Maximal Cohen–Macaulay modules over singularities of type $\mathbf{x^t + y^3}$

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Introduction

Let K be a field of characteristic $\neq 3$, $K[[X]]$ be the formal power series ring in the variables $X = (X_1, \ldots, X_n)$, f a non–zero, non–invertible formal power series, $R := K[[X]]/(f)$, $R' := K[[X,Y]]/(f + Y^3)$ and $\tilde{R} = R[Y]/(Y^2)$.

By [Po], or [HP, 2.8], every maximal Cohen–Macaulay (in short MCM) R' module N is a direct summand of the first szyzgy $\Omega^1_{R'}(T)$ of an infinitesimal deformation (or a lifting to $\widetilde{R}$ cf. [ADS]) T $(T = N/Y^2N!)$ of a maximal Cohen–Macaulay R module M $(M = N/YN!)$. If MCM(R) is known then we may describe MCM(R') using this result. The purpose of this paper is to show that this can be really a way to describe MCM(R'), when $n = 1$ and $f = X^t$. Certainly, when $t \leq 5$ then $X^t + Y^3$ defines a simple singularity and MCM(R') is already known (see for example [GK], ([Yo, Ch. 9]), but if $t \geq 6$ then nothing seems to be known except the description of A. Schappert [Sc] of the maximal Cohen–Macaulay R' modules of rank one for some special t.

Let M be a finitely generated R module, which has no free direct summands. Then $M \cong \bigoplus_{i=1}^{t-1} P_i^{d_i}$, $P_i := K[[X]]/(X^i)$, $1 \leq i < t$ for some integers $d_i \geq 0$. What are the maximal Cohen–Macaulay R' modules N (if any?) such that $N/YN \cong M$? If the d_i are all zero but one, then the problem is completely solved in [HP, 4.2], [EPP, 3.1], but, in general, this seems to be a very difficult problem. By [HP, 3.1], [EPP, 1.7] it is almost enough (see Section 5) to describe just all indecomposable infinitesimal deformations which are liftable to $R'/(Y^4)$ (see [ADS], [HP], or here 1.3), but this is not at all easy. For this aim we develop some important tools as Theorem 1.2, Theorem 1.6, Proposition 2.3. An illustration on the cases $t = 3, 4$ is given in Section 3. In the Sections 4 and 5 we try to describe the maximal Cohen–Macaulay R' modules N, such that $N/YN \cong P_i^{d_i} \oplus P_j^{d_j}$ for some $1 \leq i < j < t$. When $i = 1$, $j = t - 1$ the answer is complete by Theorem 5.2. When $t > 6$ then Proposition 5.3 and Theorem 5.4 give also answers when $j = t - 2$ and either $i = 2$, or $i = 1$ (or chancing N by $\Omega^1_{R'}(N)$, $i = 2$, $j = t - 1$).

In order to test whether the infinitesimal deformations (see 2.3, 4.3), respectively maximal Cohen–Macaulay R' modules are really indecomposable, we used a criterion for indecomposability of a module presented in Section 6. It is based on the following remark: if a module M is decomposable, then a presentation matrix A_0 of M has a block form $\left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$. Given any other presentation matrix A of M, if the modules generated by the entries of A belonging to the rows of B, respectively C, are minimal with input to those of all presentation matrices, then this matrix may be transformed by a standard base computation of its columns to a block form. Using this property, it seems possible to obtain an algorithmic procedure splitting modules over local rings.

Another result (see theorem 5.1) shows that for every maximal Cohen–Macaulay R' module N we may choose for $N \oplus \Omega^1_{R'}(N)$ a matrix factorization (see [Ei], [PP]) whose entries are polynomials in X, Y of degree $t - 1$.

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1 Infinitesimal deformations of modules

Let K be a field with char $K \neq 3$, X, Y two variables, $R := K[X]/(X^t)$, $\widetilde{R} := R[Y]/(Y^2)$, $R' := K[[X,Y]]/(X^t + Y^3)$. the non–free indecomposable (finitely generated) R modules are $P_i := K[[X]]/(X^i)$, $1 \leq i < t$. The reduced matrix factorization of P_i is (X^i, X^{t-i}) and P_{t-i} is the first syzygy of P_i.

Let M be an R module, say $M = \overset{t-1}{\underset{i=1}{\oplus}} P_i^{d_i}$ for some non–negative integers d_i, $1 \leq i < t$. Then

$$(\varphi, \psi) := \left(\begin{pmatrix} XI_{d_1} & & 0 \\ & \ddots & \\ 0 & & X^{t-1} I_{d_{t-1}} \end{pmatrix}, \begin{pmatrix} X^{t-1} I_{d_1} & & 0 \\ & \ddots & \\ 0 & & XI_{d_{t-1}} \end{pmatrix} \right)$$

is a reduced matrix factorization of M. Let T be an $\widetilde{R}$ module which deforms M to $\widetilde{R}$ (in other words, T is a lifting of M to $\widetilde{R}$ (see [ADS])). By [HP, §3] (see also [Ru], [PP, 2.8]) T corresponds to a reduced matrix factorization $(\varphi + Y\alpha,\ \psi - Y\beta)$, where $\alpha, \beta \in M(d \times d, K[[X]])$, $d = \sum_{i=1}^{t-1} d_i$ with $\varphi\psi = \varphi\beta$.

Lemma 1.1 ([EPP, 2.2, 2.3])

(i) If U, V are two invertible d matrices over $K[[X]]$ such that $U\varphi V = \varphi$, then $\alpha' := U\alpha V$, $\beta' := V^{-1}\beta U^{-1}$ gives also a matrix factorization of T.

(ii) If $\alpha'' := \alpha + \eta\varphi + \varphi w$, $\beta'' := \beta + w\psi + \psi\eta$ for two d matrices η, w over $K[[X]]$, then (α', β') gives also a matrix factorization of T.

Theorem 1.2
α, β can be chosen such that their entries are polynomials of degree $\leq t-2$. Moreover, α, β can be chosen to have the following form:

$$\alpha = \left(\begin{array}{c|c|c|c|c} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,t-2} & \alpha_{1,t-1} \\ \hline \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,t-2} & X\beta_{2,t-1} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \alpha_{t-1,1} & X\beta_{t-1,2} & \cdots & X^{t-3}\beta_{t-1,t-2} & X^{t-2}\beta_{t-1,t-2} \end{array} \right)$$

$$\beta = \left(\begin{array}{c|c|c|c|c} X^{t-2}\alpha_{11} & X^{t-3}\alpha_{12} & \cdots & X\alpha_{1,t-2} & \alpha_{1,t-1} \\ \hline X^{t-3}\alpha_{21} & X^{t-4}\alpha_{22} & \cdots & \cdots & \beta_{2,t-1} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \alpha_{t-1,1} & \beta_{t-1,2} & \cdots & \beta_{t-1,t-2} & \beta_{t-1,t-1} \end{array} \right)$$

where α_{ij} is a $d_i \times d_j$ matrix, $1 \leq i, j < t$, $i + j \leq t$ (respectively β_{rs} is a $d_r \times d_s$ matrix, $1 \leq r, s < t$, $r + s > t$), whose entries are polynomials of degree $< \min\{i, j\}$ (respectively $< \min\{t - r, t - s\}$).

Proof: Let $M = \overset{t-1}{\underset{i=1}{\oplus}} P_i^{d_i}$ for some integers $d_i \geq 0$ with $\sum_{i=1}^{t-1} d_i = d$. Clearly (φ, ψ) is a matrix factorization of X^t corresponding to M and T is given by $(\varphi + Y\alpha, \psi - Y\beta)$ for two d matrices α, β over $K[[X]]$ with $\alpha\psi = \varphi\beta$. Express α, β in blocks $(\alpha_{ij})_{1\leq i,j<t}, (\beta_{rs})_{1\leq r,s<t}$ and note that $X^{t-j}\alpha_{ij} = X^i\beta_{ij}$ for all $1 \leq i, j < t$ since $\alpha\psi = \varphi\beta$. Thus, $\beta_{ij} = X^{t-i-j}\alpha_{ij}$ if $i + j \leq t$ and $\alpha_{rs} = X^{r+s-t}\beta_{rs}$ if $r + s \geq t$.

Let (φ) be the two–sided ideal of $M(d \times d, K[[X]])$ generated by φ. The ideal (φ) contains exactly the matrices

$$a = \left(\begin{array}{c|c|c} a_{11} & \cdots & a_{1,t-1} \\ \hline \cdots & \cdots & \cdots \\ \hline a_{t-1,1} & \cdots & a_{t-1,t-1} \end{array}\right)$$

such that $a_{ij} \equiv 0 \bmod (X^i, X^j)$, a_{ij} being blocks of order $d_i \times d_j$. Thus, $\alpha \equiv \alpha' \bmod (\varphi)$ if and only if α' is given by some blocks $(\alpha'_{ij})_{i+j\leq t}, (\beta'_{rs})_{r+s>t}$ such that

1. $\alpha'_{ij} \equiv \alpha_{ij} \bmod (X^i, X^j)$ and
2. $\beta'_{rs} \equiv \beta_{rs} \bmod (X^{t-r}, X^{r-s})$.

If α' satisfies 1. then by Lemma 1.1 we see that $(\varphi + Y\alpha', \psi - Y\beta')$ defines also T. Choosing the entries of α'_{ij} to be polynomials of degree $< \min\{i, j\}$ and similarly the entries of β'_{rs}, we are done. $\square$

An infinitesimal deformation T of M is **liftable** to $R'/(Y^j)$, $j > 2$, if there exists an $R'/(Y^j)$ module E such that $E/Y^2E \cong T$ and the complex

$$E \xrightarrow{Y^2} E \xrightarrow{Y^{j-2}} E \xrightarrow{Y^2} E$$

is exact (it follows $\mathrm{Tor}_\ell^{R'/(Y^j)}(\widetilde{R}, E) = 0$ for all $\ell \geq 1$, see, for example, [ADS], [HP]).

The following theorem shows why it is necessary to study infinitesimal deformations of M liftable to $R'(Y^4)$ in the description of the maximal Cohen–Macaulay R' modules (in short $\mathrm{MCM}(R')$).

Theorem 1.3 (Herzog–Popescu [HP, 3.1])
For every indecomposable MCM R' module N there exists an indecomposable $\widetilde{R}$ module T such that

(i) T is an infinitesimal deformation of an MCM R module M to $\widetilde{R}$,

(ii) T is liftable to $R'/(Y^4)$,

(iii) N is a direct summand of the first syzygy $\Omega^1_{R'}(T)$ of T over R'.

Next we try to describe the integers $d_1, \ldots, d_{t-1}$ for which $M = \overset{t-1}{\underset{i=1}{\oplus}} P_i^{d_i}$ has an infinitesimal deformation T to $\widetilde{R}$ liftable to $R'/(Y^4)$. Our next two Lemmas are important tools for this purpose.

Lemma 1.4
With the notation of Theorem 1.2 suppose that T is liftable to $R'/(Y^3) \cong R[Y]/(Y^3)$. Then, for all j, r, $1 \leq j, r < T$

(i) $\sum\limits_{i=1}^{t-r-1} X^{t-i-r}\alpha_{ji}\alpha_{ir} + \sum\limits_{i=t-r}^{t-j} \alpha_{ji}\beta_{ir} + \sum\limits_{i>t-j}^{t-1} X^{i+j-t}\beta_{ji}\beta_{ir} \equiv 0 \; mod(X^j, X^{t-r})$
if $j \leq r$, and

(ii) $\sum\limits_{i=1}^{t-j-1} X^{t-j-i}\alpha_{ji}\alpha_{ir} + \sum\limits_{i=t-j}^{t-r} \beta_{ji}\alpha_{ir} + \sum\limits_{i>t-r}^{t-1} X^{i+r-t}\beta_{ji}\beta_{ir} \equiv 0 \; mod(X^r, X^{t-j})$
if $j \geq r$.

In particular

(i') $\sum\limits_{i=t-r}^{t-j} \alpha_{ji}\beta_{ir} \equiv 0 \; modX$ *if $j \leq r$, and*

(ii') $\sum\limits_{i=t-j}^{t-r} \beta_{ji}\alpha_{ir} \equiv 0 \; modX$ *if $j \geq r$.*

Proof: By [HP, 3.4], T is liftable to $R[Y]/(Y^3)$ if and only if $\alpha\beta \in E := M(d \times d, K[[X]])\psi + \varphi M(d \times d, K[[X]])$. A matrix a, as in the proof of Theorem 1.2, belongs to E if and only if the entries of $a_{u\nu}$ are in $(X^u, X^{t-\nu})$ for all u, ν, $1 \leq u, \nu < t$. Multiplying the line j of α with the column $r \geq j$ of β we see that (i) holds. Multiplying the line j of α with the column $r \leq j$ of β, we obtain

1) $\sum\limits_{i=1}^{t-j-1} X^{t-i-r}\alpha_{ji}\alpha_{ir} + X^{j-r}\sum\limits_{i=t-j}^{t-r} \beta_{ji}\alpha_{ir} + \sum\limits_{i>t-r}^{t-1} X^{i+j-t}\beta_{ji}\beta_{ir} \equiv 0 \bmod(X^j, X^{t-r})$.

Dividing 1) by X^{j-r} we see that (ii) holds, too. $\square$

Lemma 1.5
With the notation of Theorem 1.2, let $U_{nm}(c), c \in K[[X]]$, $1 \leq n, m \leq d$ be the invertible d matrix, whose (n,m) entry is c and coincides with I_d in all other entries, that is

$$U_{nm}(c) = \begin{array}{c} n\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & c & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ \multicolumn{2}{c}{\underbrace{\qquad\qquad}_{m}} & & & & & \end{pmatrix}$$

Then for all j, r, $1 \leq j \leq r < t$

(i) $U_{nm}(c)\varphi U_{nm}(-cX^{r-j}) = \varphi$ *if* $q_{j-1} < n \leq q_j,\ q_{r-1} < n \leq q_r$, *where* $q_s := \sum_{i=1}^{s} d_i$ *for* $1 \leq s < t,\ q_0 = 0$, *and*

(ii) $U_{nm}(cX^{r-j})\varphi U_{nm}(-c) = \varphi$ *if* $q_{j-1} < m \leq q_j,\ q_{r-1} < n \leq q_r$.

The proof is obvious.

Theorem 1.6
With the notation of Theorem 1.2 suppose T *is liftable to* $R'/(Y^3)$ *and* $\alpha_{j,t-j} \not\equiv 0\ modX$ *for a certain integer* j, $1 \leq j < t$. *Then* T *has a direct summand* T' *given by the two–matrix factorization of* X^t*:*

$$\left(\begin{pmatrix} X^j & Y \\ 0 & X^{t-j} \end{pmatrix}, \begin{pmatrix} X^{t-j} & -Y \\ 0 & X^j \end{pmatrix} \right),$$

providing K *is algebraically closed when* $j = \frac{t}{2}$.

The proof will follow from our next two lemmas, the first of them applied to T when $j < \frac{t}{2}$ or to $\Omega^1_{\widetilde{R}}(T)$ when $j > \frac{t}{2}$.

Lemma 1.7
Theorem 1.6 holds when $j < \frac{t}{2}$.

Proof: We preserve the notation of Lemma 1.5 and let $P_{n,m},\ n \neq m$ the the "permutation" transformation

$$P_{nm} = \begin{array}{cc} & \begin{array}{ccccccc} & & n & & m & & \end{array} \\ \begin{array}{c} \\ \\ n \\ \\ m \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & 1 & & 0 & & 0 & \\ & & & & 1 & & \\ & 0 & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & 0 & \\ & & 1 & & & & \\ & 0 & & 0 & & 1 & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

where the entries outside diagonal are zero, except the entries (m,n) and (n,m) which are 1. By Lemma 1.1 we may change α, β by $(P_{nm}\alpha P_{nm},\ P_{nm}\beta P_{nm})$ either if $q_{j-1} < n, m \leq q_j$, or if $q_{t-j-1} < n, m \leq q_{t-j}$.

After such permutations we may suppose that the entry (q_j, q_{t-j}) of α modulo X is not zero. By Lemmas 1.1 and 1.5 we may change α by $\alpha' := U_{q_{t-j},m}(-cX^{t-j-r})\alpha U_{q_{t-j},m}(-c)$, $c \in K[[X]]$ and β similarly by β', where r is an integer, $1 \leq r \leq t-j$, $q_{r-1} < m \leq q_r$, $m \neq q_{t-j}$.

Multplying α on the right with $U_{q_{t-j},m}(-c)$ means adding the column q_{t-j} multiplied with $-c$ to the column m, the corresponding multiplication from the left with α will affect only the line q_{t-j}. Using such transformations we may suppose that the entries (q_j, i) of α are zero for $i \leq i < q_{t-j}$.

We may also change α by $U_{q_{t-j},m}(-c)\alpha U_{q_{t-j},m}(-cX^{r+j-t}), c \in K[[X]]$ and similarly β, for an integer r, $t-j < r < t$, $q_{r-1} < m \leq q_r$. The right multiplication can make the (q_j, m) entry of α zero, (here it is important that $\alpha_{ij} = X^{i+j-t}\beta_{ij}$ if $i+j > t$ by Theorem 1.2!). The left multiplication affects only the q_{t-j}–th row of α. Thus, we may suppose that all entries of the q_j–th row of α are zero, except the entry (q_j, q_{t-j}), which is invertible.

Now change α by $U_{n,q_j}(-c)\alpha U_{n,q_j}(-cX^{j-r})$, $c \in K[[X]]$ for an integer r, $1 \leq r \leq j$, $q_{r-1} < n \leq q_r$, $n \neq q_j$. Using such transformations for suitable c the multiplication from left makes the entries (i, q_{t-j}) of α zero, $i < q_j$. The right multiplcation will affect only the column q_j of α but not the entry (n, q_j), which remains still zero $(n \neq q_j!)$.

Next change α by $U_{n,q_j}(cX^{r-j})\alpha U_{n,q_j}(-c)$, $c \in K[[X]]$ for an integer r, $j < r < t$, $q_{r-1} < n \leq q_r$, $n \neq q_{t-j}$. Using such transformations for suitable c, the multiplicaton from the left of α makes the entries (i, q_{t-j}) of α zero, $i > q_j$. The right multiplication of affects only the column q_j, but not the entry (n, q_j) which remains zero since $n \neq q_{t-j}$. Thus, we may suppose the row q_j and the column q_{t-j} of α to be zero, with just two exceptions, the entries (q_j, q_{t-j}) and (q_{t-j}, q_{t-j}).

Apply Lemma 1.4 for $j \leq r < t$. We have

$$(1) \qquad \sum_{i=1}^{t-r-1} \alpha_{ji} X^{t-i-r} \alpha_{ir} + \sum_{i=t-r}^{t-j} \alpha_{ji}\beta_{ir} + \sum_{i>t-j}^{t-1} X^{i+j-t}\beta_{ji}\beta_{ir} \equiv 0 \bmod (X^j, X^{t-r}).$$

Since the last row of α_{ji} (respectively β_{ji}) is zero if $i \neq t-j$ it follows that the last row of $\alpha_{j,t-j}\beta_{t-j,r}$ is zero $\bmod(X^j, X^{t-r})$ But the last row of $\alpha_{j,t-j}$ is zero, except the entry (q_j, q_{t-j}) which is invertible. Hence, the last row of $\beta_{t-j,r}$, $r \geq j$ must be zero modulo (X^j, X^{t-r}) and so we may suppose this row to be zero by Theorem 1.2.

Using Lemma 1.4 for $1 \leq r < j$ we obtain

$$(2) \qquad \sum_{i=1}^{t-j-1} \alpha_{ji} X^{t-i-j} \alpha_{ir} + \sum_{i=t-j}^{t-r} \beta_{ji}\alpha_{ir} + \sum_{i>t-r}^{t-1} X^{i+r-t}\beta_{ji}\beta_{ir} \equiv 0 \bmod (X^r, X^{t-j}).$$

As above, it follows that the last row of $\alpha_{j,t-j}\alpha_{t-j,r} = \beta_{j,t-j}\alpha_{t-j,r}$ is zero $\bmod(X^r, X^{t-j})$ and so the last row of $\alpha_{t-j,r}$ is zero modulo (X^r, X^{t-j}). By Theorem 1.2 we may suppose that the last row of $\alpha_{t-j,r}$ is zero. Hence, the rows q_j, q_{t-j} and the column q_{t-j} of α are zero, except the entry (q_j, q_{t-j}).

Apply Lemma 1.4 for $1 \leq r \leq t-j$. We have

$$(3)\quad \sum_{i=1}^{j-1} X^{j-i}\alpha_{ri}\alpha_{i,t-j} + \sum_{i=j}^{t-r}\alpha_{ri}\beta_{i,t-j} + \sum_{i>t-r}^{t-1} X^{i+r-t}\beta_{ri}\beta_{i,t-j} \equiv 0 \bmod (X^r, X^j).$$

Since the last column of $\alpha_{i,t-j}$ (respectively $\beta_{i,t-j}$) is zero for $i \neq j$ it follows that the last column of $\alpha_{r,j}\beta_{j,t-j}$ is zero modulo (X^r, X^j), as above, the last column of $\alpha_{r,j}$ must be zero modulo (X^r, X^j). By Theorem 1.2 we may suppose that the last column of $\alpha_{r,j}$ is zero if $n \leq t-j$.

Finally, apply Lemma 1.4 for $r > t-j$. We have

(4)
$$\sum_{i=1}^{t-r-1} \alpha_{ri} X^{t-i-r}\alpha_{i,t-j} + \sum_{i=t-r}^{j} \beta_{ri}\alpha_{i,t-j} + \sum_{i>j}^{t-1} X^{i-j}\beta_{ri}\beta_{i,t-j} \equiv 0 \bmod (X^{t-r}, X^{t-j}).$$

As above, we conclude that the last column of β_{rj} is zero modulo (X^{t-r}, X^{t-j}) and zero applying Theorem 1.2. Hence, we may suppose that the column q_j of α is zero. Consequently $\varphi + Y\alpha$ has as a direct summand a two–square matrix $\sigma := \left(\begin{smallmatrix} X^j & \lambda Y \\ 0 & X^{t-j}\end{smallmatrix}\right)$ for a certain invertible element $\lambda \in K[[X]]$. Thus, T has as a direct summand an infinitesimal deformation T' of $P_j \oplus P_{t-j}$ given by $\left(\sigma, \left(\begin{smallmatrix} X^{t-j} & -\lambda Y \\ 0 & X^j\end{smallmatrix}\right)\right)$. Applying Lemma 1.1 to T', we see that changing σ by $U\sigma U^{-1}$ for $U = \left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)$ we may assume that $\lambda = 1$. $\square$

Remark 1.8
T' is indecomposable (cf. [HP, 5.3]).

Lemma 1.9
Theorem 1.6 holds for $j = \frac{t}{2}$.

Proof: Using transformations of type $U, V = U^{-1}$ we may suppose that α_{jj} is modulo X in the Jordan normal form (K is algebraically closed!), $\alpha_{jj} \equiv \bigoplus_{\mu=1}^{e} \varepsilon_\mu \bmod X$, where $\varepsilon_\mu = \begin{pmatrix} \lambda_\mu & & 0 \\ 1 & \ddots & \\ & \ddots & \\ 0 & 1 & \lambda_\mu \end{pmatrix}$, $\lambda_\mu \in K$ is a s_μ Jordan cell, $s_\mu \geq 1$, $\sum_{\mu=1}^{e} s_\mu = d_\mu$. By Lemma 1.4 applied for $j = r$ we obtain $\alpha_{jj}^2 \equiv 0 \bmod X$. It follows $\lambda_\mu = 0$ and $s_\mu \leq 2$ for all $\mu = 1, \ldots, e$. Since $\alpha_{jj} \not\equiv 0 \bmod X$ it follows $s_\mu = 2$ for a certain μ, we may assume $\mu = 1$.

Changing α by $U_{q_{j-1}+1,m}(cX^{j-r})\alpha U_{q_{j-1}+1,m}(-c)$, $c \in K[[X]]$ (as in the proof of Lemma 1.7), where r is an integer $1 \leq r \leq j$, $q_{r-1} < m \leq q_r$, $m \neq q_{j-1}+1$, we may suppose that the entries $(q_{j-1}+2, i)$ of α are zero for $1 \leq i \leq q_j$, $i \neq q_{j-1}+1$. The left multiplication will affect only the row $q_{j-1}+1$. Since we need $c \in (X)$ when either $r < j$, or $q_{j-1} < m < q_j$, $m \neq q_{j-1}+1$ (the first row of α_{jj} is zero modulo X!), we may still suppose that the first row of α_{jj} is zero modulo X after these transformations.

We may also change α by $U_{q_{j-1}+1,m}(c)\alpha \cdot U_{q_{j-1}+1,m}(-cX^{r-j})$, $c \in K[[X]]$ for an integer r, $j < r < t$, $q_{r-1} < m \leq q_r$. The right multiplication of can make the $(q_{j-1}+2, m)$ entry of α zero. The left multiplication affects only the row $q_{j-1}+1$ with terms from (X) ($\alpha_{jr} = X^{r-j}\beta_{jr}$!).

Thus, we may suppose that all entries of the $(q_{j-q}+2)$–th row of α to be zero except the entry $(q_{j-1}+2, q_{j-1}+1)$ which is 1 (we still have α_{jj} modulo X in the above Jordan normal form!).

Now change α by $U_{n,q_{j-1}+2}(c)\alpha U_{n,q_{j-1}+2}(-cX^{j-r})$ $c \in K[[X]]$ for an integer r, $1 \leq r \leq j$, $q_{r-1} < n \leq q_r$, $n \neq q_{j-1}+2, q_{j-1}+1$. Using such transformations, the multiplication from the left makes the entries $(i, q_{j-1}+1)$, $i \leq q_j$ zero, $i \neq q_{j-1}+1$, $q_{j-1}+2$, the right multiplication affects only the column $q_{j-1}+2$, but not the entry $(q_{j-1}+2, q_{j-1}+2)$ which is still zero.

Next change α by $U_{n,q_{j-1}+2}(cX^{r-j})\alpha U_{n,q_{j-1}+2}(-c)$, $c \in K[[X]]$ for an integer r, $j < r < t$, $q_{r-1} < n \leq q_r$. Using such transformations we obtain the column $q_{j-1}+2$ to be zero, except the entries $(q_{j-1}+1, q_{j-1}+1)$, $(q_{j-1}+2, q_{j-1}+1)$.

As in the proof of Lemma 1.7, apply Lemma 1.4. We have for $j \leq r < t$ ($t - j = j$!)

$$1)\quad \sum_{i=1}^{t-r-1} \alpha_{ji} X^{t-i-r} \alpha_{ir} + \sum_{i=t-r}^{j} \alpha_{ji}\beta_{ir} + \sum_{i>j}^{t-1} X^{i-j}\beta_{ji}\beta_{ir} \equiv 0 \bmod X^{t-r}.$$

Since the second row of α_{ji} (respectively β_{ji}) is zero, if $i \neq j$, it follows that the last row of $\alpha_{jj}\beta_{jr}$ is zero $\bmod X^{t-r}$. Then we obtain the first line of β_{jr} to be zero $\bmod X^{t-r}$, $r \geq j$, which can be supposed to be zero by Theorem 1.2.

Using Lemma 1.4 for $1 \leq r < j$ we obtain $\sum_{i=1}^{j-1} \alpha_{ji} X^{j-i}\alpha_{ir} + \sum_{i=j}^{t-r} \beta_{ji}\alpha_{ir} + \sum_{i>t-r}^{t-1} X^{i+r-t}\beta_{ji}\beta_{ir} \equiv 0 \bmod X^r$.

As above it follows that the first row of α_{jr} is zero $\bmod X^r$, $j < r$, which can be supposed to be zero by Theorem 1.2. Thus, the rows $q_{j-1}+1$, $q_{j-1}+2$ and the column $q_{j-1}+1$ are zero except the entry $(q_{j-1}+2, q_{j-1}+1)$.

Apply again Lemma 1.4 for $1 \leq r \leq j$. We have $\sum_{i=1}^{j-1} X^{j-i}\alpha_{ri}\alpha_{ij} + \sum_{i=j}^{t-r} \alpha_{ri}\beta_{ij} + \sum_{i>t-r}^{t-1} X^{i+r-t}\beta_{ri}\beta_{ij} \equiv 0 \bmod X^r$. Since the first column of α_{ij} (respectively β_{ij}) is zero if $i \neq j$, it follows that the first column of $\alpha_{rj}\beta_{jj}$ is zero $\bmod X^r$. Thus, the second column of α_{rj} is zero $\bmod X^r$ and we may suppose it to be zero by Theorem 1.2. Consequently, the lines and columns $q_{j-1}+1, q_{j-1}+2$ of α are zero except the entry $(q_{j-1}+2, q_{j-1}+1)$, which is 1.

Thus, $\varphi + Y\alpha$ has as a direct summand a 2 matrix $\sigma'' = \left(\begin{smallmatrix} X^j & 0 \\ Y & X^j \end{smallmatrix}\right)$ and so T has a direct summand T'' given by (σ'', τ''). By Lemma 1.1 it is enough to change now σ'' by $\sigma' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \sigma'' \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\square$

2 Liftability to $\mathbf{R'/(Y^4)}$

In the notations of Section 1, assume now that T is liftable to $R'/(Y^4)$. Then by [HP, 3.4] there exist $\gamma, \gamma', \tau, \tau' \in M(d \times d, K[[X]])$ such that

$$(*) \begin{cases} \alpha\beta & = \gamma\psi + \varphi\tau \\ I_d & = -\gamma\beta + \gamma'\psi + \alpha\tau + \varphi\tau' \equiv -\gamma\beta + \alpha\tau \bmod X \end{cases}$$

because $\varphi, \psi \equiv 0 \bmod X$, I_d being the unit matrix.

Lemma 2.1
Let $\tilde{\gamma}, \tilde{\tau} \in M(d \times d, K[[X]])$ be two matrices such that $\alpha\beta = \tilde{\gamma}\psi + \varphi\tilde{\tau}$. If T is liftable to $R'/(Y^4)$ and $\alpha_{i,t-i} \equiv 0 \bmod X$ for all $1 \leq i < t$, then

$$\delta_{jr} I_{d_j} \equiv - \sum_{i=t-r+1}^{t-1} \tilde{\gamma}_{ji}\beta_{ir} + \sum_{s=1}^{t-j-1} \alpha_{js}\tilde{\tau}_{sr} \bmod X$$

for all $1 \leq r \leq j < t$, δ_{jr} being Kronecker's symbol.

Proof: As above, there exist $\gamma = (\gamma_{ji}), \tau = (\tau_{sr})$ such that $(*)$ holds. Then, by Theorem 1.2

$$(5) \quad \delta_{jr} I_{d_j} \equiv - \sum_{i=1}^{d} \gamma_{ji}\beta_{ir} + \sum_{s=1}^{d} \alpha_{js}\tau_{sr} \equiv - \sum_{i=t-r+1}^{t-1} \gamma_{ji}\beta_{ir} + \sum_{s=1}^{t-j-1} \alpha_{js}\tau_{sr} \bmod X,$$

because $\beta_{t-r,r} \equiv \alpha_{j,t-j} \equiv 0 \bmod X$. But $\varepsilon := \gamma - \tilde{\gamma}$, $\eta := \tau - \tilde{\tau}$ satisfy $\varepsilon\psi + \varphi\eta \equiv 0 \bmod X$ and thus

$$(6) \qquad \varepsilon_{uv} X^{t-v} + X^u \eta_{uv} \equiv 0 \bmod X,$$

for all $1 \leq u, v < t$. If $u < t - v$ then we obtain from (6) $\eta_{uv} \equiv 0 \bmod X$ and so $\tau_{uv} \equiv \tilde{\tau}_{uv} \bmod X$. Thus, if $r \leq j$ then $s < t - r$ and so $\tau_{sr} \equiv \tilde{\tau}_{sr} \bmod X$ for all $s < t - j$. Similarly, we obtain $j > t - i$ and so $\gamma_{ji} \equiv \tilde{\gamma}_{ji} \bmod X$ for all $i > t - r$. Substituting these congruences in (5), we are done. $\square$

Proposition 2.2
Let $\alpha\beta = (p_{uv})_{1 \leq u,r \leq d}, p_{uv} \in K[[X]]$ and suppose $\alpha_{i,t-i} \equiv 0 \bmod X$ for all $1 \leq i < t$. If T is liftable to $R'/(Y^4)$ then

$$X^{t-1}\delta_{jr} I_{d_j} \equiv - \sum_{i=t-r+1}^{t-1} X^{i-1} p_{ji}\beta_{ir} + \sum_{s=1}^{t-j-1} X^{t-s-1}\alpha_{js} p_{sr} \bmod X^t,$$

for all $1 \leq r \leq j < t$, δ_{jr} being Kronecker's symbol.

Proof: To apply Lemma 2.1, we need to find a solution to the equation

$$Z\psi + \varphi W = \alpha\beta \tag{7}$$

in $M(d \times d, K[[X]])$. Fix $j \geq r$. Since T is liftable to $R'/(Y^3)$ then by $(*)$ we have $p_{uv} \in (X^u, X^{t-v})$. If $u > t-v$ then $p_{uv} = X^{t-v}\tilde{\gamma}_{uv}$ for some $\tilde{\gamma}_{uv} \in K[[X]]$ and we set $\tilde{\tau}_{uv} = 0$. If $u \leq t-v$ then $p_{uv} = X^u\tilde{\tau}_{uv}$ for some $\tilde{\tau}_{uv} \in K[[X]]$ and we set $\tilde{\gamma}_{uv} = 0$. Then, $(\tilde{\gamma}_{uv})$, $(\tilde{\tau}_{uv})$ is a solution of (7). If $i > t-r$ then $i > t-j$ and so $p_{ji} = X^{t-i}\tilde{\gamma}_{ji}$. If $s < t-j$ then $r \leq t-s$ and so $p_{sr} = X^s\tilde{\sigma}_{sr}$. The proof ends substituting p_{ji} and p_{sr} in the congruence given by Lemma 2.1. $\square$

Proposition 2.3
Suppose that $t \geq 4$, $\alpha_{i,t-i} \equiv 0 \, mod X$ for all i, $1 \leq i < t$, $\alpha_{1,t-2} \not\equiv 0 \, mod X$ and the rank modulo X of $(\alpha_{12}|\cdots|\alpha_{1,t-2})$ is smaller than d_1. If T is liftable to $R'/(Y^4)$ then T has an indecomposable direct summand T' which deforms $P_1^2 \oplus P_{t-2}$ and is given by

$$\left(\left(\begin{array}{cc|c} X & 0 & Y \\ Y & X & 0 \\ \hline 0 & Y & X^{t-2} \end{array} \right), \left(\begin{array}{cc|c} X^{t-1} & 0 & -XY \\ -X^{t-2}Y & X^{t-1} & 0 \\ \hline 0 & -XY & X^2 \end{array} \right) \right).$$

Proof: Using transformations of type $\alpha \to P_{np}\alpha P_{np}$ and $\alpha \to U_{nm}(a)\alpha U_{nm}(-a)$, $a \in K$, $1 \leq n, m, p \leq d_1$, or $q_{t-3} < n, m, p \leq q_{t-2}$ (see the proof of 1.7), we may suppose that $\alpha_{1,t-2}$ in diagonal form $\bmod X$. By Theorem 1.2, we may even consider $\alpha_{1,t-2}$ to be in diagonal form, let us say $\alpha_{1,t-2} = \left(\begin{array}{c|c} I_{e_{t-2}} & 0 \\ \hline 0 & 0 \end{array} \right)$ for an integer e_{t-2}, $1 \leq e_{t-2} \leq d_1$.

Next, using transformations of type $\alpha \to U_{nm}(aX^{t-r-2})\alpha U_{nm}(-a)$, $a \in K$, $q_{t-3} < n \leq q_{t-3} + e_{t-2}$, $1 \leq r \leq t-3$, $q_{r-1} < m \leq q_r$ we may supose the first e_{t-2} rows of α_{1i} to be zero, $i \leq t-3$. The left multiplication affects only the rows $q_{t-3}+1, \ldots, q_{t-3}+e_{t-2}$, that is the rows $\alpha_{t-2,i}$, $1 \leq i < t$ $(t-2 \geq 2!)$. By similar transformations we may suppose

$$\left(\alpha_{11} \,\middle|\, \ldots \,\middle|\, \alpha_{1,t-2} \right) = \begin{array}{c} \\ \\ \\ e_1 \{ \end{array} \left(\begin{array}{c|cc|c|cc} 0 & 0 & & \ldots & I_{e_{t-2}} & \multicolumn{1}{|c}{0} \\ \hline 0 & 0 & & \ldots & 0 & \\ \hline 0 & I_{e_2} & \multicolumn{1}{|c|}{0} & \ldots & 0 & \\ \hline ///// & 0 & & \ldots & 0 & \end{array} \right)$$

for some integers $e_i \geq 0$ (by hypothesis $e_{t-2}, e_1 > 0$).

Apply Lemma 1.4 for $r > 1$. We have

$$\sum_{i=t-r}^{t-1} \alpha_{1i}\beta_{ir} \equiv 0 \bmod X. \tag{1}$$

Since the first e_{t-2} rows of α_{1i} are zero if $i < t-2$ and $\alpha_{1,t-1} \equiv 0 \bmod X$ by hypothesis we see that the first e_{t-2} rows of $\alpha_{1,t-2}\beta_{t-2,r}$ are zero $\bmod X$. However, the other rows of $\alpha_{1,t-2}$ and so of $\alpha_{1,t-2}\beta_{t-2,r}$ are already zero. Thus, $\alpha_{1,t-2}\beta_{t-2,r} = X\nu_{t-2,r}$ for a $\nu_{t-2,r} \in M(d \times d, K[[X]])$, $r \geq e$. Similarly, we obtain $\alpha_{1s}\beta_{sr} = X\nu_{sr}$ for a $\nu_{sr} \in M(d \times d, K[[X]])$, $r \geq t-s$.

By Proposition 2.2 it follows

$$X^{t-1} I_{d_1} \equiv \sum_{s=1}^{t-2} X^{t-s-1}\alpha_{1s}p_{s1} \bmod X^t, \tag{2}$$

where $p_{s1} = X^{s-1}(\sum\limits_{i=1}^{t-s-1} X^{t-s-i}\alpha_{si}\alpha_{i1} + \sum\limits_{i=t-s}^{t-1}\beta_{si}\alpha_{i1})$ (see the proof of 1.4). Note that $p_{11} \equiv X\alpha_{1,t-2}\alpha_{t-2,1} \bmod X^2$ and $\beta_{1,t-1} \equiv \alpha_{t-1,1} \equiv 0 \bmod X$. Substituting p_{s1} in (2) we obtain

$$I_{d_1} \equiv \sum_{s=1}^{t-2}\alpha_{1s}\alpha_{1,t-s-1}\alpha_{t-s-1,1} + \sum_{s=2}^{t-2}\ \sum_{i=t-s}^{t-1}\nu_{si}\alpha_{i1} \bmod X. \tag{3}$$

Express α_{11} and $\alpha_{t-2,1}$ in blocks as follows:

$$\alpha_{11} = \left(\begin{array}{c|c} \multicolumn{2}{c}{0} \\ \hline A & B \end{array}\right) \quad \alpha_{t-2,1} = \left(\begin{array}{c|c} C & D \\ \hline \multicolumn{2}{c}{E} \end{array}\right),$$

A an $e_1 \times e_{t-2}$ matrix and D an $e_{t-2} \times e_1$ matrix.

By (3) we see that $\alpha_{11}\alpha_{1,t-2}\alpha_{t-2,1}$ has the form $\left(\begin{array}{c|c} \multicolumn{2}{c}{0} \\ \hline 0 & I_{e_1} \end{array}\right)$ modulo X. It follows

$$AC = 0,\ AD = I_{e_1}, \tag{4}$$

after applying Theorem 1.2. Thus $e_1 \leq e_{t-2}$ and $d_1 \geq 2e_1$.

After some transformations we may suppose $D = \left(\frac{I_{e_1}}{0}\right)$. Indeed, we may operate on the last e_1 columns of α_{11} using $\alpha \longrightarrow U_{nm}(c)\alpha U_{nm}(-c)$, $c \in K[[X]]$, $d_1 - e_1 < n, m \leq d_1$. We may also operate on the rows of D using $\alpha \longrightarrow U_{nm}(c)\alpha U_{nm}(-c)$, $q_{t-3} < n, m \leq q_{t-3} + e_{t-2}$. The right multiplication will change the block $I_{e_{t-2}}$ of $\alpha_{1,t-2}$ by an invertible matrix F. Multiplying the first e_{t-2} rows of α by F^{-1} and corresponding the first e_{t-2} columns of α by F (these transformations do not change the structure of other blocks of α), we may again suppose $F = I_{t-2}$.

Then by (4) we obtain $A = (I_{e_1} \mid ////)$. Subtracting multiples of the rows $q_{t-3} + 1, \ldots, q_{t-3} + e_1$ from the rows $q_{t-3} + e_{t-2} + 1, \ldots, q_{t-2}$ and correspondingly adding multiples of the columns $q_{t-3}+e_{t-2}+1,\dots,q_{t-2}$ to the columns $q_{t-3}+1,\dots,q_{t-3}+e_1$ (it does not harm $\alpha_{1,t-2}$!), we may suppose $E = (//// \mid \underbrace{0}_{e_1})$. Changing α by $U_{m,q_{t-3}+1}\ (a)\alpha U_{m,q_{t-3}+1}\ (-aX^{t-2-r}),\ a\in K[[X]], 1\le r<t-2,\ q_{r-1}<m\le q_r,\ d_1-e_1<m$, we may suppose the column d_1-e_1+1 to be zero, except the entry $(q_{t-3}+1, d_1-e_1+1)$ which is 1 (by theorem 1.2 we may suppose $\alpha_{t-1,1}=0$). The right multiplication modifies the column $q_{t-3}+1$ with terms from (X), but $\alpha_{1,t-2}$ remains unchanged by using 1.2. Similarly we may suppose the columns $(d_1-e_1+i)_{i\le e_1}$ of α_{i1} for $i\ne t-2$ to be zero.

Apply again Lemma 1.4 for $r>1$. We have $\sum\limits_{i=t-r}^{t-1}\beta_{ri}\alpha_{i1}\equiv 0 \bmod X$ (for $r=1$ this was clear because $\alpha_{t-1,1}\equiv 0 \bmod X$). Since the columns $\{d_1-j\}_{1\le j\le e}$ of α_{i1} are zero if $i\ne t-2$ we see that the columns $\{d_1-j\}_{j\le e}$ of $\beta_{r,t-2}\alpha_{t-2,1}$ are zero $\bmod X$. But $D=\binom{I_e}{0}$ and it follows that the first e columns of $\beta_{r,t-2}$ are zero $\bmod X$, $r>1$. Changing α by $U_{m1}(aX^{r-1})\alpha U_{m1}(-a),\ a\in K[[X]],\ 1<r<t,\ q_{r-1}+e_r<m\le q_r$ (take $e_{t-1}=0$), we may suppose the first column of $\alpha_{r,t-2}(1<r<t)$ to be zero, except its first e_r entries (here it was important that the first e columns of $\beta_{r,t-2}$ are zero $\bmod X$ if $r>1$!). The right multiplication does not really affect the first column of α_{11} since the last (d_r-e_r) columns of α_{1r}, $1<r<t$ are zero.

We show that, in fact, the whole first column of $\beta_{r,t-2}(1<r<t)$ is zero.

By (3) we have

$$\alpha_{1r}\alpha_{r,t-r-1}\alpha_{t-r-1,1}+\sum_{i=t-r}^{t-2}\nu_{ri}\alpha_{i1}\equiv \left(\begin{array}{c|c} & 0 \\ \cline{2-2} \underbrace{0}_{f_r} & \begin{array}{c|c} I_{e_r} & 0\end{array} \\ \cline{2-2} & 0 \end{array}\right)\}f_r \quad \bmod X, \tag{5}$$

where $f_r := \sum\limits_{i=1}^{e_{r-1}} e_i$.

The last e_1 columns of α_{i1} are zero, if $i\ne t-2$ and the last e_1 columns of $\alpha_{1r}I_{d_1}$ are also zero if $r>1$. Then we obtain from (5) that the last e_1 columns of $\nu_{r,t-2}\alpha_{t-2,1}$ are zero $\bmod X$, $1<r<t$. Since the last e_1 columns of of $\alpha_{t-2,1}$ have the form $\binom{I_{e_1}}{0}$ it follows that the first e_1 columns of $\nu_{r,t-2}$ are zero $\bmod X$. Then $\alpha_{1r}\beta_{r,t-2}$ has the first e_1 columns zero $\bmod X^2$, which may be considered even zero by Theorem 1.4. Thus, the first column of $\beta_{r,t-2}$ is zero $1<r<t$ and similarly we see that the first e_1 columns of $\beta_{r,t-2}$ are zero, that is the columns $\{q_{t-3}+i\}_{1\le i\le e_1}$ of α are zero, except the entries $(i, q_{t-3}+i)$, which are 1.

Apply Lemma 1.4 for $2\le r\le t-2$ and we have

$$X\alpha_{r1}\alpha_{1,t-2}+\sum_{i=2}^{t-r}\alpha_{ri}\beta_{i,t-2}+X\alpha_{r,t-r+1}\beta_{t-r+1,t-2}\equiv 0 \bmod X^2. \tag{6}$$

Since the first e_1 columns of $\beta_{i,t-2}$ are zero if $i > 1$ it follows that the first e_1 columns of $\alpha_{r1}\alpha_{1,t-2}$ are zero $\mod X$, that is the first e_1 columns of $\alpha_{r1}\alpha_{1,t-2}$ are zero because we may suppose by 1.2 the entries of $\alpha_{r1}, \alpha_{1,t-2}$ to be in K. Thus, the first e_1 columns of α_{r1} are zero, $2 \leq r < t$.

At this moment $(A|B)$ has the form $(I_{e_1}|G|\underbrace{0}_{e_1})$ and we shall reduce the case to $G = 0$, subtracting multiples of the first e_1 columns from the next (d_1-2e_1) columns of α together with adding multiples of the rows $e_1+1, \ldots, d_1-e_1$ to the first e_1 rows of α. This will affect the first e_1 rows of α_{1i}, $2 \leq i \leq t-2$. To correct this , we shall subtract multiples of the first $q_{t-3}+1, \ldots, q_{t-3}+e_1$ columns from the columns $d_1+1, \ldots, q_{t-3}, q_{t-3}+e_1+1, \ldots, q_{t-3}+e_{t-2}$ of α and, necessarily, we shall add the rows $d_1+1, \ldots, q_{t-3}$, $q_{t-3}+e_1+1, \ldots, q_{t-3}+e_{t-2}$ multiplied by the corresponding factors to the rows $q_{t-3}+1, \ldots, q_{t-3}+e_1$ of α (these operations do not change the previous structure, the first e_{t-2} rows of $\beta_{t-2,r}, r \geq 2$ are still zero $\mod X$ by (1)!). In particular, we now have $A = (I_{e_1}|0)$. By (4) $AC = 0$ and so the first e_1 rows of C are zero, too. Thus,

$$\alpha_{t-2,1} = \left(\begin{array}{c|c|c} & 0 & I_{e_1} \\ \cline{2-3} \underbrace{0}_{e_1} & //// & 0 \end{array}\right) \quad \text{and } \alpha_{11} = \left(\begin{array}{c|c} \multicolumn{2}{c}{0} \\ \hline I_{e_1} & 0 \end{array}\right).$$

Now, changing α by $U_{d_1-i,m}(a)\alpha U_{d_1-i,m}(-aX^{r-1})$, $a \in K[[X]]$, $2 \leq r < t$, $q_{r-1} < m \leq q_r$, $m \neq q_{t-3}+1, \ldots, q_{t-3}+e_1$ $1 \leq i \leq e_1$ we may suppose that the first e_1 rows of $\alpha_{t-2,s}$ are zero for $2 \leq s < t$ (here it is important that the first e_1 rows of $\beta_{t-e,s}$ are zero $\mod X$!).

Unfortunately, the left multiplication seems to affect the rows $\{d_1-i\}_{1 \leq i \leq e_1}$ of α. Apply Lemma 1.4 for $2 \leq r \leq t-2$ and we obtain

$$(7) \qquad X\alpha_{t-2,1}\alpha_{1r} + \sum_{i=2}^{t-r}\beta_{t-2,i}\alpha_{ir} + X\beta_{t-2,t-r+1}\beta_{t-r+1,r} \equiv 0 \mod X^2.$$

Since we now have the first e_1 rows of $\beta_{t-2,i}$ as zero, for $i \geq 2$ it follows that the first e_1 rows of $\alpha_{t-2,1}\alpha_{1r}$ are zero $\mod X$. But $\alpha_{t-2,1}$ has the nice form above and so the last e rows of α_{1r} must be zero $\mod X$, even zero using 1.2.

What really happens with α_{11}? Let $\rho_{t-2,i}$ be the block of $\nu_{t-2,i}$ given by its first e_1 rows. In order to make the first e_1 rows of $\beta_{t-2,s}$ above be zero, we subtract, in fact, from $\beta_{t-2,s}$ the last e_1 columns of $\alpha_{t-2,1}$ multiplied by $\rho_{t-2,s}$. Then the left multiplication above will add to $(A|B)$ the matrix $\sum_{s=2}^{t-2}\rho_{t-2,s}\alpha_{s1}$. Note that the last e_1 columns of $\rho_{t-2,t-2}\alpha_{t-2,1}$ are zero, since the first e_1 columns of $\beta_{t-2,t-2}$ are zero and $(\frac{I_{e_1}}{0})$ is the block of $\alpha_{t-2,1}$ formed by its last e_1 columns. Thus, the rows $q_{t-3}+1, \ldots, q_{t-3}+e_1$, do not contribute really to the sum (we also had $m \neq q_{t-3}+1, \ldots, q_{t-3}+e_1$!). Using (5) for $r = t-2$ we obtain

$$(8) \qquad \alpha_{1,t-2}\alpha_{t-2,1}\alpha_{11} + \sum_{i=2}^{t-2}\nu_{t-2,i}\alpha_{i1} \equiv \left(\begin{array}{c|c} I_{e_{t-2}} & 0 \\ \hline 0 & 0 \end{array}\right) \bmod X,$$

since $\alpha_{1,t-2}\alpha_{t-2,1}\alpha_{11} = \left(\begin{smallmatrix} I_{e_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ we see that the first e_1 rows of $\sum\limits_{i=2}^{t-2}\nu_{t-2,i}\alpha_{i1}$ (that is $\sum\limits_{i=2}^{t-2}\rho_{t-2,i}\alpha_{i1}!$) are zero $\bmod X$. Using again 1.2 we conclude that the left multiplication does not affect also α_{11}.

Consequently $\varphi + Y\alpha$ has as a direct summand, a direct sum of e_1 copies of the 3–matrix

$$\left(\begin{array}{cc|c} X & 0 & Y \\ Y & X & 0 \\ \hline 0 & Y & X^{t-2} \end{array}\right),$$

which defines an indecomposable module by 6.4 and 6.10. $\square$

Remark 2.4

(i) If $e_1 = 0$ then (as in the proof of 2.3) we may reduce the matrix to the case $\alpha_{11} = 0$ and by (8) it follows

(9) $\sum\limits_{i=2}^{t-2}\nu_{t-2,i}\alpha_{i1} \equiv \left(\begin{smallmatrix} I_{e_{t-2}} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *mod* X.

(ii) If rank $((\alpha_{12}|\dots|\alpha_{1,t-2})$ mod $X) < d_1$ then the rank $(\alpha_{21}|\dots|\alpha_{t-2,1})$ mod X is also $< d_1$.

Proposition 2.5

Suppose that $t \geq 4$, $\alpha_{i,t-i} \equiv 0 \bmod X$ for all i, $1 \leq i < t$, $\beta_{t-1,2} \not\equiv 0 \bmod X$ and rank $((\beta_{t-1,2}|\dots|\beta_{t-1,t-2})$ mod $X) < d_{t-2}$. If T is liftable to $R'/(Y^4)$ then T has a direct summand T' which deforms $P_{t-1}^2 \oplus P_2$ and is given by

$$\left(\left(\begin{array}{c|cc} X^2 & 0 & -XY \\ \hline -XY & X^{t-1} & 0 \\ 0 & -X^{t-2}Y & X^{t-1} \end{array}\right), \left(\begin{array}{c|cc} X^{t-2} & 0 & Y \\ \hline Y & X & 0 \\ 0 & Y & X \end{array}\right)\right).$$

For the proof, apply Proposition 2.3 for $\Omega^1_{\tilde{R}}(T)$.

Remark 2.6

If rank $((\beta_{t-1,2}|\dots|\beta_{t-1,t-2})$ mod $X) = d_{t-1}$ then we may reduce the matrix to the case $\beta_{t-1,t-1} = 0$.

3 An illustration on cases $\mathbf{t = 3, 4}$

Let T be an infinitesimal deformation of M liftable to $R'/(Y^4)$.

First **suppose** $\mathbf{t = 3}$. Then $\alpha = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & X\beta_{22} \end{smallmatrix}\right)$ and we may reduce to the case $\alpha_{12} = 0,\ \alpha_{21} = 0$ applying Theorems 1.6, 1.2. This means T splits into direct summands given either by

$$(\sigma, \rho) := \left(\begin{pmatrix} X & Y \\ 0 & X^2 \end{pmatrix}, \quad \begin{pmatrix} X^2 & -Y \\ 0 & X \end{pmatrix} \right)$$

or by (ρ, σ). When $\alpha_{12} = 0,\ \alpha_{21} = 0$ then T is a direct sum of two $\widetilde{R}$ modules T', T'' such that T' is an infinitesimal deformation of $P_1^{d_1}$ and T'' is an infinitesimal deformation of $P_2^{d_2}$, both of them being liftable to $R'/(Y^4)$. But these are completely described in [EPP, 2.1] if K is algebraically closed. If this is the case, then T' (respectively T'') is a direct sum of copies of three types of cyclic infinitesimal deformations given by the matrix factorizations $(\sigma_i, \rho_i)_{1 \le i \le 3}$ (respectively $(\rho_i, \sigma_i)_{1 \le i \le 3}$), where $\sigma_i := X + \lambda_i Y$, $\rho_i := X^2 - \lambda_i XY$ and $\lambda_1, \lambda_2, \lambda_3$ are the third roots of unity.

Now **suppose** $\mathbf{t = 4}$. Then $\alpha = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{smallmatrix}\right)$ and we may reduce to the case $\alpha_{13} = 0,\ \alpha_{31} = 0,\ \alpha_{22} \equiv \alpha \bmod X$ using Theorems 1.6, 1.2. This means T splits into direct smmands given by

$$(\tau, \eta) = \left(\begin{pmatrix} X & Y \\ 0 & X^3 \end{pmatrix}, \quad \begin{pmatrix} X^3 & -Y \\ 0 & X \end{pmatrix} \right), \quad (\eta, \tau) \text{ and}$$

$$(\tau', \eta') = \left(\begin{pmatrix} X^2 & Y \\ 0 & X^2 \end{pmatrix}, \quad \begin{pmatrix} X^2 & -Y \\ 0 & X^2 \end{pmatrix} \right),$$

the last one in the case, when K is algebraically closed.

Thus $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & X\alpha'_{22} & X\beta_{23} \\ 0 & X\beta_{32} & X^2\beta_{33} \end{pmatrix}$ and the following identities hold by Lemma 1.4:

$$\alpha_{21}\alpha_{12} + \beta_{23}\beta_{32} \equiv 0 \bmod X, \tag{1}$$

$$\alpha_{12}\beta_{23} \equiv 0 \bmod X, \tag{2}$$

$$\beta_{32}\alpha_{21} \equiv 0 \bmod X. \tag{3}$$

Now apply Proposition 2.2. We have

$$\begin{cases} X^3 I_{d_1} & \equiv X^2\alpha_{11}P_{11} + X\alpha_{12}P_{21} \bmod X^4 \\ X^3 I_{d_2} & \equiv -X^2 P_{23}\beta_{32} + X^2\alpha_{21}P_{12} \bmod X^4 \\ X^3 I_{d_3} & \equiv -XP_{32}\beta_{23} - X^2 P_{33}\beta_{33} \bmod X^4, \end{cases} \tag{4}$$

where

$$(5)\qquad \begin{cases} P_{11} &= X^2\alpha_{11}^2 + X\alpha_{12}\alpha_{21},\ P_{21} = X^2(\alpha_{21}\alpha_{11} + \alpha'_{22}\alpha_{21}) \\ P_{23} &= X\alpha'_{22}\beta_{23} + X\beta_{23}\beta_{33},\ P_{12} = X(\alpha_{11}\alpha_{12} + \alpha_{12}\alpha'_{22}) \\ P_{32} &= X^2(\beta_{32}\alpha'_{22} + \beta_{33}\beta_{32})\ P_{33} = X\beta_{32}\beta_{23} + X^2\beta_{33}^2. \end{cases}$$

Combining (4), (5) it follows

$$(6)\qquad I_{d_1} \equiv \alpha_{11}\alpha_{12}\alpha_{21} + \alpha_{12}\alpha_{21}\alpha_{11} + \alpha_{12}\alpha'_{22}\alpha_{21} \bmod X,$$

$$(7)\qquad I_{d_2} \equiv -\alpha'_{22}\beta_{23}\beta_{32} - \beta_{23}\beta_{32}\beta_{33} + \alpha_{21}\alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{12}\alpha'_{22} \bmod X,$$

$$(8)\qquad I_{d_3} \equiv -\beta_{32}\alpha'_{22}\beta_{23} - \beta_{33}\beta_{32}\beta_{23} - \beta_{32}\beta_{23}\beta_{33} \bmod X.$$

If $\alpha_{12} \equiv 0 \bmod X$ then we obtain $d_1 = 0$ by (6). If $d_2 = 0$ then T is an infinitesimal deformation of $P_3^{d_3}$ which contradicts [HP, 4.1] (see also [EPP, 2.1]). Then $d_2 \neq 0$ and so $\beta_{32} \not\equiv 0 \bmod X$ by (7). If rank $(\beta_{32} \bmod X) < d_3$, then by Proposition 2.5 we may split a direct summand T' off from T, given by

$$(\tau'', \eta'') = \left(\begin{pmatrix} X^2 & 0 & -XY \\ -XY & X^3 & 0 \\ 0 & -X^2Y & X^3 \end{pmatrix}, \quad \begin{pmatrix} X^2 & 0 & Y \\ Y & X & 0 \\ 0 & Y & X \end{pmatrix} \right).$$

Thus, the rank modulo X of β_{32} is d_3 and by Remark 2.6 we may suppose $\beta_{33} = 0$. Then by (7), (8) we see that $d_2 = d_3$ and β_{32}, β_{23} are invertible, which contradicts (1).

Similarly, we treat the case $\beta_{32} \equiv 0 \bmod X$ the only type of module T'' which we may split by Proposition 2.3 being given by (η'', τ''). Finally, suppose $\alpha_{12} \not\equiv 0,\ \beta_{32} \not\equiv 0 \bmod X$. If the rank modulo X of α_{12} (respectively β_{32}) is smaller than d_1 (respectively d_3) then we may split as above a module T'' (respectively T'). By Theorem 1.2 we may suppose the entries of $\alpha_{12}, \alpha_{21}, \beta_{23}, \beta_{22}, \alpha'_{22}$ to be from K and so $\mathrm{rank}_K \alpha_{12} = d_1$, $\mathrm{rank}_K \beta_{32} = d_3$, $\alpha_{11} = 0$ and $\beta_{33} = 0$ by Remarks 2.4, 2.6. The identities (6) – (8) become

$$\begin{aligned} (9)\qquad && I_{d_1} &= \alpha_{12}\alpha'_{22}\alpha_{21}, \\ (10)\qquad && I_{d_2} &= -\alpha'_{22}\beta_{23}\beta_{32} + \alpha_{21}\alpha_{12}\alpha'_{22} \\ (11)\qquad && I_{d_3} &= -\beta_{32}\alpha'_{22}\beta_{23}. \end{aligned}$$

Then $\mathrm{rank}_K \beta_{23} = d_3$, $\mathrm{rank}_K \alpha_{21} = d_1$ by (9), (11).

As in the proof of Proposition 2.3, we may suppose,after some transformations, $\alpha_{12} = (I_{d_1}|0)$. By (2) the first d_1 rows of β_{23} are zero and it follows that the first d_1 rows of $\alpha_{21}\alpha_{12}$ are zero by (1). Thus, α_{21} has the form $\begin{array}{c} \left(\begin{array}{c} 0 \\ \hline A \end{array}\right) \end{array}\begin{array}{l} \}d_1 \\ \end{array}$ and so $\alpha_{21}\alpha_{12} = \left(\begin{array}{c|c} \multicolumn{2}{c}{0} \\ \hline A & 0 \end{array}\right).\}d_1$

Substituting (1) in (10) we obtain

$$I_{d_2} = \alpha'_{22}\alpha_{21}\alpha_{12} + \alpha_{21}\alpha_{12}\alpha'_{22}. \tag{12}$$

Since $\mathrm{rank}_K(\alpha'_{22}\alpha_{21}\alpha_{12}) \leq d_1$, $\mathrm{rank}_K(\alpha_{21}\alpha_{12}\alpha'_{22}) \leq d_1$ if follows by (12) that $d_2 \leq 2d_1$. In fact, $d_2 = 2d_1$, because $\mathrm{rank}_K A = \mathrm{rank}_K \alpha_{21} = d_1$. Hence, A is invertible.

Multiplying with A^{-1} the rows $2d_1+1, \ldots, 3d_1$ of α (and with A the columns $2d_1+1, \ldots, 3d_1$ of α), we may suppose $A = I_{d_1}$. Thus, we have

$$\alpha = \left(\begin{array}{c|cc|c} 0 & I_{d_1} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$X\alpha'_{22}$}} & \multirow{2}{*}{$X\beta_{23}$} \\ \cline{1-1} I_{d_1} & & & \\ \hline 0 & \multicolumn{2}{c|}{X\beta_{32}} & 0 \end{array}\right).$$

Let $\alpha'_{22} = \left(\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right)$ where B, C, D, E are $d_1 \times d_1$ matrices.

By (12) we obtain

$$\left(\begin{array}{c|c} C & 0 \\ \hline E & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline B & C \end{array}\right) = I_{d_2},$$

that is $C = I_{d_1}$ and $E + B = 0$.

Subtracting suitable F multiples of the columns $2d_1+1, \ldots, 3d_1$ from the columns $d_1+1, \ldots, 2d_1$ of α (and correspondingly adding these multiples of the rows $d_1+1, \ldots, 2d_1$ to the rows $2d_1+1, \ldots, 3d_1$ of α), we may suppose $B = 0$ (both operations do not change the previous structure). Thus, $E = 0$, too.

Subtracting D multiples of the first d_1 rows from the rows $2d_1+1, \ldots, 3d_1$ of α (and correspondingly adding XD multiples of the rows $2d_1+1, \ldots, 3d_1$ to the first d_1 columns), we may suppose also $D = 0$ (the operation on the columns adds terms from (X) to the entries of the first d_1 columns, which can be removed by 1.2!). Thus, $\alpha'_{22} = \left(\begin{smallmatrix} 0 & I_{d_1} \\ 0 & 0 \end{smallmatrix}\right)$.

By (2) and (3) we have $\beta_{23} = \left(\begin{array}{c} 0 \\ \hline F \end{array}\right)\}d_1$, $\beta_{32} = (G|\underbrace{0}_{d_1})$. Then $GF = -I_{d_3}$ by (11) and so $d_3 \leq d_1$ because G is a $d_3 \times d_1$ matrix. But $\mathrm{rank}_K(\beta_{23}\beta_{32}) = \mathrm{rank}_K(\alpha_{21}\alpha_{12}) = d_1$ by (1) and, consequently, $d_3 = d_1$. Then F, G are invertible.

Now, multiplying the columns $3d_1+1, \ldots, 4d_1$ of α by G (and correspondingly multiplying the lines $3d_1+1, \ldots, 4d_1$ by F), we may suppose $G = -F = I_{d_1}$. Then

$\varphi + Y\alpha$ is a direct sum of d_1 copies of the following matrix:

$$\tilde{\sigma} := \left(\begin{array}{c|c|c|c} X & Y & 0 & 0 \\ \hline 0 & X^2 & XY & 0 \\ \hline Y & 0 & X^2 & -XY \\ \hline 0 & XY & 0 & X^3 \end{array}\right).$$

Then T is a direct sum of d_1 copies of an infinitesimal deformation T' of $P_1 \oplus P_2^2 \oplus P_3$ given by $(\tilde{\sigma}, \tilde{\eta})$ for

$$\tilde{\eta} = \left(\begin{array}{c|cc|c} X^3 & -XY & 0 & 0 \\ \hline 0 & X^2 & -XY & 0 \\ -XY & 0 & X^2 & Y \\ \hline 0 & -Y & 0 & X \end{array}\right).$$

4 Infinitesimal deformations of a direct sum of copies of two indecomposable R modules

Let $d, d' \geq 0$ be two integers and $1 \leq i < \frac{t}{2}$.

Lemma 4.1
Let T be an infinitesimal deformation of $M = P_i^{d'} \oplus P_{t-i}^{d''}$ liftable to $R'/(Y^4)$. Then T is given by $(\varphi + Y\alpha, \psi - Y\beta)$, where

$$\varphi = \left(\begin{array}{c|c} X^i I_{d'} & 0 \\ \hline 0 & X^{t-i} I_{d''} \end{array} \right), \quad \alpha = \left(\begin{array}{c|c} \alpha_{ii} & \alpha_{i,t-i} \\ \hline \alpha_{t-i,i} & X^{t-2i}\beta_{t-i,t-i} \end{array} \right)$$

satisfy

(1) $X^{t-2i}\alpha_{ii}^2 + \alpha_{i,t-i}\alpha_{t-i,i} \equiv 0 \; mod \, X^i$,

(2) $X^{t-2i}\beta_{t-i,t-i}^2 + \alpha_{t-i,i}\alpha_{i,t-i} \equiv 0 \; mod \, X^i$,

(3) $\alpha_{ii}\alpha_{i,t-i} + \alpha_{i,t-i}\beta_{t-i,t-i} \equiv 0 \; mod \, X^i$,

(4) $\alpha_{t-i,i}\alpha_{ii} + \beta_{t-i,t-i}\alpha_{t-i,i} \equiv 0 \; mod \, X^i$,

(5) $X^i I_{d'} \equiv \alpha_{ii}(X^{t-2i}\alpha_{ii}^2 + \alpha_{i,t-i}\alpha_{t-i,i}) \; mod \, X^{i+2}$,

(6) $X^i I_{d''} \equiv -(X^{t-2i}\beta_{t-i,t-i}^2 + \alpha_{t-i,i}\alpha_{i,t-i})\beta_{t-i,t-i} \; mod \, X^{i+1}$.

The proof follows by Lemma 1.4 and Proposition 2.2.

Proposition 4.2
Suppose $t \geq 4$. Then

(i) There exists an infinitesimal deformation of $P_1^{d'} \oplus P_{t-1}^{d''}$ liftable to $R'/(Y^4)$ if and only if $d' = d''$.

(ii) Every infinitesimal deformation of $P_1^{d'} \oplus P_{t-1}^{d'}$ liftable to $R'/(Y^4)$ is a direct sum of d' copies of two non–isomorphic infinitesimal deformations: one is given by

$$(\gamma, \eta) = \left(\begin{pmatrix} X & Y \\ 0 & X^{t-1} \end{pmatrix}, \begin{pmatrix} X^{t-1} & -Y \\ 0 & X \end{pmatrix} \right)$$

and the other is given by (η, γ).

Proof: Let T be an infinitesimal deformation of $P_1^{d'} \oplus P_{t-1}^{d''}$ liftable to $R'/(Y^4)$. Then T is given by $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{1,t-1} \\ \alpha_{t-1,1} & X^{t-2}\beta_{t-1,t-1} \end{pmatrix}$. Using Theorem 1.6 and Theorem 1.2 we may reduce to the case $\alpha_{1,t-1} = 0$, $\alpha_{t-1,1} = 0$ by splitting from T (let us say e copies of) two types of infinitesimal deformations: one is given by (γ, η) and the other by (η, γ). The remaining part $\widetilde{T}$ deforms now $P_1^{d'-e} \oplus P_{t-1}^{d''-e}$ and splits in the form $\widetilde{T} = T' \oplus T''$, where T' deforms $P_1^{d'-e}$ and T'' deforms $P_{t-1}^{d''-e}$. But this is possible only when $d' = e = d''$ by [HP, 4.1]. $\square$

Remark 4.3
The above proposition describes completely the infinitesimal deformations from Lemma 4.1 for the case $i = 1$. When $i > 1$ the problem is much more difficult. For example, when $t = 14$, $i = 5$, the matrices $\alpha_{55} = 1$, $\alpha_{59} = -X + (1 + \lambda X)X^2$, $\lambda \in K$, $\alpha_{95} = X$, $\beta_{99} = -1$ satisfy (1) – (6) from Lemma 4.1 and

$$\varphi + Y\alpha = \begin{pmatrix} X^5 + Y & (-1 + X + \lambda X^3)XY \\ XY & X^9 - X^4Y \end{pmatrix}$$

defines an infinite set of indecomposable infinitesimal deformations of $P_5 \oplus P_9$, providing K is infinite (see 6.4 and 6.10 (iii)).

Proposition 4.4
Suppose $t > 6$ and K algebraically closed. Then

(i) there exists an infinitesimal deformation of $P_2^{d'} \oplus P_{t-2}^{d''}$ liftable to $R'/(Y^4)$ if and only if $d' = d''$.

(ii) Every infinitesimal deformation of $P_2^{d'} \oplus P_{t-2}^{d'}$ liftable to $R'/(Y^4)$ is a direct sum of d' copies of two infinitesimal deformations defined by

$$(\gamma, \eta) = \left(\begin{pmatrix} X^2 & Y \\ 0 & X^{t-2} \end{pmatrix}, \begin{pmatrix} X^{t-2} & -Y \\ 0 & X^2 \end{pmatrix} \right), \ (\eta, \gamma)$$

and some infinitesimal deformations defined by $(\sigma_{\lambda r}, \rho_{\lambda r})_{\substack{\lambda \in K^* \\ r \in \mathbb{N}}}$

$$\sigma_{\lambda r} = \left(\begin{array}{c|c} X^2 I_r + YA_{\lambda r} & XYI_r \\ \hline XYA_{\lambda r}^{-1} & X^{t-2}I_r - X^{t-4}YA_{\lambda r} \end{array} \right)$$

$$\rho_{\lambda r} = \left(\begin{array}{c|c} X^{t-2}I_r - X^{t-4}YA_{\lambda r} & -XYI_r \\ \hline -XYA_{\lambda r}^{-1} & X^2 I_r + YA_{\lambda r} \end{array} \right),$$

where $A_{\lambda r} = \begin{pmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{pmatrix}$ *is an $r \times r$ Jordan cell.*

Proof: Let T be an infinitesimal deformation of $P_2^{d'} \oplus P_{t-2}^{d''}$ liftable to $R'/(Y^4)$. As in the proof of Proposition 4.2, we may reduce to the case $\alpha = \begin{pmatrix} \alpha_{22} & X\alpha'_{2,t-2} \\ X\alpha'_{t-2,2} & X^{t-4}\beta_{t-2,t-2} \end{pmatrix}$ by splitting off some copies of the infinitesimal deformations defined by (γ, η) from T, or (η, γ). By 1.2 we may suppose the entries of $\alpha'_{2,t-2}$, $\alpha'_{t-2,2}$ to be from K.

Operating on the first d' rows (and correspondingly on the first d' columns) and on the last d'' columns (and correspondingly on the last d'' rows) of α, we may suppose $\alpha'_{2,t-2} = \left(\begin{array}{c|c} I_e & 0 \\ \hline \multicolumn{2}{c}{0} \end{array} \right)$ for an e, $0 \le e \le d'$.

Let $\alpha'_{t-2,2} = \left(\begin{array}{c} B \\ \hline //// \end{array} \right)\}e$. By (5) in Lemma 4.1 we obtain

$$I_{d'} \equiv \alpha_{22} \left(\begin{array}{c} B \\ \hline 0 \end{array} \right) \mod X.$$

Thus, $e = d' \leq d''$, $\alpha'_{2,t-2} = (I_{d'}|0)$ and α_{22} is invertible. Similarly by (6) in lemma 4.1 it follows

$$I_{d''} \equiv -\alpha'_{t-2,2}\alpha'_{2,t-2}\beta_{t-2,t-2} \bmod X.$$

Hence, $d'' = d'$, $\alpha'_{t-2,2} = B$ is invertible and $\beta_{t-2,t-2} \equiv -B^{-1} \bmod X$, $\alpha_{22} \equiv B^{-1} \bmod X$. Subtracting multiples of the last d' rows from the first d' ones (and correspondingly adding X^{t-4} multiples of the first d' columns to the last d' ones), we may suppose $\alpha_{22} = B^{-1}$. These operations add to $\alpha_{2,t-2}$ terms from (X^{t-4}), which may be removed by 1.2 ($t > 6$). Similarly, we may suppose $\beta_{t-2,t-2} = -B^{-1}$.

Now, let U be a $d' \times d'$ invertible matrix. Multiplying the first d' rows by U and correspondingly the first d' columns by U^{-1}, we see that $\alpha'_{2,t-2} = I_{d'}$ is not changed. Thus, we may change B by $U^{-1}BU$. In this way we may suppose B to be a direct sum of some Jordan cells of type $A_{\lambda r}$. Clearly, T is a direct sum of infinitesimal deformations defined by $(\sigma_{\lambda r}, \rho_{\lambda r})$. $\square$

Proposition 4.5
Suppose $t > 6$. Then

(i) *there exists an infinitesimal deformation of $P_1^{d'} \oplus P_{t-2}^{d''}$ liftable to $R'/(Y^4)$ if and only if $d' = 2d''$.*

(ii) *Every infinitesimal deformation of $P_1^{2d''} \oplus P_{t-2}^{d''}$ liftable to $R'/(Y^4)$ is a direct sum of d'' copies of a unique indecomposable infinitesimal deformation given by*

$$(\gamma, \eta) = \left(\left(\begin{array}{cc|c} X & 0 & Y \\ Y & X & 0 \\ \hline 0 & Y & X^{t-2} \end{array} \right), \left(\begin{array}{cc|c} X^{t-1} & 0 & -XY \\ -X^{t-2}Y & X^{t-1} & 0 \\ \hline 0 & -XY & X^2 \end{array} \right) \right).$$

Proof: Let T be an infinitesimal deformation of $P_1^{d'} \oplus P_{t-2}^{d''}$. By Thorem 1.2 we may suppose the entries of $\alpha_{11}, \alpha_{1,t-2}, \alpha_{t.2,2}$ to be in K. Splitting from T some (let us say e) direct summands given by (γ, η) (see Proposition 2.3) we may suppose $\mathrm{rank}_K \alpha_{1,t-2} = \mathrm{rank}_K \alpha_{t-2,1} = d'$ and $\alpha_{11} = 0$. The remaining part $\widetilde{T}$ deforms now $P_1^{d'-2e} \oplus P_{t-2}^{d''-e}$. We can see that this is possible only when $\bar{d} := d' - 2e = 0$, $\tilde{d} := d'' - e = 0$.

By Proposition 2.2 we have

$$X^{t-1}I_{\tilde{d}} \equiv -X^{t-3}P_{t-2,t-2}\beta_{t-2,t-2} + X^{t-2}\alpha_{t-2,1}P_{1,t-2} \bmod X^t,$$

where $P_{1,t-2} = \alpha_{1,t-2}\beta_{t-2,t-2}$ and $P_{t-2,t-2} = X\alpha_{t-2,1}\alpha_{1,t-2} + X^{t-4}\beta^2_{t-2,t-2}$. Thus,

$$XI_{\tilde{d}} \equiv -\alpha_{t-2,1}\alpha_{1,t-2}\beta_{t-2,t-2} + \alpha_{t-2,1}\alpha_{1,t-2}\beta_{t-2,t-2} = 0 \bmod X^2$$

because $X^{2t-7} \equiv 0 \bmod X^t$ for $t > 6$. This happens only when $\tilde{d} = 0$. $\square$

Remark 4.6
The above proposition has a variant for infinitesimal deformations of $P_2^{d'} \oplus P_{t-1}^{d''}$. For the proof, change T by $\Omega^1_{\tilde{R}}(T)$.

5 Maximal Cohen–Macaulay modules over $\mathbf{K[[X,Y]]/(X^t+Y^3)}$

Suppose $t \geq 3$.

Theorem 5.1
Let M be a finitely generated R module without free summands. Then, for every maximal Cohen–Macaulay R' module N there exists a matrix factorization $(\hat{\varphi}, \hat{\psi})$ of $X^t + Y^3$ corresponding to $N \oplus \Omega^1_{R'}(N)$ such that the entries of $\hat{\varphi}, \hat{\psi}$ are polynomials in X, Y of degree $t-1$.

Proof: By Theorem 1.2 every infinitesimal deformation T of M is given by a matrix factorization $(\varphi + Y\alpha, \psi - Y\beta)$ of X^t over $K[[X,Y]]/(Y^2)$, where

$$\varphi = \begin{pmatrix} XI_{d_1} & & 0 \\ & \ddots & \\ 0 & & X^{t-1}I_{d_{t-1}} \end{pmatrix}, \ \psi = \begin{pmatrix} X^{t-1}_{I_{d_1}} & & 0 \\ & \ddots & \\ 0 & & XI_{d_{t-1}} \end{pmatrix}$$

and $\alpha\beta$ are matrices over $K[[X]]$ given by some blocks $(\alpha_{ij})_{i+j\leq t}, (\beta_{rs})_{r+s>t}$, the entries of α_{ij} (respectively β_{rs}) being polynomials in X of degree $< \min\{i,j\}$ (respectively $\min\{t-r, t-s\}$). Take $T := N/Y^2N$. Then $\Omega^1_{R'}(T) \cong N \oplus \Omega^1_{R'}(N)$ by [Po] or [HP, 2.8].

The matrix factorization of $X^t + Y^3$ corresponding to $\Omega^1_{R'}(T)$ is given after [HP, 3.2] by

$$\left(\left(\begin{array}{cc} \psi - Y\beta & -Y^2 I_d \\ \hline \alpha\beta + YI_d & \varphi + Y\alpha \end{array} \right), \left(\begin{array}{cc} \varphi + Y\alpha & Y^2 I_d \\ \hline -(\alpha\beta + YI_d) & \psi - Y\beta \end{array} \right) \right).$$

We claim that the entries of these matrices are polynomials in X, Y of degree $t-1$. Indeed, by Theorem 1.2 the entries of α, β are polynomials in X of degree $\leq t-2$ and it is only enough to show that the entries of $\alpha\beta, \beta\alpha$ are polynomials of degree $\leq t-1$. We shall show even that the degree of the entries of $\alpha\beta, \beta\alpha$ is $\leq t-2$.

As in Lemma 1.4 (i) the block (j,r) of $\alpha\beta$ when $j \leq r$ has the form

$$\sum_{i=1}^{t-r-1} X^{t-i-r}\alpha_{ji}\alpha_{ir} + \sum_{i=t-r}^{t-j} \alpha_{ji}\beta_{ir} + \sum_{i>t-j}^{t-1} X^{i+j-t}\beta_{ji}\beta_{ir}.$$

An entry from $X^{t-i-r}\alpha_{ji}\alpha_{ir}$ is a polynomial in X of degree $\leq t-i-r+\min\{j,i\} - 1 + \min\{i,r\} - 1 \leq t-i-r+i-1+r-1 = t-2$. An entry of $\alpha_{ji}\beta_{ir}$ has degree in $X \leq \min\{j,i\} - 1 + \min\{t-i, t-r\} - 1 \leq i + t - i - 2 = t-2$. An entry of $X^{i+j-t}\beta_{ji}\beta_{ir}$ has the degree $\leq i+j-t+\min\{t-j,t-i\}-1+\min\{t-i,t-r\}-1 \leq i+j-t+t-j-1+t-i-1 = t-2$.

If $j > r$ then the block j, r of $\alpha\beta$ has the form

$$\sum_{i=1}^{t-j-1} X^{t-i-r}\alpha_{ji}\alpha_{ir} + \sum_{i=t-j}^{t-r} X^{i+j-t}\beta_{ji}X^{t-i-r}\alpha_{ir} + \sum_{i>t-r}^{t-1} X^{i+j-t}\beta_{ji}\beta_{ir}.$$

An entry of $X^{j-r}\beta_{ji}\alpha_{ir}$ has degree $\leq j-r+\min\{t-j,t-i\}-1+\min\{i,r\}-1\leq j-r+t-j+r-2=t-2$. Thus, all entries of $\alpha\beta$ are polynomials of degree $\leq t-2$. Similarly, it holds for $\beta\alpha$. $\square$

Theorem 5.2
Let $d',d''\geq 0$ be two integers, and suppose $t>3$. Then

(i) there exists a maximal Cohen–Macaulay R' module N such that $N/YN\cong P_1^{d'}\oplus P_{t-1}^{d''}$ if and only if $d'=d''$;

(ii) every maximal Cohen–Macaulay R' module N such that $N/YN\cong P_1^{d'}\oplus P_{t-1}^{d'}$ is a direct sum of d' copies of the ideals $(X,Y),(X^{t-1},Y)$.

Proof: If N is a maximal Cohen–Macaulay R' module such that $M=N/YN\cong P_1^{d'}\oplus P_{t-1}^{d''}$, then $T=N/Y^2N$ is an infinitesimal deformation of M, liftable to $R'/(Y^4)$ and by Proposition 4.2 we must have $d'=d''$. If this is the case then T is a direct sum of d' copies of two non–isomorphic deformations given by

$$(\lambda,\eta):=\left(\begin{pmatrix} X & Y \\ 0 & X^{t-1}\end{pmatrix},\begin{pmatrix} X^{t-1} & -Y \\ 0 & X\end{pmatrix}\right)$$

and (η,λ). Thus, $\Omega^1_{R'}(T)$ is a direct sum of d' copies of maximal Cohen–Macaulay R' modules corresponding to the matrix factorization

$$(\hat{\lambda},\hat{\eta}):=\left(\left(\begin{array}{cc|cc} X^{t-1} & -Y & -Y^2 & 0 \\ 0 & X & 0 & -Y^2 \\ \hline Y & 0 & X & Y \\ 0 & Y & 0 & X^{t-1}\end{array}\right),\left(\begin{array}{cc|cc} X & Y & Y^2 & 0 \\ 0 & X^{t-1} & 0 & Y^2 \\ \hline -Y & 0 & X^{t-1} & -Y \\ 0 & -Y & 0 & X\end{array}\right)\right).$$

Clearly, $\hat{\lambda}$ is the direct sum of $\lambda':=\begin{pmatrix} X^{t-1} & -Y^2 \\ Y & X\end{pmatrix}$ with $\lambda''=\begin{pmatrix} X & -Y^2 \\ Y & X^{t-1}\end{pmatrix}$. Since Coker $\lambda'\cong(X,Y)$ and Coker $\lambda''=(X^{t-1},Y)$ we see that $N\oplus\Omega^1_{R'}(N)\cong\Omega^1_{R'}(T)$ is the direct sum of d' copies of the ideals $(X,Y),(X^{t-1},Y)$. $\square$

Proposition 5.3
Let $d',d''\geq 0$ be two integers and suppose $t>6$. Then

(i) there exists a maximal Cohen–Macaulay R' module N such that $N/YN\cong P_2^{d'}\oplus P_{t-2}^{d''}$ if and only if $d'=d''$;

(ii) every maximal Cohen–Macaulay R' module N such that $N/YN\cong P_2^{d'}\oplus P_{t-2}^{d''}$ is a direct sum of d' copies of the the ideals $(X^2,Y),(X^{t-2},Y)$ and of direct summands of the modules given by

$$\left(\begin{array}{c|c} \rho_{\lambda r} & -Y^2 I_{2r} \\ \hline B_{\lambda r}+YI_{2r} & \sigma_{\lambda r}\end{array}\right),\ \left(\begin{array}{c|c} \sigma_{\lambda r} & Y^2 I_{2r} \\ \hline B_{\lambda r}-YI_{2r} & \rho_{\lambda r}\end{array}\right),$$

where

$$B_{\lambda r}=\left(\begin{array}{c|c} X^{t-4}A^2_{\lambda r}+X^2A^{-1}_{\lambda r} & 0 \\ \hline 0 & X^{t-4}A^2_{\lambda r}+X^2A^{-1}_{\lambda r}\end{array}\right)$$

and $(\sigma_{\lambda r},\rho_{\lambda r},A_{\lambda r})_{\substack{\lambda\in K^* \\ r\in\mathbb{N}}}$ are defined in Proposition 4.4.

The proof is as in Theorem 5.2, using Proposition 4.4.

Theorem 5.4
Suppose $t > 6$. Then

(i) there exists a maximal Cohen–Macaulay R' module N such that $N/YN \cong P_1^{d'} \oplus P_{t-2}^{d''}$ if and only if $d' = 2d''$;

(ii) every maximal Cohen–Macaulay R' module N such that $N/YN \cong P_1^{2d''} \oplus P_{t-2}^{d''}$ is a direct sum of d'' copies of a unique maximal Cohen–Macaulay R' module given by

$$(\gamma, \eta) = \left(\begin{pmatrix} X & 0 & Y \\ Y & X & 0 \\ 0 & Y & X^{t-2} \end{pmatrix}, \begin{pmatrix} X^{t-1} & Y^2 & -XY \\ -X^{t-2}Y & X^{t-1} & Y^2 \\ Y^2 & -XY & X^2 \end{pmatrix} \right).$$

The proof is as in Theorem 5.2, using Proposition 4.5.

Remark 5.5

(i) Let T be the unique infinitesimal deformation defined in Proposition 4.5. Then $\Omega^1_{\widetilde{R}}(T)$ is given by a matrix factorization of $X^t + Y^3$, which must split into a sum of two matrix factorizations by [EPP, 1.7, ii)] because obviously the module defined by (γ, η) from the above theorem lifts T to R'. This avoids a difficult computation in the proof of Theorem 5.4.

(ii) As in Theorem 5.2, we may describe all indecomposable maximal Cohen–Macaulay modules in the case $t = 3, 4$ using Section 3. Since these are already known (see, for example, [GK], or [Yo]) we shall not do it but we invite the reader to compare modulo Y^2 to the matrix factorizations given in [Yo] (there X corresponds to Y!) with our infinitesimal deformations given in Section 3.

(iii) Using Remark 4.6 we obtain a variant of Theorem 5.4 for maximal Cohen–Macaulay R' modules N such that $N/YN \cong P_2^{d'} \oplus P_{t-1}^{d''}$.

6 Indecomposability of modules

Let $(R, \underline{m})$ be a local ring and M a finitely generated R module of rank $r := \dim(M/\underline{m}M)$. We wish to present a finite algorithm, testing whether M is indecomposable. This can be done, if we are able to detect rank 1 summands of M. The algorithm is programmed in the computer algebra system SINGULAR.

Consider all possible presentations of M by matrices with r rows:

$$\text{Pres } (M) := \{A \in M(r \times n, R) | M \cong \text{ coker } A\},$$

n not necessarily fixed. Consider the following equivalence on Pres (M):

$$A \sim A' \text{ if and only if span } (A) = \text{ span } (A'),$$

span $(A) \subseteq R^r$ the submodule generated by the columns of A. Obviously we have for $A \in$ Pres (M):

$$A' \in \text{ Pres } (M) \text{ if and only if } A' \sim UA \text{ for some } U \in \text{ Gl}_r(R).$$

For simplicity we wish to split only rank 1 summands of M, which is the case if and only if Pres (M) contains a 1 row block matrix

$$A_0 = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & B \end{array}\right) \qquad \begin{array}{l} a = (a_1, \ldots, a_k) \\ B \in M(r-1 \times n-k; R). \end{array}$$

Let $J(A)$ be the ideal generated by the first two entries of A and $\mathcal{M}(A) \subset R^{r-1}$ be the module spanned by the columns of A, but omitting the first row.

$J(A)$ and $\mathcal{M}(A)$ depend only on the quivalence class of A.

Note that for $A \in$ Pres (M) there is a surjection

$$M \twoheadrightarrow R/J(A) \oplus R/\mathcal{M}(A),$$

which is an isomorphism for a matrix equivalent to A_0.

The idea of the algorithm is as follows: find in Pres (M) a matrix A having minimal $J(A)$ and $\mathcal{M}(A)$, then a reduced standard basis of A will be such a block matrix, in case of the existence of a rank 1 summand. More precisely: consider the following subgroups of $\text{Gl}_r(R)$:

$$\mathbf{T} = \left\{ \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D \end{array}\right) \middle| \lambda, \det D \in R^* \right\}$$

$$\mathbf{U}_+ = \left\{ U_\alpha := \left(\begin{array}{c|c} 1 & \alpha \\ \hline 0 & I_{r-1} \end{array}\right) \middle| \alpha = (\alpha_{21}, \ldots, \alpha_r) \right\}$$

$$\mathbf{U}_- = \left\{ V_\beta := \left(\begin{array}{c|c} 1 & 0 \\ \hline \beta & I_{r-1} \end{array}\right) \middle| \beta = (\beta_2, \ldots, \beta_r) \right\}$$

and let G be the subset of $\mathrm{Gl}_r(R)$:

$$\mathbf{G} = \left\{ \left(\begin{array}{c|c} \lambda & \alpha \\ \hline \beta & D \end{array} \right) \Big| \lambda, \det(D) \in R^* \right\}.$$

It is not difficult to check the following properties.

(i) $\mathbf{G} = \mathbf{G}^{-1}$

(ii) $\mathbf{G} \subset \mathbf{T}\mathbf{U}_+\mathbf{U}_- = \mathbf{U}_+\mathbf{U}_-\mathbf{T}$
$\mathbf{G} \subset \mathbf{T}\mathbf{U}_-\mathbf{U}_+ = \mathbf{U}_-\mathbf{U}_+\mathbf{T}$

Lemma 6.1
If M has a rank 1 summand and $A \in$ Pres(M), then permuting at most two rows of A we may assume: $A_0 \sim UA$ for some block matrix A_0 and

$$U \in \mathbf{G} \cap \mathbf{U}_+ \cdot \mathbf{U}_- \cdot A.$$

Proof: By assumption there exists $A_0 \sim UA$, $U \in \mathrm{Gl}_r(R)$. For some k, $1 \leq k \leq r$, u_{1k} and the $(1,k)$ adjoint of U being units. Let P_{1k} be the permutation matrix associated to the transposition $(1,k)$, then $UP_{1k} \in \mathbf{G}$ and take the decomposition

$$UP_{1k} = U_0 U_\alpha V_\beta$$

then $U_0^{-1}A_0 \sim U_\alpha V_\beta P_{1k} A$. Replacing A by $P_{1k}A$ and A_0 by $U_0^{-1}A_0$ we are finished. $\square$

Lemma 6.2
(i) Let $A_0 \sim UA$, $U \in \mathbf{G}$, then $J(A_0) \subseteq J(A)$.

(ii) Let $A_0 \sim UA$, $U \in \mathbf{U}$, then $\mathcal{M}(A_0) \subseteq \mathcal{M}(A)$.

Proof:

(i) $A \sim U^{-1}A_0 = \left(\begin{smallmatrix} \lambda & \alpha \\ \beta & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda a & \alpha B \\ \beta a & DB \end{smallmatrix}\right)$,
hence $J(A) = (\lambda a) + (\alpha B) \supseteq (a) = J(A_0)$.

(ii) If $D = I_{r-1}$ from the above equation we obtain $\mathcal{M}(A) = \mathrm{span}(\beta a) + \mathrm{span}(B) \supseteq \mathrm{span}(B) = \mathcal{M}(A_0)$.

Lemma 6.3
Assume $A_0 \sim UA$ and $U \in \mathbf{G} \cap \mathbf{U}_+\mathbf{U}_-$

(i) $J(A) = J(A_0)$ then $\binom{0}{x} \in$ span(A) if and only if $x \in \mathcal{M}(A_0)$.

(ii) $\mathcal{M}(A) = \mathcal{M}(A_0)$ then $\binom{y}{0} \in$ span(A) if and only if $y \in J(A_0)$.

(iii) If assumptions from (i) and (ii) are fulfilled, then $A \sim A_0$ and a reduced standard base with respect to a suitable ordering of the module span (A) has a 1 row block form.

Proof:

(i) We may assume
$A = V_\beta U_\alpha A_0 = \left(\begin{smallmatrix} 1 & \alpha \\ \beta & I+\beta a \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & B \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \alpha B \\ \beta a & DB \end{smallmatrix}\right)$.
$D := I_{r-1} + \beta a$, hence by assumption

$$(a) = (a) + (\alpha B), \text{ that is } (\alpha B) \subseteq (a).$$

$\binom{0}{x} =$ span(A) if and only if there exists $g \in R^k,\ h \in R^{n-k}$, such that

$$0 = ag + \alpha Bh$$

and

$$x = \beta ag + DBh = -\beta aBh - DBh = Bh,$$

that is $x \in$ span$(B) = \mathcal{M}(A_0)$. On the other hand, $x \in \mathcal{M}(A_0)$ determines a $h \in R^{n-k}$ and because of $(\alpha B) \subseteq (a)$ we shall find $g \in R^k$ such that $ag = -\alpha Bh$.

(ii) $\binom{y}{0} \in$ span(A) if and only if

$$0 = \beta ag + DBh$$

and

$$y = ag + \alpha Bh = ag - \alpha D^{-1}\beta ag = (\underbrace{1 - \alpha D^{-1}\beta}_{\lambda})ag$$

(Note: D is regular because $V_\beta U_\alpha \in \mathbf{G}$ by assumption.)

Now we obtain

$$\begin{aligned}(1+\alpha\beta)\lambda &= (1+\alpha\beta)(1-\alpha D^{-1}\beta) \\ &= 1 + a\beta - \alpha D^{-1}\beta - \alpha\beta\alpha D^{-1}\beta \\ &= 1 + \alpha\beta - \alpha(I+\beta\alpha)D^{-1}\beta = 1 + \alpha\beta - \alpha\beta = 1,\end{aligned}$$

hence $\lambda \neq 0$, that is the assertion holds.

(iii) From (i) and (ii) we obtain $\mathrm{span}(A_0) \subseteq \mathrm{span}(A)$, the other inclusion is trivial by assumption:
$\binom{y}{x} \in \mathrm{span}(A) \Rightarrow y \in J(A) = (a)$ and $x \in \mathcal{M}(A) = \mathrm{span}(B)$, hence, $\binom{y}{x} = \binom{y}{0} + \binom{0}{x} \in \mathrm{span}(A_0)$.

Take any monomial ordering in R and a module ordering first by components, then a reduced standard base of A coincides with the union of reduced standard bases of $\binom{a}{0}$ and $\binom{0}{b}$, hence base a 1 row block form.

Define $J_\ell(A) = J(P_{1\ell}A)$ and $\mathcal{M}_\ell(A) = J(P_{1\ell}A)$. As a consequence of 6.1 – 6.3 we obtain

Proposition 6.4
If $J_\ell(A)$ for $\ell = 1, \ldots, m$ and $\mathcal{M}_\ell(A)$ are minimal in the set of all $J_\ell(UA)$, $U \in \mathbf{G}$, respectively $\mathcal{M}_\ell(V_\beta A)$, $V_\beta \in \mathbf{U}_-$ and the reduced standard basis of span(A) does not have a 1 row block, then coker(A) does not have a rank 1 summand.

Corollary 6.5
If the assumptions of 6.4 are fulfilled for $A, \Lambda^2 A, \ldots, \Lambda A^s$, $s \leq [\frac{r}{2}]$, then coker(A) is indecomposable.

Proof of 6.4: We may assume $A \sim U_\alpha V_\beta A_0$, $U_\alpha V_\beta \in \mathbf{G}$. Then $J(A) = J\left(\left(\begin{smallmatrix} 1+\alpha\beta & \alpha \\ \beta & I \end{smallmatrix}\right) A_0\right) = \lambda(\mathbf{a}) + (\alpha B)$, where $\lambda = 1 + \alpha\beta \neq 0$ by assumption, hence, $J(A_0) \subset J(A)$. On the other hand

$$J(A) \subseteq J(U_{-\alpha}A) = J(V_\beta A_0) = J(A_0) = (a).$$

The same argument will work, showing $\mathcal{M}(A) = \mathcal{M}(A_0)$. Using 6.3 (iii) we are done. $\square$

Next we want to replace A by UA, $U \in \mathbf{G}$, such that $J(A)$ becomes smaller. We may assume $U = U_\alpha \in \mathbf{U}_+$. Then $J(U_\alpha A)$ is generated by $(a_{11} + b_{11}, \ldots, a_{1n} + b_{1n})$, where $b_{1j} = \sum_{i=2}^{r} \alpha_i a_{ij}$, that is $\mathbf{b}$ is a linear combination of the rows $\mathbf{z}_i$, $i = 2, \ldots, r$, of A. Moreover, $J(U_\alpha A) \subseteq J(A)$ if and only if $b_{ij} \in J(A)$ for $j = 1, \ldots, n$, that is α belongs to a certain syzygy module of the rows of A, hence we obtain:

Lemma 6.6
Denote by $\mathbf{z}_i$ the i–th row of A, then $J(U_\alpha A) \subseteq J(A)$ if and only if $\alpha \in \mathcal{N}(A)$, $\mathcal{N}(A) := \{\alpha | \alpha_2 \mathbf{z}_2 + \cdots + \alpha_r \mathbf{z}_r \equiv 0 \mod J(A)\}$.

We wish to decide whether there is an $\alpha \in \mathcal{N}(A)$, such that the inclusion $J(U_\alpha A) \subset J(A)$ is strict.

Lemma 6.7
Let $\alpha \in N(A)$, then $J(U_\alpha A) = J(A)$, if $\alpha \equiv 0 \mod J(A)$ or $\alpha \in m \cdot \mathcal{N}(A)$.

Proof: We may assume all entries of A are in the maximal ideal m, because otherwise A obviously splits. From the assumption on α we obtain $J(A) \subseteq J(U_\alpha A) + mJ(A)$ and we apply Nakayama's lemma. $\square$

A sufficient criterion for the minimality of $J(A)$ will be as follows: we may replace A by an equivalent matrix, such that the first k non–zero entries of the first row form a minimal base of $J(A)$. Denote by $J^{(e)}(A)$ the ideal generated by

$$a_{11}, \ldots, a_{1\ell-1}, ma_{1\ell}, \ldots, ma_{1k}.$$

Associate to $\alpha \in \mathcal{N}(A)$ the first row of $U_\alpha A - A : b^{(\alpha)} = (b_1^{(\alpha)}, \ldots, b_n^{(\alpha)})$.

Lemma 6.8
If for all entries α from a base B of $\mathcal{N}(A)$ holds: $b_\ell^{(\alpha)} \in J^{(\ell)}(A)$; $\ell = 1, \ldots, k$, then $J(A)$ is minimal.

Proof: If there exists $\alpha \in \mathcal{N}(A)$ such that $J(U_\alpha A) \subsetneq J(A)$, where $\alpha = \sum\limits_{\alpha^{(j)} \in B} \lambda_j \alpha^{(j)}$, then we obtain a strict inclusion of the first "row" ideals already for some summands $\lambda_j \alpha^{(j)}$ and $c = \lambda_j(0) \neq 0$ by 6.7: $J(U_{c\alpha^{(j)}} A) \subsetneq J(A)$.
There exists the smallest index ℓ, such that

$$a_{1\ell} \notin (a - cb^{(\alpha_j)}),$$

the ideal of the first row of $U_{c\alpha^{(j)}} A$.
Computing modulo $J^{(\ell)}(A)$ we have by assumption

$$a_{1i} - cb_i^{(\alpha_j)} \equiv \begin{cases} 0 & i < \ell \\ a_{1i} & i \geq \ell, \end{cases}$$

hence we have a contradiction, knowing $a_{1\ell} \notin J^{(\ell)}(A)$:

$$0 \neq \overline{a}_{1\ell} \notin (a - cb^{(\alpha_j)}) \bmod J^{(\ell)}(A) = (a_{1\ell}, \ldots, a_{1n}) \bmod J^{(\ell)}(A)!$$

Remarks 6.9

1. A similar sufficient criterion could be stated for the minimality of $\mathcal{M}(A)$.
2. In order to minimize $J(A)$ we have to use certain constant multiples of the base elements $\alpha^{(j)}$ from $J(A)$. These reductions of the first row could be found by certain standard base computations. The reduction process is finite, because the regularity of the Hilbert functions of $J(A)$, $A \in$ Pres(M) is bounded by the regularity of hilb(M), (for details and the following algorithm cf. [MS]).
3. There is an algorithm for splitting rank 1 summands, based on the following idea: start with any $A \in$ Pres(M);
   (a) replace A by $U_\alpha A$, such that $J(A)$ becomes minimal,
   (b) replace A by a standard basis and sort the columns to obtain the following block form:
   $$A = \left(\begin{array}{c|c} a & 0 \\ \hline c & b \end{array}\right), a \in M(1 \times k),$$
   being the base of $J(A)$.

(c) test if the equation $c - \beta a \equiv 0 \bmod (b)$ has a solution $\beta \in R^{r-1}$. If the answer is "yes", the $V_\beta A \sim \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and M splits.
If the answer is "no", then there is no matrix of type A_0, such that $A_0 \sim UA$, $U \in \mathbf{G}$, by 6.3 (i).
We repeat the algorithm for the next rows, that is we start with $A = P_{1k}A$, $k = 2, 3, \ldots$.

Examples 6.10
The following matrices present indecomposable modules, checked by applying 6.4:

(i) $A = \begin{pmatrix} x & -y & 0 \\ 0 & x^2 & -y \\ y & 0 & x^2 \end{pmatrix}$.

- *$J(A) = (x, y)$ is minimal because we have $\mathcal{N}(A) = R^2$ and $(x + \mu y, -y + \lambda x^2, -\mu y + \lambda x^2) = (x, y)$.*
- *$\mathcal{M}(A) = span\{\binom{0}{y}, \binom{x^2}{0}, \binom{-y}{x^2}\}$ is minimal: Let $\mathcal{N}'(A) := \{\binom{\beta_1}{\beta_2} | \mathcal{M}(V_\beta A) \subseteq \mathcal{M}(A)\}$. Obviously $\mathcal{M}(A) \subseteq \mathcal{N}'(A)$. If we check equality, we obtain $\mathcal{M}(A) \subseteq \mathcal{M}(V_\beta A) + m\mathcal{M}(A)$ for $\beta \in \mathcal{N}'(A)$, hence, $\mathcal{M}(A) = \mathcal{M}(V_\beta A)$, (6.7).*
If $\binom{\beta_1}{\beta_2} \in \mathcal{N}'(A)$ $y\binom{\beta_1}{\beta_2}$, $x\binom{\beta_1}{\beta_2} \in \mathcal{M}(A)$.
Let $y\binom{\beta_1}{\beta_2} = \lambda_1\binom{0}{y} + \lambda_2\binom{x^2}{0} + \lambda_3\binom{-y}{x^2}$, then $\lambda_2 = \lambda_2' \cdot y$, $\lambda_3 = \lambda_3' y$ and $\beta_1 = \lambda_2' x^2 - \lambda_3' y$, $\beta_2 = \lambda_1 + \lambda_3' x^2$.
Because $x\binom{\beta_1}{\beta_2} \in \mathcal{M}(A)$, we obtain
$$\begin{array}{rll} \textit{for} & \lambda_1 = y, \lambda_2 = \lambda_3 = 0 & \beta = \binom{0}{y} \\ \textit{for} & \lambda_2' = 1, \lambda_1 = \lambda_3 = 0 & \beta = \binom{x^2}{0} \\ \textit{and for} & \lambda_3' = 1, \lambda_1 = \lambda_1 = 0 & \beta = \binom{-y}{x^2}. \end{array}$$
- *The same is checked by similar computations for $P_{12}A$ and $P_{13}A$.*
- *The columns of A form a reduced standard basis and do not have a block form, hence by 6.4 coker(A) is indecomposable.*

(ii) $A = \begin{pmatrix} x & 0 & y \\ y & x & 0 \\ 0 & y & x^t \end{pmatrix}$.

One easily checks the indecomposability as in the first example.

(iii) $A = \begin{pmatrix} y+x^5 & -xy(1-x-\lambda x^2) \\ xy & x^9 - x^4 y \end{pmatrix}$.

Reduce the 2nd column of A by the first one, then
$A \sim A' = \begin{pmatrix} x^5+y & \varepsilon x^6 \\ xy & x^2 y \varepsilon' + x^9 \end{pmatrix}$,
$\varepsilon = (-1 + x + \lambda x^2)$, $\varepsilon' = (1 - x - x^2 - \lambda x^2)$.

To apply 6.4 we have only to check that the two "row" ideals of A' are minimal. But $J(A) = J_1(A) = (x^5 + y, x^6), \mathcal{N}(A) = R$ and $(x^5 + y + \lambda xy, x^6\varepsilon + \lambda x^2 y \varepsilon' + \lambda x^9) = (x^5 + y, x^6)$; and $J_2(A) = J(P_{12}A) = (xy, x^9)$ and $N_2(A) := \mathcal{N}(P_{12}A) = (x^4)$ and $(xy + \lambda(x^9 + x^4 y), x^2 y \varepsilon' + x^9 + \lambda x^{10}\varepsilon) = (xy, x^9)$.

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