= \{0,3,6,...,3k,3k+1,...,6k-2,6k,...\}$ with conductor [2] $6k$.

Lemma 4. *If* $p \nmid 3(l-k)+1$ *then*

$$(t^3, t^{3k+1}+t^{3l+2}+\sum_{i>3l+2} a_i t^{3(l+i)+2}) \sim_{\mathcal{A}} (t^3, t^{3k+1}+t^{3l+2}).$$

Proof. Let s be minimal with $a_s \neq 0$ and consider $\varphi(t) = t + \alpha t^{3s+1}$. We have

$$(t+\alpha t^{3s+1})^3 = t^3 + 3\alpha t^{3s+3} + 3\alpha^2 t^{6s+3} + \alpha^3 t^{9s+3}.$$
$$(t+\alpha t^{3s+1})^{3k+1} = t^{3k+1} + \alpha(3k+1)t^{3(k+s)+1} + \alpha^2 \binom{3k+1}{2} t^{3(k+2s)+1} + ... + \alpha^{3k+1} t^{(3k+1)(3s+1)}.$$
$$(t+\alpha t^{3s+1})^{3l+2} = t^{3l+2} + \alpha(3l+2)t^{3l+3s+2} + ... \ .$$

Since all exponents of $(t+\alpha t^{3s+1})^3$ are divisible by 3 there exist $H \in K[[x]]$ such that $H((t+\alpha t^{3s+1})^3) = t^3$. Since all exponents of $(t+\alpha t^{3s+1})^{3k+1}$ are congruent to 1 modulo 3 there exist $L \in K[[x,y]]$ such that $L((t+\alpha t^{3s+1})^3, (t+\alpha t^{3s+1})^{3k+1}) = t^{3k+1}$, $L = y - \alpha(3k+1)x^s y + ...$ .We obtain

$$L(\varphi(t)^3, \varphi(t)^{3k+1} + \varphi(t)^{3l+2} + \Sigma a_i \varphi(t)^{3(l+i)+2}) =$$
$$t^{3k+1} + t^{3l+2} + (\alpha(3(l-k)+1) + a_s)t^{3(l+s)+2} + ... \ .$$

Now choose $\alpha = -\frac{a_s}{3(l-k)+1}$ to obtain

$$(t^3, t^{3k+1}+t^{3l+2}+\sum_{i>3l+2} a_i t^{3(l+i)+2}) \sim_{\mathcal{A}} (t^3, t^{3k+1}+t^{3l+2}+\sum_{i>s} b_i t^{3(l+i)+2}).$$

The lemma follows using induction. $\square$

Lemma 5. *If* $p \mid 3(l-k)+1$ *and* $l \leq 2k-3$ *then* $(t^3, t^{3k+1}+t^{3l+2})$ *is not simple.*

Proof. Assume that $\varphi(t) = t + \sum_{i>1} a_i t^i$, $H, L \in < x, y > K[[x,y]]$ are given such that $\det(\frac{\partial(H,L)}{\partial(x,y)}(0,0)) \neq 0$ and

$$\varphi(t)^3 = H(t^3, t^{3k+1}+t^{3l+2}+at^{3l+5})$$
$$\varphi(t)^{3k+1} + \varphi(t)^{3l+2} + b\varphi(t)^{3l+5} = L(t^3, t^{3k+1}+t^{3l+2}+at^{3l+5}).$$

The first equation implies $a_2 = a_3 = a_5 = \cdots = a_{3k-4} = a_{3k-3} = 0$. This implies that $\varphi(t) = t + a_4 t^4 + \cdots$ and the first possible term with exponent not congruent 1 mod 3 is t^{3k-1}. This implies that the first possible term with exponent in $\varphi(t)^{2k+1}$ not congruent to 1 mod 3 is t^{6k-1}. Now $\varphi(t)^{3l+2} = t^{3l+2} + a_4(3l+2)t^{3l+5} + \cdots$. This implies that the coefficient of t^{3l+5} in $\varphi(t)^{3k+1} + \varphi(t)^{3l+2} + b\varphi(t)^{3l+5}$ is $b + a_4(3l+2)$ and the coefficient of t^{3k+4} is $(3k+1)a_4$. Now consider the coefficient of t^{3l+5} in $L(t^3, t^{3k+1}+t^{3l+2}+at^{3l+5})$. Obviously $L = y + \sum_{v \geq k+1} a_v x^v + \sum_{v \geq 1} b_v x^v y + y^2 L_2$ since $L(t^3, t^{3k+1}+t^{3l+2}+at^{3l+5})$ has order $k+1$. Now

$$L(t^3, t^{3k+1}+t^{3l+2}+at^{3l+5}) =$$
$$t^{3k+1}+t^{3l+2}+at^{3l+5} + \sum_{v \geq k+1} a_v t^{3v} + \sum_{v \geq 1} b_v t^{3v}(t^{3k+1}+t^{3l+2}+at^{3l+5}) \bmod t^{6k}.$$

This implies that b_1 is the coefficient of t^{3k+4} and $a + b_1$ is the coefficient of t^{3l+5}. Therefore $b_1 = (3k+1)a_4$ and $b + (3l+2)a_4 = a + (3k+1)a_4$. But $3l+2 = 3k+1$ in K implies $a = b$. Now obviously

[2]The conductor of a semigroup is the minimum of all c in the semigroup such that all integers greater than c are in the semigroup.

$$(t^3, t^{3k+1} + t^{3l+2} + at^{3l+5}) \sim_{\mathcal{A}} (t^3, t^{3k+1} + t^{3l+2} + r^4 at^{3l+5})$$

for $r \in K, r^{3(l-h)+1} = 1$ induced by $t \to rt$. This implies

$$(t^3, t^{3k+1} + t^{3l+2} + at^{3l+5}) \sim_{\mathcal{A}} (t^3, t^{3k+1} + t^{3l+2} + bt^{3l+5})$$

if and only if $a = r^4 b$ for $r \in K$, $r^{3(l-k)+1} = 1$, i.e. the parametrization $(t^3, t^{3k+1} + t^{3l+2})$ is not simple. □

Lemma 6. *If $l > 2k - 3$ then $(t^3, t^{3k+1} + t^{3l+2} + \sum_{i>3l+2} a_i t^i) \sim_{\mathcal{A}} (t^3, t^{3k+1} + t^{3l+2})$.*

Proof. If $p \nmid 3(l-k)+1$ then the lemma is consequence of lemma 4. If $l = 2k-1$ the lemma follows from Theorem 1 (2) and the fact that $3l + 3 = 6k$ is the conductor of the semigroup. Assume now that $l = 2k - 2$ and $p \mid 3(l - k) + 1 = 3k - 5$. First of all

$$(t^3, t^{3k+1} + t^{3l+2} + \sum_{i>3l+2} a_i t^i) \sim_{\mathcal{A}} (t^3, t^{3k+1} + t^{6k-4} + a_{6k-1} t^{6k-1})$$

since i is in the semigroup for $i > 3l + 2$ and $i \neq 6k - 1$. Now consider the automorphism $\varphi(t) = t + \alpha t^{3k-1}$. We obtain

$$\varphi(t)^3 = t^3 + 3\alpha t^{3k+1} + 3\alpha^2 t^{6k-1} mod\ (t^{6k}),$$
$$\varphi(t)^{3k+1} = t^{3k+1} + \alpha(3k+1)t^{6k-1} mod\ (t^{6k}),\ \varphi(t)^{3l+2} = t^{3l+2} \bmod (t^{6k}).$$

This implies

$$\varphi(t^{3k+1} + t^{6k-4} + a_{6k-1}t^{6k-1}) = t^{3k+1} + t^{6k-4} + (\alpha(3k+1) + a)t^{6k-1} \bmod (t^{6k}).$$

Since $p \mid 3k - 5$ and $p > 3$ we have $p \nmid 3k + 1$ and we can choose $\alpha = -\frac{a}{3k+1}$ and obtain

$$(t^3, t^{3k+1} + t^{6k-4} + \sum_{i>6k-4} a_i t^i) \sim_{\mathcal{A}} (t^3 + 3\alpha t^{3k+1} + 3\alpha^2 t^{6k-1}, t^{3k+1} + t^{6k-4}) \sim_{\mathcal{A}}$$
$$(t^3 - 3\alpha t^{6k-4} + 3\alpha^2 t^{6k-1}, t^{3k+1} + t^{6k-4}).$$

Now consider the map $\psi(t) = t + t^{6k-6} + \alpha t^{6k-3}$. We obtain

$$\psi(t^3 - 3\alpha t^{6k-4} + 3\alpha^2 t^{6k-1}) = t^3 \bmod (t^{6k}),\ \psi(t^{3k+1} + t^{6k-4}) = t^{3k+1} + t^{6k-4} mod\ (t^{6k}).$$

This proves the lemma. □

Remark 2. *Using same arguments as in proof of lemma 6 we can prove that $(t^3, t^{3k+1} + at^{6k-1}) \sim_{\mathcal{A}} (t^3, t^{3k+1})$ if $p \nmid 3k + 1$.*

Corollary 2. *The parametrizations $(t^3, t^{p+9} + t^{2p+9})$ resp. $(t^3, t^{2p+9} + t^{4p+9})$ are not simple if $p \equiv 1 \bmod 3$ resp. $p \equiv 2 \bmod 3$.*

Proof. In the first case $k = \frac{p+8}{3}$ and $l = \frac{2p+7}{3} = 2k - 3$ and $3(\frac{2p+7}{3} - \frac{p+8}{3}) + 1 = p$, the results follows from lemma 5. Similarly in the second case $k = \frac{2p+8}{3}, l = \frac{4p+7}{3}$ and $3(l - k) + 1 = 2p$. □

Corollary 3.

(1) *For $p \equiv 1 \bmod 3$ the parametrizations $(t^3, t^{3k+1} + \sum_{i>3k+1} a_i t^i)$ are simple if and only if $k \leq \frac{p+5}{3}$.*
(2) *For $p \equiv 2 \bmod 3$ the parametrizations $(t^3, t^{3k+1} + \sum_{i>3k+1} a_i t^i)$ are simple if and only if $k \leq \frac{2p+5}{3}$.*

Proof. We will proof the first case, the second case is similar. If $3k-8 \geq p$ then $(t^3, t^{p+9}+t^{2p+9})$ is in a deformation of $(x(t), y(t))$ and therefore not simple. If $3k-8 < p$ then $p \nmid 3(l-h)+1$ for all l, $k \leq l < 2k-2$. The corollary is a consequence of lemma 6. □

Similarly we can treat the case of the semigroup $< 3, 3k+2 >$. The conductor of the semigroup is $6k+2$. We obtain the following lemmas.

Lemma 7. *Let* $1 \leq k < l \leq 2k-1$. *If* $p \nmid 3(l-k)-1$ *then* $(t^3, t^{3k+2}+t^{3l+1}+\sum a_i t^{3(l+i)+1}) \sim_{\mathcal{A}} (t^3, t^{3k+2}+t^{3l+1})$.

Lemma 8. *If* $p \mid 3(l-k)-1$ *and* $l \leq 2k-2$ *then* $(t^3, t^{3k+2}+t^{3l+1})$ *is not simple.*

Corollary 4. *The parametrizations* $(t^3, t^{p+9}+t^{2p+9})$ *resp.* $(t^3, t^{2p+9}+t^{4p+9})$ *are not simple if* $p \equiv 2 \mod 3$ *resp.* $p \equiv 1 \mod 3$.

Corollary 5.

(1) *For* $p \equiv 1 \mod 3$ *the parametrizations* $(t^3, t^{3k+2}+\sum_{i>3k+2} a_i t^i)$ *are simple if and only if* $k \leq \frac{2p+4}{3}$.
(2) *For* $p \equiv 2 \mod 3$ *the parametrizations* $(t^3, t^{3k+2}+\sum_{i>3k+2} a_i t^i)$ *are simple if and only if* $k \leq \frac{p+4}{3}$.

4. Parametrizations with semigroup $< 4, 6, 2k+7 >$

In this section we assume that the characteristic $p \geq 3$. Let $(x(t), y(t))$ be a parametrization such that

$$dim_K K[[t]]/K[[x(t), y(t)]] < \infty \text{ and } ord_t x(t) = 4,\ ord_t y(t) = 6.$$

Assume that the semigroup $\Gamma = < 4, 6, 2k+7 >$.

The aim of this section is to find depending on the characteristic a minimal k_0 such that $(t^4, t^6+t^{2k+1}+\sum_{i \geq 2k+3} a_i t^i)$ is not simple for all $k \geq k_0$.

Lemma 9. *If* $p \nmid 2k+7$ *then* $(x(t), y(t)) \sim_{\mathcal{A}} (t^4, t^6+t^{2k+1})$.

Proof. First of all it is not difficult to see that

$$(x(t), y(t)) \sim_{\mathcal{A}} (t^4, t^6+t^{2k+1}+\sum_{i \geq 2k+3} a_i t^i).$$

Since $2k+10$ is the conductor of Γ, the even integers ≥ 4 are in Γ and $2k+9+4-6 = 2k+7 \in \Gamma$, we obtain using Theorem 1 (2)

$$(x(t), y(t)) \sim_{\mathcal{A}} (t^4, t^6+t^{2k+1}+a_{2k+3}t^{2k+3}+a_{2k+5}t^{2k+5}).$$

Consider the map $\varphi(t) = t+\alpha(t^3+t^{2k-2})$. we obtain

$$\varphi(t)^4 = t^4+4\alpha(t^6+t^{2k+1}+a_{2k+3}t^{2k+3})+6\alpha^2(t^8+2t^{2k+3})+4\alpha^3 t^{10}+\alpha^4 t^{12} \mod t^{2k+4}$$
$$\varphi(t)^6 = t^6+6\alpha(t^8+t^{2k+3})+15\alpha^2 t^{10}+20\alpha^3 t^{12}+15\alpha^4 t^{14}+6\alpha^5 t^{16}+\alpha^6 t^{18} \mod t^{2k+4}$$
$$\varphi(t)^{2k+1} = t^{2k+1}+\alpha(2k+1)t^{2k+3} \mod t^{2k+4}.$$

This implies that

$$(t^4, t^6 + t^{2k+1} + a_{2k+3}t^{2k+3} + a_{2k+5}t^{2k+5}) \sim_{\mathcal{A}}$$
$$(t^4, t^6 + t^{2k+1} + (\alpha(2k+7) + a_{2k+3})t^{2k+3} + t^{2k+4}b),\ b \in K[[t]].$$

Since $p \nmid 2k+7$ we can choose $\alpha = -\frac{a_{2k+3}}{2k+7}$ and obtain

$$(t^4, t^6 + t^{2k+1} + a_{2k+3}t^{2k+3} + a_{2k+5}t^{2k+5}) \sim_{\mathcal{A}} (t^4, t^6 + t^{2k+1} + \overline{a}_{2k+5}t^{2k+5}).$$

Now we consider the map $\psi(t) = t + \beta t^5$ and obtain similarly

$$(t^4, t^6 + t^{2k+1} + \overline{a}_{2k+5}t^{2k+5}) \sim_{\mathcal{A}} (t^4, t^6 + t^{2k+1} + (\beta(2k+7) + \overline{a}_{2k+5})t^{2k+5} + t^{2k+6}c),$$
$$c \in K[[t]].$$

Again we choose $\beta = -\frac{\overline{a}_{2k+5}}{2k+7}$ and obtain $(x(t), y(t)) \sim_{\mathcal{A}} (t^4, t^6 + t^{2k+1})$. □

Lemma 10. *If $p \mid 2k+7$ and $(t^4, t^6 + t^{2k+1} + \sum_{i>2k+1} a_i t^i) \sim_{\mathcal{A}} (t^4, t^6 + t^{2k+1} + \sum_{i>2k+1} b_i t^i)$, then $a_{2k+3} = b_{2k+3}$. Especially $(t^4, t^6 + t^{2k+1})$ is not simple.*

Proof. We may assume that $a_{2k+2} = b_{2k+2} = 0$. Let $\varphi(t) = t + \sum_{i\geq 2} c_i t^i \in K[[t]]$ and $H, L \in <x, y> K[[x, y]]$, $H = \sum_{i,j} H_{ij}x^i y^j$ and $L = \sum_{i,j} L_{ij}x^i y^j$ such that
(i) $\varphi(t)^4 = H(t^4, t^6 + t^{2k+1} + \sum_{i\geq 2k+3} a_i t^i)$.
(ii) $\varphi(t)^6 + \varphi(t)^{2k+1} + \sum_{i\geq 2k+3} b_i\varphi(t)^i = L(t^4, t^6 + t^{2k+1} + \sum_{i\geq 2k+3} a_i t^i)$.

We obtain from (i) $H_{10} = 1$, $H_{01} = 4c_3$ and $c_2 = 0$. (ii) implies that $L_{10} = 0$, $L_{01} = 1$ and $L_{20} = 6c_3$. Moreover we obtain $c_i = 0$ for all even $i \leq 2k-3$. This implies that
$\varphi(t)^6 + \varphi(t)^{2k+1} + \sum_{i\geq 2k+3} b_i\varphi(t)^i = t^6 + ... + ((2k+1)c_3 + 6c_{2k-2} + b_{2k+3})t^{2k+3} + ...$
and
$L(t^4, t^6 + t^{2k+1} + \sum_{i\geq 2k+3} a_i t^i) = t^6 + ... + a_{2k+3}t^{2k+3} + ...$
We obtain $a_{2k+3} = b_{2k+3} + (2k+1)c_3 + 6c_{2k-2}$. Now we use again (i) looking at the coefficients of t^{2k+1} to see that $4c_{2k-2} = H_{01}$. This implies that $a_{2k+3} = b_{2k+3} + (2k+7)c_3 = b_{2k+3}$ since the characteristic $p \mid 2k+7$. □

The following corollary gives the minimal value of k such that $(t^4, t^6 + t^{2k+1})$ is not simple.

Corollary 6. *The minimal k_0 such that $(t^4, t^6 + t^{2k+1} + \sum_{i\geq 2k+3} a_i t^i)$ is not simple for $k \geq k_0$ depends as follows on the characteristic p.*

1. *If $p = 3$ or $p = 5$ then $k_0 = 4$.*
2. *If $p = 7$ then $k_0 = 7$.*
3. *If $p = 11$ then $k_0 = 13$.*
4. *If $p \geq 13$ then $k_0 = \frac{p-7}{2}$.*

Proof. According to lemma 10 we have to find the minimal $k \geq 3$ such that $p \mid 2k+7$. Obviously k is minimal if $p = 2k+7$. This is possible for $p \geq 13$. The remaining cases can be checked easily. □

5. Characteristic 2

In this section we assume that the characteristic of K is $p = 2$. We will prove that parametrizations with $n \geq 4$ or $n = 3$ and $m \geq 8$ are not simple and give normal forms for the remaining cases.

Lemma 11. *Let $x(t), y(t) \in K[[t]]$ such that $dim_K K[[t]]/K[[x(t), y(t)]] < \infty$. Assume $ord_t x(t) = 2$, $ord_t y(t) = m, m > 2$ odd. Then $(x(t), y(t)) \sim_{\mathcal{A}} (t^2, t^m)$ or there exist k, $2 < k < m$ and k odd such that $(x(t), y(t)) \sim_{\mathcal{A}} (t^2 + t^k, t^m)$.*

Proof. If $x(t) \in K[[t^2]]$, we obtain $(x(t), y(t)) \sim_{\mathcal{A}} (t^2, \overline{y}(t))$ and $\text{ord}_t \overline{y}(t) = m$. Then obviously $(x(t), y(t)) \sim_{\mathcal{A}} (t^2, t^m)$. We may assume now that $x(t) = t^2 + \sum_{i>2} a_i t^i$ and k odd and minimal such that $a_k \neq 0$. If $k \geq m$, we obtain $(x(t), y(t)) \sim_{\mathcal{A}} (t^2, t^m)$. If $k < m$, we obtain $(x(t), y(t)) \sim_{\mathcal{A}} (t^2 + \sum_{i\ even} a_i t^i + t^k, t^m)$. Now there is a power series $H(x)$ such that $H(t^2 + \sum_{i\ even} a_i t^i + t^k) = t^2 + t^k$ and we obtain $(x(t), y(t)) \sim_{\mathcal{A}} (t^2 + t^k, t^m)$. □

Corollary 7. *Parametrization $(x(t), y(t))$ with $ord_t x(t) = 2$ and semigroup $< 2, m >$, m odd, are simple with normal form (t^2, t^m) or $(t^2 + t^k, t^m)$, $3 \leq k < m$, k odd.*

Lemma 12. *Parametrizations with semigroup $< 3, 8 >$ or semigroup $< 4, 5 >$ are not simple.*

Proof. We have to prove that

(1) $(t^3, t^8 + t^{10} + \sum_{i \geq 11} a_i t^i) \sim_{\mathcal{A}} (t^3, t^8 + t^{10} + \sum_{i\geq 11} b_i t^i)$ implies $a_{11} + a_{13} = b_{11} + b_{13}$
(2) $(t^4, t^6 + \sum_{i\geq 7} a_i t^i) \sim_{\mathcal{A}} (t^4, t^6 + \sum_{i \geq 7} b_i t^i)$ implies $a_7 = b_7$.

This can be proved similarly to the corresponding cases before. □

As an example we show how (2) in lemma 12 could also checked by a computer. Consider the following Singular code (cf. [5], [9]):

```
ring R=(2,a,b,c,d,e,f,g,H1,H2,H3,H4,H5,H6,H7,H8,H9,L1,L2,L3,
       L4,L5,L6,L7,L8,L9,u,v),(x,y,t),ds;
poly p=t+a*t2+b*t3+c*t4+d*t5+e*t6;
poly H=H1*x+H2*y+H3*x2+H4*xy+H5*y2+H6*x3+H7*x2y+H8*xy2+H9*x4;
poly L=L1*x+L2*y+L3*x2+L4*xy+L5*y2+L6*x3+L7*x2y+L8*xy2+L9*x4;

jet(p^4+p^6+v*p^7-subst(H,x,t4+t6+u*t7,y,t5),7);

(H1+1)*t^4+(H2)*t^5+(H1+1)*t^6+(H1*u+v)*t^7

This implies H1=-1 and u=v.
```

Lemma 13. *Parametrizations with semigroup $< 3, 4 >$, $< 3, 5 >$ or $< 3, 7 >$ have normal forms (t^3, t^4), $(t^3, t^4 + t^5)$, (t^3, t^5), (t^3, t^7) or $(t^3, t^7 + t^8)$.*

Proof. The proof is similar to the corresponding proof in section 3. □

Corollary 8. *Parametrization* $(x(t), y(t))$ *with semigroup* $< 3, 4 >$, $< 3, 5 >$ *or* $< 3, 7 >$ *are simple.*

6. Proof of the Classification

In this section we will prove Theorem 3.

Lemma 14. *Let* $(x(t), y(t))$ *be a parametrization with semigroup* Γ *generated by at most* 3 *elements. Then the semigroups of parametrizations in a deformation of* $(x(t), y(t))$ *are smaller or equal to* Γ.

Proof. Let $x(t) = \sum_{i \geq n} a_i t^i$ and $y(t) = \sum_{i \geq m} b_i t^i$, $a_n = b_m = 1$. Then $n = ord_t x(t)$ and $m = ord_t y(t)$ are the first two elements[3] in the ordered set of minimal generators of Γ. If n and m are coprime then $\Gamma =< n, m >$. If $c := gcd(n, m) > 1$ then Γ is generated by three elements. To find third generator we have to compute a minimal sagbi basis of $K[[x(t), y(t)]]$. This sagbi basis has three elements, $x(t)$, $y(t)$, $w(t)$ and $\Gamma =< n, m, r >$ with $r = ord_t w(t)$. According to the algorithm to compute sagbi bases (cf. [10]) we obtain $w(t)$ as the final reduction [4] of $x(t)^{\frac{m}{c}} - y(t)^{\frac{n}{c}}$ with respect to $x(t)$ and $y(t)$. Let $w(t) = \sum_{i \geq r} c_i t^i$, $c_r \neq 0$. The coefficients $c_i = c_i(a, b)$ are polynomials in the coefficients $a = \{a_i\}$ and $b = \{b_j\}$.
Let $(X(z, t), Y(z, t))$ be a deformation of $(x(t), y(t))$, i.e. $X(z, t) = \sum_{i \geq \bar{n}} A_i t^i$, $Y(z, t) = \sum_{i \geq \bar{m}} B_i t^i \in C[[t]]$ with $C = K[z]/J$ an affine K-algebra, $z = (z_1, \dots, z_s)$, such that $X(0, t) = x(t)$ and $Y(0, t) = y(t)$. Let $U \subset V(J) \subset K^s$ be defined [5] by $A_n \neq 0$, $A_{\bar{n}} \neq 0$,$B_m \neq 0$, $B_{\bar{m}} \neq 0$. Since $A_n(0) = 1$ and $B_m(0) = 1$ we have $\bar{n} \leq n$ and $\bar{m} \leq m$. If one of the inequalities is strict choose $a \in U$ and consider the parametrization $(X(a, t), Y(a, t))$ with semigroup $\bar{\Gamma}$. The semigroup $\bar{\Gamma}$ of the deformation is smaller than Γ, the semigroup of $(x(t), y(t))$. We may assume now that $\bar{n} = n$ and $\bar{m} = m$. If n and m are coprime we are done since $\Gamma =< n, m >= \bar{\Gamma}$. If n and m are not coprime we have to find as above a sagbi basis to obtain generators of $\bar{\Gamma}$. As above we can compute the next element $W(z, t)$ in the sagbi basis by the normal form of $B_m^{\frac{n}{c}} X(z, t)^{\frac{m}{c}} - A_n^{\frac{m}{c}} Y(z, t)^{\frac{n}{c}}$ with respect to $X(z, t)$ and $Y(z, t)$. This can be done simultaneously over the localization of C by $A_n B_m$. Let $W(z, t) = \sum_{i \geq s} C_i t^i$, $C_s \neq 0$. We define U by $A_n \neq 0$, $B_m \neq 0$ and $C_s \neq 0$. Choose $a \in U$ and consider the parametrization $(X(a, t), Y(a, t))$ with semigroup $\bar{\Gamma}$. The semigroup $\bar{\Gamma}$ of the deformation will be smaller than Γ since $W(0, t)$ is either the normal form of $x(t)^{\frac{m}{c}} - y(t)^{\frac{n}{c}}$ with respect to $x(t)$ and $y(t)$ or an element occuring during the normal form computation. This implies that $s = ord_t W(a, t) \leq r$ and proves the lemma. □

Proof. (of Theorem 3)
Let us first consider the case of characteristic 2. Lemma 12 implies that the candidates for simple parametrizations must have semigroups smaller than $< 3, 8 >$ and

[3] We assume as always that $n < m$ and $n \nmid m$.

[4] also called sagbi normal form

[5] According to the definition of a deformation we are allowed to choose U with the property $0 \in U$ as small as we need.

$< 4, 5 >$. Normal forms for these candidates are given in Corollary 7 and Lemma 13. Lemma 14 implies that those parametrizations are simple.
Now assume that the characteristic $p \geq 3$. Lemma 1 implies that the candidates for simple parametrization must have semigroups smaller than $< 4, 9 >$ and $< 5, 6 >$. In case of semigroup $< 2, 2k + 1 >$ we may assume (Theorem 1 (1)) that the parametrization is of the form $(t^2, y(t))$ and obtain the normal form of the parametrization (t^2, t^{2k+1}) similarly to Lemma 11. The cases of the semigroups $< 3, 3k + 1 >$ resp. $< 3, 3k + 2 >$ are consequences of Corollary 3 resp. Corollary 5 in case of characteristic $p \geq 5$. The case of characteristic $p = 3$ is a consequence of Lemma 2. The case of the semigroup $< 4, 6, 2k + 7 >$ is a consequence of Corollary 6. The semigroups $< 4, 5 >$ resp. $< 4, 7 >$ behave in characteristic $p \neq 5$ resp. $p \neq 7$ as in charcteristic zero. The case of characteristic $p = 7$ is a consequence of Lemma 3. In case of characteristic $p = 5$ we obtain as expected an additional case. □

References

[1] Arnold, V.I.: Normal form of functions near degenerate critical points. Russian Math. Survays 29, 10-50 (1995).

[2] Boubakri, Y., Greuel, G-M., Markwig, T.: Invariants of hypersurface singularities in positive characteristic. Rev. Mat. Complut. 25, No. 1, 61-85 (2012).

[3] Boubakri, Y., Greuel, G-M., Markwig, T.: Normal forms of hypersurface singularities in positive characteristic. Mosc. Math. J. 11, No. 4, 657-683 (2011).

[4] Bruce, J.W., Gaffney, T.J.: Simple singularities of mappings $(C, 0) - > (C^2, 0)$. J. London Math. Soc. (2) 26 , 465-474 (1982).

[5] Decker, W., Greuel, G.-M., Pfister, G.: Schönemann, H.: Singular 4-1-0 – A computer algebra system for polynomial computations. `http://www.singular.uni-kl.de` (2017).

[6] Gibson, C.G., Hobbs, C.A.:Simple Singularities of Space Curves. Math.Proc. Comb.Phil.Soc. 113, 297-310 (1993).

[7] Greuel, G.-M., Kröning, H.: Simple Singularities in Positive Characteristic, Math. Z. 203, 339-354 (1990).

[8] Greuel, G.-M., Hong Duc Nguyen.: Right simple singularities in positive characteristic. J. Reine Angew. Math. 712, 81-106 (2016).

[9] Greuel, G.-M., Pfister, G.: A Singular Introduction to Commutative Algebra. Second edition, Springer (2007).

[10] Hefez, A., Hernandes, M.E.:Standard bases for local rings of branches and their modules of differentials. Journal of Symbolic Computation 42, 178-191 (2007).

[11] Hefez, A., Hernandes, M.E.:The Analytic Classification of Plane Branches. Bull. Lond. Math. Soc.43. 2, 289-298 (2011).

[12] Mehmood, K., Pfister, G.: Unimodal Singularities of Parametrized Plane Curves. Bull. Math. Soc. Sci. Math. Roumanie 60(108) 4, 417-424 (2017).

[13] Zariski,O.: The Moduli Problem for Plane Branches (with an appendix by Bernard Teissier). University Lecture Series 39. AMS (2006).

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